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An Iteratively Regularized Projection Method with Quadratic Convergence for Nonlinear Ill-posed Problems

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Abstract

An iteratively regularized projection method, which converges quadratically, has been considered for obtaining stable approximate solution to nonlinear ill-posed operator equations $F(x) = y$ where $F : D(F) \subseteq X \longrightarrow X$ is a nonlinear monotone operator defined on the real Hilbert space X. We assume that only a noisy data y^δ with $\|y - y^\delta\| \leq \delta$ are available. Under the assumption that the Fréchet derivative F' of F is Lipschitz continuous, a choice of the regularization parameter using an adaptive selection of the parameter and a stopping rule for the iteration index using a majorizing sequence are presented. We prove that under a general source condition on $x_0 - \hat{x}$, the error $\|x_{n,\alpha}^{h,\delta} - \hat{x}\|$ between the regularized approximation $x_{n,\alpha}^{h,\delta}, (x_{0,\alpha}^{h,\delta} := P_h x_0$ where P_h is an orthogonal projection on to a finite dimensional subspace X_h of $X)$ and the solution $\hat{x}$ is of optimal order.

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1 Introduction

Let $F : D(F) \subseteq X \mapsto X$ be a nonlinear monotone operator (see [13]) defined on a real Hilbert space X. We consider the problem of approximately solving the nonlinear ill-posed operator equation;

$$F(x) = y. \tag{1}$$

Throughout this paper we shall denote the inner product and the corresponding norm on X by $\langle ., . \rangle$ and $\|.\|$ respectively. And we assume that (1) has a solution, namely $\hat{x}$ and that $y^\delta \in X$ are the available noisy data with

$$\|y - y^\delta\| \leq \delta. \tag{2}$$

Then the problem of recovery of $\hat{x}$ from noisy equation $F(x) = y^\delta$ is ill-posed, in the sense that the Fr$\acute{e}$chet derivative $F'(.)$ is not boundedly invertible (see [1], page 26).

A well known method for regularizing (1), when F is monotone is the method of Lavrentiev regularization (see [13]). In this method approximation x_α^δ is obtained by solving the singularly perturbed operator equation

$$F(x) + \alpha(x - x_0) = y^\delta. \tag{3}$$

In practice, one has to deal with some sequence $(x_{n,\alpha}^\delta)_{n=1}^\infty$, converging to the solution x_α^δ of (3). Many authors considered such sequences (see [2, 3, 6, 7, 9, 8]).

In [11], George and Elmahdy considered an iterative regularization method;

$$x_{n+1,\alpha}^\delta = x_{n,\alpha}^\delta - (F'(x_{n,\alpha}^\delta) + \alpha I)^{-1}(F(x_{n,\alpha}^\delta) - y^\delta + \alpha(x_{n,\alpha}^\delta - x_0)), \tag{4}$$

where $x_{0,\alpha}^\delta = x_0$ and proved that (4) converges quadratically to the unique solution x_α^δ of (3).

Recall that, a sequence (x_n) is said to be converges quadratically to x^*, if there exist a positive number M, not necessarily less than 1, such that

$$\|x_{n+1} - x^*\| \leq M\|x_n - x^*\|^2,$$

for all n sufficiently large. And the convergence of (x_n) to x^*, is said to be linear if there exist a positive number $M_0 \in (0, 1)$, such that

$$\|x_{n+1} - x^*\| \leq M_0\|x_n - x^*\|.$$

Note that regardless of the value of M quadratic convergent sequence will always eventually converges faster than a linearly convergent sequence.

The following Assumptions are used for proving the results in [11] as well as the results in this paper.

Assumption 1.1 *There exists* $r_0 > 0$ *such that* $B_{r_0}(\hat{x}) \subseteq D(F)$ *and* F *is Fréchet differentiable at all* $x \in B_{r_0}(\hat{x})$.

Assumption 1.2 *There exists a constant* $k_0 > 0$ *such that for every* $x, u \in B_{r_0}(\hat{x})$ *and* $v \in X$, *there exists an element* $\Phi(x, u, v) \in X$ *satisfying*

$$[F'(x) - F'(u)]v = F'(u)\Phi(x, u, v), \quad \|\Phi(x, u, v)\| \le k_0\|v\|\|x - u\|$$

for all $x, u \in B_{r_0}(\hat{x})$ *and* $v \in X$.

Assumption 1.3 *There exists a continuous, strictly monotonically increasing function* $\varphi : (0, a] \to (0, \infty)$ *with* $a \ge \|F'(\hat{x})\|$ *satisfying* $\lim_{\lambda \to 0} \varphi(\lambda) = 0$ *and* $v \in X$ *with* $\|v\| \le 1$ *such that*

$$x_0 - \hat{x} = \varphi(F'(\hat{x}))v$$

and

$$\sup_{\lambda \ge 0} \frac{\alpha\varphi(\lambda)}{\lambda + \alpha} \le c_\varphi \varphi(\alpha), \quad \forall \alpha \in (0, a].$$

The analysis in [11] as well as in this paper is based on majorizing sequences. Recall (see [5], Definition 1.3.11) that a nonnegative sequence (t_n) is said to be a majorizing sequence of a sequence (x_n) in X if

$$\|x_{n+1} - x_n\| \le t_{n+1} - t_n, \quad \forall n \ge 0.$$

The majorizing sequence gives an a priori error estimate which can be used to determine the number of iterations needed to achieve a prescribed solution accuracy before actual computation take place.

The plan of this paper is as follows. In section 2 we collect some results from [11] for the error estimates in this paper. In section 3 we considered an iteratively regularized projection method for obtaining a sequence $(x_{n,\alpha}^{h,\delta})$ in a finite dimensional subspace X_h of X and proved that $x_{n,\alpha}^{h,\delta}$ converges to x_α^δ. Also in section 3 we obtained an estimate for $\|x_{n,\alpha}^{h,\delta} - x_\alpha^\delta\|$.

Using an error estimate for $\|x_\alpha^\delta - \hat{x}\|$ (see [11, 13]), we obtained an estimate for $\|x_{n,\alpha}^{h,\delta} - \hat{x}\|$ in section 4. The error analysis for the order optimal result using an adaptive selection of the parameter α and a stopping rule using a majorizing sequence are also given in section 4. Implementation of the adaptive choice of the parameter and the choice of the stopping rule are given in section 5. Finally the paper ends with some concluding remarks in section 6.

2 Preliminaries

In [11] the following majorizing sequence (t_n) defined iteratively by, $t_0 = 0, t_1 = \eta$, and

$$t_{n+1} = t_n + \frac{3k_0}{2}(t_n - t_{n-1})^2 \tag{5}$$

where $k_0, \eta,$and $q \in [0, 1)$ are nonnegative numbers such that

$$\frac{3k_0}{2}\eta \leq q \tag{6}$$

were used for proving the quadratic convergence of the sequence $(x^{\delta}_{n,\alpha})$ to the unique solution x^{δ}_{α} of equation (3).

For proving the results in [11] as well as the results in this paper we use the following Lemma on majorization, which is a reformulation of Lemma 1.3.12 in [5].

Lemma 2.1 *Let (t_n) be a majorizing sequence for (x_n). If $\lim_{n\to\infty} t_n = t^*$, then $x^* = \lim_{n\to\infty} x_n$ exists and*

$$\|x^* - x_n\| \leq t^* - t_n, \quad \forall n \geq 0. \tag{7}$$

The following Lemma based on the Assumption 1.2 will be used in due course.

Lemma 2.2 *For $u, v \in B_{r_0}(x_0)$*

$$F(u) - F(v) - F'(u)(u - v) = F'(u)\int_0^1 \Phi(v + t(u - v), u, u - v)dt.$$

Proof. *Using the Fundamental Theorem of Integration, for $u, v \in B_{r_0}(x_0)$ we have*

$$F(u) - F(v) = \int_0^1 F'(v + t(u - v))(u - v)dt$$

and

$$F'(u)(u - v) = \int_0^1 F'(u)(u - v)dt$$

so,

$$F(u) - F(v) - F'(u)(u - v) = \int_0^1 [F'(v + t(u - v)) - F'(u)](u - v)dt$$

so by Assumption 1.2 we have

$$F(u) - F(v) - F'(u)(u - v) = F'(u)\int_0^1 \Phi(v + t(u - v), u, u - v)dt.$$

This completes the proof of the Lemma.

Here after we assume that $\|x_0 - \hat{x}\| \leq \rho$ and

$$\frac{k_0}{2}\rho^2 + \rho + \frac{\delta}{\alpha} \leq \eta \leq min\{\frac{2q}{3k_0}, r_0(1 - q)\}. \tag{8}$$

Theorem 2.3 *([11], Theorem 2.1) Suppose Assumption 1.2 holds. Let* $0 < t^* \leq \frac{\eta}{1-q} := t^{**}$ *where* η *as in (8) and let (5) and (6) be satisfied. Then the sequence* $(x^\delta_{n,\alpha})$ *defined in (4) is well defined and* $x^\delta_{n,\alpha} \in B_{t^*}(x_0)$ *for all* $n \geq 0$*. Further* $(x^\delta_{n,\alpha})$ *is a Cauchy sequence in* $B_{t^*}(x_0)$ *and hence converges to* $x^\delta_\alpha \in \overline{B_{t^*}(x_0)} \subset B_{t^{**}}(x_0)$ *and* $F(x^\delta_\alpha) = y^\delta + \alpha(x_0 - x^\delta_\alpha)$*.*

Moreover, the following estimate hold for all $n \geq 0$*,*

$$\|x^\delta_{n+1,\alpha} - x^\delta_{n,\alpha}\| \leq t_{n+1} - t_n, \tag{9}$$

$$\|x^\delta_{n,\alpha} - x^\delta_\alpha\| \leq t^* - t_n \leq \frac{q^n\eta}{1 - q}, \tag{10}$$

and

$$\|x^\delta_{n+1,\alpha} - x^\delta_\alpha\| \leq \frac{3k_0}{2}\|x^\delta_{n,\alpha} - x^\delta_\alpha\|^2. \tag{11}$$

Remark 2.4 *Note that (11) implies* $(x^\delta_{n,\alpha})$ *converges quadratically to* x^δ_α*.*

3 Iteratively Regularized Projection Method

Let H be a bounded subset of positive reals such that zero is a limit point of H, and let $\{P_h\}_{h\in H}$ be a family of orthogonal projections from X into itself. We assume that

$$b_h := \|(I - P_h)x_0\| \to 0 \tag{12}$$

as $h \to 0$. The above assumption is satisfied if $P_h \to I$ pointwise. Let

$$x^{h,\delta}_{n+1,\alpha} = x^{h,\delta}_{n,\alpha} - (P_hF'(x^{h,\delta}_{n,\alpha}) + \alpha I)^{-1}P_h(F(x^{h,\delta}_{n,\alpha}) - y^\delta + \alpha(x^{h,\delta}_{n,\alpha} - x_0)), \tag{13}$$

where $x^{h,\delta}_{0,\alpha} := P_hx_0$.

Let

$$\Gamma_{n,h} := \|(I - P_h)F'(x^\delta_{n,\alpha})\|, \tag{14}$$

$$\varrho := \|F'(P_hx_0)\|,$$

$$\gamma_{n,h} := \|F'(x_{n,\alpha}^{h,\delta})(I - P_h)\| \tag{15}$$

and let $(\tilde{t}_{n,h}), n \geq 0$ be defined iteratively by $\tilde{t}_{0,h} = 0, \tilde{t}_{1,h} = \eta_h > 0$,

$$\tilde{t}_{n+1,h} = \tilde{t}_{n,h} + (1 + \frac{k_0 \varrho \tilde{t}_{n,h} + \gamma_{0,h}}{\alpha})\frac{3k_0}{2}(\tilde{t}_{n,h} - \tilde{t}_{n-1,h})^2 \tag{16}$$

where k_0, α, η_h and $r_h \in [0, 1)$ are nonnegative numbers.

Lemma 3.1 *Assume there exist nonnegative numbers k_0, α, η_h and $\tilde{r}_h \in [0, \frac{1}{1+\frac{3k_0^2}{2\alpha}\varrho\tilde{\eta}_h^2})$ such that for all $n \geq 0$*

$$(1 + \frac{k_0 \varrho \eta_h + \gamma_{0,h}}{\alpha})\frac{3k_0}{2}\eta_h \leq r_h. \tag{17}$$

Then the sequence $(\tilde{t}_{n,h})$ defined in (16) is increasing, bounded above by $\tilde{t_h}^{**} := \frac{\eta_h}{1-r_h}$, and converges to some $\tilde{t_h}^{*}$ such that $0 < \tilde{t_h}^{*} \leq \frac{\eta_h}{1-r_h}$. Moreover, for $n \geq 0$;

$$0 \leq \tilde{t}_{n+1,h} - \tilde{t}_{n,h} \leq r_h(\tilde{t}_{n,h} - \tilde{t}_{n-1,h}) \leq r_h^n \eta_h, \tag{18}$$

and

$$\tilde{t_h}^{*} - \tilde{t}_{n,h} \leq \frac{r_h^n}{1 - r_h}\eta_h. \tag{19}$$

Proof. Since the result is true for $\eta_h = 0, k_0 = 0$ or $r_h = 0$, we assume that $\eta_h \neq 0, k_0 \neq 0$ and $r_h \neq 0$. Observe that $\tilde{t}_{i,h} - \tilde{t}_{i-1,h} \geq 0$ for all $i \geq 1$. If

$$(1 + \frac{k_0 \varrho \tilde{t}_{i,h} + \gamma_{0,h}}{\alpha})\frac{3k_0}{2}(\tilde{t}_{i,h} - \tilde{t}_{i-1,h}) \leq r_h \tag{20}$$

then the estimate (18) follows from (16). Thus we shall prove (20) by induction on $i \geq 1$.

For $i = 1$, (20) holds by (17). Suppose (20) holds for all $i \leq k$ for some k. Then by (16) we have

$$\begin{aligned}
(1 + \frac{k_0 \varrho \tilde{t}_{k+1,h} + \gamma_{0,h}}{\alpha})\frac{3k_0}{2}(\tilde{t}_{k+1,h} - \tilde{t}_{k,h}) &\leq (1 + \frac{k_0 \varrho \tilde{t}_{k+1,h} + \gamma_{0,h}}{\alpha})\frac{3k_0}{2} \\
&\quad \times (1 + \frac{k_0 \varrho \tilde{t}_{k,h} + \gamma_{0,h}}{\alpha})\frac{3k_0}{2}(\tilde{t}_{k,h} - \tilde{t}_{k-1,h})^2 \\
&\leq (1 + \frac{k_0 \varrho (\tilde{t}_{k+1,h} - \tilde{t}_{k,h} + \tilde{t}_{k,h}) + \gamma_{0,h}}{\alpha})\frac{3k_0}{2} \\
&\quad \times (1 + \frac{k_0 \varrho \tilde{t}_{k,h} + \gamma_{0,h}}{\alpha})\frac{3k_0}{2}(\tilde{t}_{k,h} - \tilde{t}_{k-1,h})^2 \\
&\leq [(1 + \frac{k_0 \varrho \tilde{t}_{k,h} + \gamma_{0,h}}{\alpha})\frac{3k_0}{2}(\tilde{t}_{k,h} - \tilde{t}_{k-1,h})]^2
\end{aligned}$$

$$
\begin{aligned}
&+\frac{k_0\varrho(\tilde{t}_{k+1,h}-\tilde{t}_{k,h})}{\alpha}\frac{3k_0}{2} \\
&\times(1+\frac{k_0\varrho\tilde{t}_{k,h}+\gamma_{0,h}}{\alpha})\frac{3k_0}{2}(\tilde{t}_{k,h}-\tilde{t}_{k-1,h})^2 \\
\leq\ & [(1+\frac{k_0\varrho\tilde{t}_{k,h}+\gamma_{0,h}}{\alpha})\frac{3k_0}{2}(\tilde{t}_{k,h}-\tilde{t}_{k-1,h})]^2 \\
&+\frac{3k_0^2\varrho}{2\alpha}[(1+\frac{k_0\varrho\tilde{t}_{k,h}+\gamma_{0,h}}{\alpha})\frac{3k_0}{2}(\tilde{t}_{k,h}-\tilde{t}_{k-1,h})^2]^2 \\
\leq\ & r_h^2+\frac{3k_0^2\varrho}{2\alpha}r_h^2(\tilde{t}_{k,h}-\tilde{t}_{k-1,h})^2 \\
\leq\ & r_h^2+\frac{3k_0^2\varrho}{2\alpha}r_h^2(r_h^{k-1}\eta_h)^2 \\
\leq\ & r_h. \qquad (21)
\end{aligned}
$$

Thus by induction (20) holds for all $i \geq 1$ and for $k \geq 0$,

$$
\begin{aligned}
\tilde{t}_{k+1,h} \leq \tilde{t}_{k,h}+r_h(\tilde{t}_{k,h}-\tilde{t}_{k-1,h}) &\leq \cdots \leq \eta_h+r_h\eta_h+\cdots+r_h^k\eta_h \qquad (22)\\
&= \frac{1-r_h^{k+1}}{1-r_h}\eta_h < \frac{\eta_h}{1-r_h} \qquad (23)
\end{aligned}
$$

Hence the sequence $(\tilde{t}_{n,h}), n \geq 0$ is bounded above by $\frac{\eta_h}{1-r_h}$ and nondecreasing, so it converges to some $t_h^* \leq \frac{\eta_h}{1-r_h}$. Further

$$
t_h^*-\tilde{t}_{n,h} = \lim_{i\to\infty}\tilde{t}_{n+i,h}-\tilde{t}_{n,h} \leq \lim_{i\to\infty}\sum_{j=0}^{i-1}(\tilde{t}_{n+1+j,h}-\tilde{t}_{n+j,h}) \leq \frac{r_h^n\eta_h}{1-r_h}.
$$

This completes the proof of the Lemma.

Hereafter we assume;

$$
\begin{aligned}
(1+\frac{\gamma_{0,h}}{\alpha})(\frac{k_0}{2}(b_h+\rho)^2+b_h+\rho)+\frac{\delta}{\alpha} &\leq \eta_h \\
&\leq min\{C, r_0(1-r_h)\} \qquad (24)
\end{aligned}
$$

where $C := \frac{1}{2k_0\varrho}[-(\alpha+\gamma_{0,h})+\sqrt{(\alpha+\gamma_{0,h})^2+\frac{8\alpha\varrho\tilde{r}_h}{3}}]$.

Remark 3.2 *We need the above assumption because:*

- *The assumption (24) implies* $t_h^{**} = \frac{\eta_h}{1-r_h} \leq r_0$ *and hence we can apply Assumption 1.1.*

- *Equation (17) implies*

$$
\tilde{\eta}_h \leq \frac{1}{2k_0\varrho}[-(\alpha+\gamma_{0,h})+\sqrt{(\alpha+\gamma_{0,h})^2+\frac{8\alpha\varrho\tilde{r}_h}{3}}] = C. \qquad (25)
$$

Theorem 3.3 *Let the assumptions in Lemma 3.1 with η_h as in (24) and Assumption 1.2 be satisfied. Then the sequence $(\tilde{t}_{n,h})$ defined in (16) is a majorizing sequence of sequence $(x_{n,\alpha}^{h,\delta})$ defined in (13) and $x_{n,\alpha}^{h,\delta} \in B_{\tilde{t_h}^*}(P_h x_0)$ for all $n \geq 0$.*

Proof. Let

$$G(x) = x - R_\alpha(x)^{-1}[F(x) - y^\delta + \alpha(x - x_0)]$$

where $R_\alpha(x)^{-1} = (P_h F'(x) P_h + \alpha P_h)^{-1}$. Then for $u, v \in B_{\tilde{t_h}^*}(P_h x_0)$, we have

$$\begin{aligned}
G(u) - G(v) &= u - v - R_\alpha(u)^{-1}[F(u) - y^\delta + \alpha(u - x_0)] \\
&\quad + R_\alpha(v)^{-1}[F(v) - y^\delta + \alpha(v - x_0)] \\
&= R_\alpha(u)^{-1}[R_\alpha(u)(u - v) - (F(u) - F(v)) - \alpha(u - v)] \\
&\quad + (R_\alpha(v)^{-1} - R_\alpha(u)^{-1})(F(v) - y^\delta + \alpha(v - x_0)) \\
&= R_\alpha(u)^{-1}[P_h F'(u) P_h(u - v) - (F(u) - F(v))] \\
&\quad + R_\alpha(u)^{-1}(R_\alpha(u) - R_\alpha(v)) R_\alpha(v)^{-1}(F(v) - y^\delta + \alpha(v - x_0)) \\
&= R_\alpha(u)^{-1}[P_h F'(u) P_h(u - v) - (F(u) - F(v))] \\
&\quad - R_\alpha(u)^{-1}(R_\alpha(u) - R_\alpha(v))(G(v) - v).
\end{aligned}$$

Now since $G(x_{n,\alpha}^{h,\delta}) = x_{n+1,\alpha}^{h,\delta}$, $R_\alpha(x)^{-1} = R_\alpha(x)^{-1} P_h = P_h R_\alpha(x)^{-1}$ and $P_h(x_{n,\alpha}^{h,\delta} - x_{n-1,\alpha}^{h,\delta}) = (x_{n,\alpha}^{h,\delta} - x_{n-1,\alpha}^{h,\delta})$ we have

$$\begin{aligned}
(x_{n+1,\alpha}^{h,\delta} - x_{n,\alpha}^{h,\delta}) &= G(x_{n,\alpha}^{h,\delta}) - G(x_{n-1,\alpha}^{h,\delta}) \\
&= R_\alpha(x_{n,\alpha}^{h,\delta})^{-1}[F'(x_{n,\alpha}^{h,\delta})(x_{n,\alpha}^{h,\delta} - x_{n-1,\alpha}^{h,\delta}) \\
&\quad - (F(x_{n,\alpha}^{h,\delta}) - F(x_{n-1,\alpha}^{h,\delta}))] \\
&\quad - R_\alpha(x_{n,\alpha}^{h,\delta})^{-1}(R_\alpha(x_{n,\alpha}^{h,\delta}) - R_\alpha(x_{n-1,\alpha}^{h,\delta}))(x_{n,\alpha}^{h,\delta} - x_{n-1,\alpha}^{h,\delta}) \\
&= R_\alpha(x_{n,\alpha}^{h,\delta})^{-1} F'(x_{n,\alpha}^{h,\delta}) \\
&\quad \times \int_0^1 \Phi(x_{n,\alpha}^{h,\delta} + t(x_{n-1,\alpha}^{h,\delta} - x_{n,\alpha}^{h,\delta}), x_{n,\alpha}^{h,\delta}, x_{n-1,\alpha}^{h,\delta} - x_{n,\alpha}^{h,\delta}) dt \\
&\quad - R_\alpha(x_{n,\alpha}^{h,\delta})^{-1}(F'(x_{n,\alpha}^{h,\delta}) - F'(x_{n-1,\alpha}^{h,\delta}))(x_{n,\alpha}^{h,\delta} - x_{n-1,\alpha}^{h,\delta}) \\
&= R_\alpha(x_{n,\alpha}^{h,\delta})^{-1} F'(x_{n,\alpha}^{h,\delta}) \\
&\quad \times \int_0^1 \Phi(x_{n,\alpha}^{h,\delta} + t(x_{n-1,\alpha}^{h,\delta} - x_{n,\alpha}^{h,\delta}), x_{n,\alpha}^{h,\delta}, x_{n-1,\alpha}^{h,\delta} - x_{n,\alpha}^{h,\delta}) dt \\
&\quad + R_\alpha(x_{n,\alpha}^{h,\delta})^{-1} F'(x_{n,\alpha}^{h,\delta}) \int_0^1 \Phi(x_{n-1,\alpha}^{h,\delta}, x_{n,\alpha}^{h,\delta}, x_{n,\alpha}^{h,\delta} - x_{n-1,\alpha}^{h,\delta}) dt \\
&= R_\alpha(x_{n,\alpha}^{h,\delta})^{-1}[F'(x_{n,\alpha}^{h,\delta}) P_h + F'(x_{n,\alpha}^{h,\delta})(I - P_h)] \\
&\quad \times \int_0^1 \Phi(x_{n,\alpha}^{h,\delta} + t(x_{n-1,\alpha}^{h,\delta} - x_{n,\alpha}^{h,\delta}), x_{n,\alpha}^{h,\delta}, x_{n-1,\alpha}^{h,\delta} - x_{n,\alpha}^{h,\delta}) dt \\
&\quad + R_\alpha(x_{n,\alpha}^{h,\delta})^{-1}[F'(x_{n,\alpha}^{h,\delta}) P_h + F'(x_{n,\alpha}^{h,\delta})(I - P_h)]
\end{aligned}$$

$$\times \int_0^1 \Phi(x_{n-1,\alpha}^{h,\delta}, x_{n,\alpha}^{h,\delta}, x_{n,\alpha}^{h,\delta} - x_{n-1,\alpha}^{h,\delta})dt$$

The last but one step follows from Lemma 2.2. So by Assumption 1.2 and the relation

$$\|R_\alpha(x_{n,\alpha}^{h,\delta})^{-1}[F'(x_{n,\alpha}^{h,\delta})P_h + F'(x_{n,\alpha}^{h,\delta})(I-P_h)]\| \leq 1 + \frac{\gamma_{n,h}}{\alpha}, \tag{26}$$

we have

$$\|x_{n+1,\alpha}^{h,\delta} - x_{n,\alpha}^{h,\delta}\| \;\; \leq \;\; (1 + \frac{\gamma_{n,h}}{\alpha})\frac{3k_0}{2}\|x_{n,\alpha}^{h,\delta} - x_{n-1,\alpha}^{h,\delta}\|^2. \tag{27}$$

Note that

$$\begin{aligned}
\gamma_{n,h} &= \|F'(x_{n,\alpha}^{h,\delta})(I-P_h)\| \\
&= \|[F'(x_{n,\alpha}^{h,\delta}) - F'(P_hx_0) + F'(P_hx_0)](I-P_h)\| \\
&\leq \|F'(P_hx_0)(I-P_h)\| + \|[F'(x_{n,\alpha}^{h,\delta}) - F'(P_hx_0)](I-P_h)\| \\
&\leq \gamma_{0,h} + k_0\|F'(P_hx_0)\|\|x_{n,\alpha}^{h,\delta} - P_hx_0\| \\
&\leq \gamma_{0,h} + k_0\varrho\|x_{n,\alpha}^{h,\delta} - P_hx_0\|. \qquad (28)
\end{aligned}$$

So by (27) we have

$$\|x_{n+1,\alpha}^{h,\delta} - x_{n,\alpha}^{h,\delta}\| \;\; \leq \;\; (1 + \frac{\gamma_{0,h} + k_0\varrho\|x_{n,\alpha}^{h,\delta} - P_hx_0\|}{\alpha})\frac{3k_0}{2}\|x_{n,\alpha}^{h,\delta} - x_{n-1,\alpha}^{h,\delta}\|^2 \tag{29}$$

Now we shall prove that the sequence $(\tilde{t}_{n,h})$ defined in (16) is a majorizing sequence of the sequence $(x_{n,\alpha}^{h,\delta})$ and $x_{n,\alpha}^{h,\delta} \in B_{\tilde{t_h}^*}(P_hx_0)$, for all $n \geq 0$. Note that $F(\hat{x}) = y$, so

$$\begin{aligned}
\|x_{1,\alpha}^{h,\delta} - P_hx_0\| &= \|(P_hF'(P_hx_0)P_h + \alpha P_h)^{-1}P_h(F(P_hx_0) - y^\delta)\| \\
&= \|(P_hF'(P_hx_0)P_h + \alpha P_h)^{-1}P_h(F(P_hx_0) - y + y - y^\delta)\| \\
&= \|(P_hF'(P_hx_0)P_h + \alpha P_h)^{-1}P_h(F(P_hx_0) - F(\hat{x}) + y - y^\delta)\| \\
&= \|(P_hF'(P_hx_0)P_h + \alpha P_h)^{-1}P_h(F(P_hx_0) - F(\hat{x}) \\
&\quad -F'(P_hx_0)(P_hx_0 - \hat{x}) + F'(P_hx_0)(P_hx_0 - \hat{x}) + y - y^\delta)\| \\
&\leq \|(P_hF'(P_hx_0)P_h + \alpha P_h)^{-1}P_h(F(P_hx_0) \\
&\quad -F(\hat{x}) - F'(P_hx_0)(P_hx_0 - \hat{x}))\| \\
&\quad +\|(P_hF'(P_hx_0)P_h + \alpha P_h)^{-1}P_hF'(P_hx_0)(P_hx_0 - \hat{x})\| \\
&\quad +\|(P_hF'(P_hx_0)P_h + \alpha P_h)^{-1}P_h(y - y^\delta)\| \\
&\leq \|(P_hF'(P_hx_0)P_h + \alpha P_h)^{-1}P_hF'(P_hx_0) \\
&\quad \times \int_0^1 \Phi(\hat{x} + t(P_hx_0 - \hat{x}), P_hx_0, (P_hx_0 - \hat{x}))dt\|
\end{aligned}$$

$$\begin{aligned}
& +\|(P_hF'(P_hx_0)P_h+\alpha P_h)^{-1}P_hF'(P_hx_0)(P_hx_0-\hat{x})\|+\frac{\delta}{\alpha}\\
\leq\; & \|(P_hF'(P_hx_0)P_h+\alpha P_h)^{-1}P_h[F'(P_hx_0)P_h+F'(P_hx_0)(I-P_h)]\\
& \times\int_0^1\Phi(\hat{x}+t(P_hx_0-\hat{x}),P_hx_0,(P_hx_0-\hat{x}))dt\|\\
& +\|(P_hF'(P_hx_0)P_h+\alpha P_h)^{-1}P_h[F'(P_hx_0)P_h\\
& +F'(P_hx_0)(I-P_h)](P_hx_0-\hat{x})\|+\frac{\delta}{\alpha}\\
\leq\; & (1+\frac{\gamma_{0,h}}{\alpha})(\frac{k_0}{2}\|P_hx_0-\hat{x}\|^2\\
& +\|P_hx_0-\hat{x}\|)+\frac{\delta}{\alpha}\\
\leq\; & (1+\frac{\gamma_{0,h}}{\alpha})(\frac{k_0}{2}(b_h+\rho)^2+b_h+\rho)+\frac{\delta}{\alpha}\\
\leq\; & \eta_h.
\end{aligned}$$

The last but one step follows from Assumption 1.2, (26) and the inequality $\|P_hx_0-\hat{x}\|\leq b_h+\rho$. So $\|x^{h,\delta}_{1,\alpha}-P_hx_0\|\leq\tilde{t}_{1,h}-\tilde{t}_{0,h}$. Assume that

$$\|x^{h,\delta}_{i+1,\alpha}-x^{h,\delta}_{i,\alpha}\|\leq\tilde{t}_{i+1,h}-\tilde{t}_{i,h},\qquad\forall i\leq k \tag{30}$$

for some k. Then

$$\begin{aligned}
\|x^{h,\delta}_{k+1,\alpha}-P_hx_0\| &\leq \|x^{h,\delta}_{k+1,\alpha}-x^{h,\delta}_{k,\alpha}\|+\|x^{h,\delta}_{k,\alpha}-x^{h,\delta}_{k-1,\alpha}\|+\cdots+\|x^{h,\delta}_{1,\alpha}-P_hx_0\|\\
&\leq \tilde{t}_{k+1,h}-\tilde{t}_{k,h}+\tilde{t}_{k,h}-\tilde{t}_{k-1,h}+\cdots+\tilde{t}_{1,h}-\tilde{t}_{0,h}\\
&= \tilde{t}_{k+1,h}\leq\tilde{t_h}^*.
\end{aligned}$$

So $x^{h,\delta}_{i+1,\alpha}\in B_{\tilde{t_h}^*}(P_hx_0)$ for all $i\leq k$. Therefore by (29) and (30) we have

$$\begin{aligned}
\|x^{h,\delta}_{k+2,\alpha}-x^{h,\delta}_{k+1,\alpha}\| &\leq \frac{3k_0}{2}(1+\frac{\gamma_{0,h}+k_0\varrho\|x^{h,\delta}_{k+1,\alpha}-P_hx_0\|}{\alpha})\|x^{h,\delta}_{k+1,\alpha}-x^{h,\delta}_{k,\alpha}\|^2\\
&\leq \frac{3k_0}{2}(1+\frac{\gamma_{0,h}+k_0\varrho\tilde{t}_{k+1,h}}{\alpha})(\tilde{t}_{k+1,h}-\tilde{t}_{k,h})^2\\
&= \tilde{t}_{k+2,h}-\tilde{t}_{k+1,h}.
\end{aligned}$$

Thus by induction $\|x^{h,\delta}_{n+1,\alpha}-x^{h,\delta}_{n,\alpha}\|\leq\tilde{t}_{n+1,h}-\tilde{t}_{n,h}$ for all $n\geq 0$ and hence $(\tilde{t}_{n,h}),n\geq 0$ is a majorizing sequence of the sequence $(x^{h,\delta}_{n,\alpha})$. In particular $\|x^{h,\delta}_{n,\alpha}-P_hx_0\|\leq\tilde{t}_{n,h}\leq\tilde{t_h}^*$, i.e., $x^{h,\delta}_{n,\alpha}\in B_{\tilde{t_h}^*}(P_hx_0)$, for all $n\geq 0$.

Hence

$$\|x^{h,\delta}_{n,\alpha}-P_hx_0\|\leq\tilde{t_h}^*\leq\frac{\eta_h}{1-r_h}. \tag{31}$$

Lemma 3.4 *Let $x^{h,\delta}_{n,\alpha}$ be as in (13) and $x^{\delta}_{n,\alpha}$ be as in (4). Let assumptions in Theorem 2.3 and Theorem(3.3) hold. Then*

$$\|x^{h,\delta}_{n-1,\alpha}-x^{\delta}_{n-1,\alpha}\|\leq\frac{\eta_h}{1-r_h}+b_h+\frac{\eta}{1-q}. \tag{32}$$

Proof. Note that

$$\begin{aligned}\|x^{h,\delta}_{n-1,\alpha}-x^{\delta}_{n-1,\alpha}\| &= \|x^{h,\delta}_{n-1,\alpha}-P_hx_0+(P_h-I)x_0+x_0-x^{\delta}_{n-1,\alpha}\|\\ &\le [\|x^{h,\delta}_{n-1,\alpha}-P_hx_0\|+\|(P_h-I)x_0\|+\|x_0-x^{\delta}_{n-1,\alpha}\|]\\ &\le \frac{\eta_h}{1-r_h}+b_h+\frac{\eta}{1-q}.\end{aligned}$$

This completes the proof.
Let

$$Q := k_0(\frac{\eta_h}{1-r_h}+b_h+\frac{\eta}{1-q}) \tag{33}$$

and

$$Q_{n,h} := \Gamma_{n,h}+Q\|F'(x^{h,\delta}_{n,\alpha})\|. \tag{34}$$

Lemma 3.5 *Let $Q_{n,h}$ be as in (34) and assumptions in Theorem 2.3 be satisfied. Then for all $n \ge 0$,*

$$Q_{n,h} \le C_h$$

where $C_h = (k_0\frac{\eta}{1-q}+1)\|F'(x_0)\|+Q(k_0\frac{\eta_h}{1-r_h}+1)\varrho$.

Proof. Note that

$$\begin{aligned}\Gamma_{n,h} &\le \|F'(x^{\delta}_{n,\alpha})\|\\ &= \sup_{\|v\|\le 1}\|F'(x^{\delta}_{n,\alpha})v\|\\ &\le \sup_{\|v\|\le 1}\|[F'(x^{\delta}_{n,\alpha})-F'(x_0)+F'(x_0)]v\|\\ &\le \sup_{\|v\|\le 1}\|[F'(x^{\delta}_{n,\alpha})-F'(x_0)]v\|+\sup_{\|v\|\le 1}\|F'(x_0)v\|\\ &\le \sup_{\|v\|\le 1}\|F'(x_0)\Phi(x^{\delta}_{n,\alpha},x_0,v)\|+\|F'(x_0)\|\\ &\le \sup_{\|v\|\le 1}k_0\|F'(x_0)\|\|x^{\delta}_{n,\alpha}-x_0\|\|v\|+\|F'(x_0)\|\\ &\le k_0\|F'(x_0)\|\|x^{\delta}_{n,\alpha}-x_0\|+\|F'(x_0)\|\\ &\le k_0\|F'(x_0)\|\frac{\eta}{1-q}+\|F'(x_0)\|\\ &\le (k_0\frac{\eta}{1-q}+1)\|F'(x_0)\|. \end{aligned} \tag{35}$$

Similarly one can proof that

$$\|F'(x^{h,\delta}_{n,\alpha})\| \le (k_0\frac{\eta_h}{1-r_h}+1)\varrho. \tag{36}$$

Now the proof follows from (35), (36) and the relation $Q_{n,h} \le \|F'(x^{\delta}_{n,\alpha})\| + Q\|F'(x^{h,\delta}_{n,\alpha})\|$.
Hereafter we assume that $1 < Q < 2$, so that $r_h < 1 < Q$ and $\frac{Q}{2} < 1$.

Theorem 3.6 *Let $x^{h,\delta}_{n,\alpha}$ be as in (13) and $x^{\delta}_{n,\alpha}$ be as in (4). Let assumptions in Theorem 3.3, Theorem 2.3 and Lemma 3.5 hold. Then we have the following estimate,*

$$\|x^{h,\delta}_{n,\alpha} - x^{\delta}_{n,\alpha}\| \leq \left(\frac{Q}{2}\right)^n b_h + \frac{C_h \eta_h}{(\frac{Q}{2} - r_h)} \frac{\left(\frac{Q}{2}\right)^n}{\alpha}.$$

Proof. *Note that*

$$\begin{aligned}
x^{h,\delta}_{n,\alpha} - x^{\delta}_{n,\alpha} &= x^{h,\delta}_{n-1,\alpha} - x^{\delta}_{n-1,\alpha} - (P_h F'(x^{h,\delta}_{n-1,\alpha}) P_h + \alpha P_h)^{-1} P_h \\
&\quad \times [F(x^{h,\delta}_{n-1,\alpha}) - y^\delta + \alpha(x^{h,\delta}_{n-1,\alpha} - x_0)] \\
&\quad + (F'(x^{\delta}_{n-1,\alpha}) + \alpha I)^{-1} [F(x^{\delta}_{n-1,\alpha}) - y^\delta + \alpha(x^{\delta}_{n-1,\alpha} - x_0)] \\
&= x^{h,\delta}_{n-1,\alpha} - x^{\delta}_{n-1,\alpha} - [(P_h F'(x^{h,\delta}_{n-1,\alpha}) P_h + \alpha P_h)^{-1} P_h \\
&\quad - (F'(x^{\delta}_{n-1,\alpha}) + \alpha I)^{-1}][F(x^{h,\delta}_{n-1,\alpha}) - y^\delta + \alpha(x^{h,\delta}_{n-1,\alpha} - x_0)] \\
&\quad - (F'(x^{\delta}_{n-1,\alpha}) + \alpha I)^{-1} [F(x^{h,\delta}_{n-1,\alpha}) - F(x^{\delta}_{n-1,\alpha}) + \alpha(x^{h,\delta}_{n-1,\alpha} - x^{\delta}_{n-1,\alpha})] \\
&= x^{h,\delta}_{n-1,\alpha} - x^{\delta}_{n-1,\alpha} - (F'(x^{\delta}_{n-1,\alpha}) + \alpha I)^{-1} \\
&\quad \times [F(x^{h,\delta}_{n-1,\alpha}) - F(x^{\delta}_{n-1,\alpha}) + \alpha(x^{h,\delta}_{n-1,\alpha} - x^{\delta}_{n-1,\alpha})] \\
&\quad - [(P_h F'(x^{h,\delta}_{n-1,\alpha}) P_h + \alpha P_h)^{-1} P_h - (F'(x^{\delta}_{n-1,\alpha}) + \alpha I)^{-1}] \\
&\quad \times [F(x^{h,\delta}_{n-1,\alpha}) - y^\delta + \alpha(x^{h,\delta}_{n-1,\alpha} - x_0)] \\
&= (F'(x^{\delta}_{n-1,\alpha}) + \alpha I)^{-1} [F'(x^{\delta}_{n-1,\alpha})(x^{h,\delta}_{n-1,\alpha} - x^{\delta}_{n-1,\alpha}) \\
&\quad - (F(x^{h,\delta}_{n-1,\alpha}) - F(x^{\delta}_{n-1,\alpha}))] - (F'(x^{\delta}_{n-1,\alpha}) + \alpha I)^{-1} \\
&\quad [F'(x^{\delta}_{n-1,\alpha}) P_h - P_h F'(x^{h,\delta}_{n-1,\alpha}) P_h](P_h F'(x^{h,\delta}_{n-1,\alpha}) P_h + \alpha P_h)^{-1} \\
&\quad \times [F(x^{h,\delta}_{n-1,\alpha}) - y^\delta + \alpha(x^{h,\delta}_{n-1,\alpha} - x_0)] \\
&= (F'(x^{\delta}_{n-1,\alpha}) + \alpha I)^{-1} [F'(x^{\delta}_{n-1,\alpha})(x^{h,\delta}_{n-1,\alpha} - x^{\delta}_{n-1,\alpha}) \\
&\quad - (F(x^{h,\delta}_{n-1,\alpha}) - F(x^{\delta}_{n-1,\alpha}))] \\
&\quad - (F'(x^{\delta}_{n-1,\alpha}) + \alpha I)^{-1} [F'(x^{\delta}_{n-1,\alpha}) - P_h F'(x^{h,\delta}_{n-1,\alpha})](x^{h,\delta}_{n,\alpha} - x^{h,\delta}_{n-1,\alpha}) \\
&= \Gamma_1 - \Gamma_2 \qquad (37)
\end{aligned}$$

where

$$\Gamma_1 = (F'(x^{\delta}_{n-1,\alpha}) + \alpha I)^{-1} [F'(x^{\delta}_{n-1,\alpha})(x^{h,\delta}_{n-1,\alpha} - x^{\delta}_{n-1,\alpha}) - (F(x^{h,\delta}_{n-1,\alpha}) - F(x^{\delta}_{n-1,\alpha}))]$$

and

$$\Gamma_2 = (F'(x^{\delta}_{n-1,\alpha}) + \alpha I)^{-1} [F'(x^{\delta}_{n-1,\alpha}) - P_h F'(x^{h,\delta}_{n-1,\alpha})](x^{h,\delta}_{n,\alpha} - x^{h,\delta}_{n-1,\alpha})$$

Thus by Lemma 2.2, Lemma 3.4 and Asssumption1.2 we have

$$\|\Gamma_1\| = \|(F'(x^{\delta}_{n-1,\alpha}) + \alpha I)^{-1} F'(x^{\delta}_{n-1,\alpha})$$

$$
\begin{aligned}
& \int_0^1 \Phi(x^{h,\delta}_{n-1,\alpha} + t(x^{\delta}_{n-1,\alpha} - x^{h,\delta}_{n-1,\alpha}), x^{\delta}_{n-1,\alpha}, x^{\delta}_{n-1,\alpha} - x^{h,\delta}_{n-1,\alpha})dt\| \\
\leq\ & k_0 \int_0^1 \|x^{\delta}_{n-1,\alpha} - x^{h,\delta}_{n-1,\alpha}\| \|x^{h,\delta}_{n-1,\alpha} + t(x^{\delta}_{n-1,\alpha} - x^{h,\delta}_{n-1,\alpha}) - x^{\delta}_{n-1,\alpha}\|dt \\
\leq\ & k_0 \int_0^1 \|x^{\delta}_{n-1,\alpha} - x^{h,\delta}_{n-1,\alpha}\| \|(t-1)(x^{\delta}_{n-1,\alpha} - x^{h,\delta}_{n-1,\alpha})\|dt \\
& \leq \frac{k_0}{2} \|x^{h,\delta}_{n-1,\alpha} - x^{\delta}_{n-1,\alpha}\|^2 \\
& \leq \frac{Q}{2} \|x^{h,\delta}_{n-1,\alpha} - x^{\delta}_{n-1,\alpha}\|. \qquad (38)
\end{aligned}
$$

and

$$
\begin{aligned}
\|\Gamma_2\| \ =\ & \|(F'(x^{\delta}_{n-1,\alpha}) + \alpha I)^{-1}((I - P_h)F'(x^{\delta}_{n-1,\alpha}) \\
& + P_h(F'(x^{\delta}_{n-1,\alpha}) - F'(x^{h,\delta}_{n-1,\alpha})))(x^{h,\delta}_{n,\alpha} - x^{h,\delta}_{n-1,\alpha})\| \\
\leq\ & \|(F'(x^{\delta}_{n-1,\alpha}) + \alpha I)^{-1}(I - P_h)F'(x^{\delta}_{n-1,\alpha})(x^{h,\delta}_{n,\alpha} - x^{h,\delta}_{n-1,\alpha})\| \\
& + \|(F'(x^{\delta}_{n-1,\alpha}) + \alpha I)^{-1}P_h(F'(x^{\delta}_{n-1,\alpha}) - F'(x^{h,\delta}_{n-1,\alpha}))(x^{h,\delta}_{n,\alpha} - x^{h,\delta}_{n-1,\alpha})\| \\
\leq\ & \frac{\|(I - P_h)F'(x^{\delta}_{n-1,\alpha})\|}{\alpha} \|x^{h,\delta}_{n,\alpha} - x^{h,\delta}_{n-1,\alpha}\| \\
& + \|(F'(x^{\delta}_{n-1,\alpha}) + \alpha I)^{-1}P_h F'(x^{h,\delta}_{n-1,\alpha})\Phi(x^{\delta}_{n-1,\alpha}, x^{h,\delta}_{n-1,\alpha}, x^{h,\delta}_{n,\alpha} - x^{h,\delta}_{n-1,\alpha})\| \\
\leq\ & \frac{\Gamma_{n-1,h} + k_0\|F'(x^{h,\delta}_{n-1,\alpha})\| \|x^{h,\delta}_{n-1,\alpha} - x^{\delta}_{n-1,\alpha}\|}{\alpha} \|x^{h,\delta}_{n,\alpha} - x^{h,\delta}_{n-1,\alpha}\| \\
\leq\ & \frac{\Gamma_{n-1,h} + Q\|F'(x^{h,\delta}_{n-1,\alpha})\|}{\alpha} \|x^{h,\delta}_{n,\alpha} - x^{h,\delta}_{n-1,\alpha}\| \\
\leq\ & \frac{Q_{n-1,h}}{\alpha} \|x^{h,\delta}_{n,\alpha} - x^{h,\delta}_{n-1,\alpha}\| \qquad (39)
\end{aligned}
$$

Therefore by (37), (38), (39) and Lemma 3.5 we have

$$
\begin{aligned}
\left\|x^{h,\delta}_{n,\alpha} - x^{\delta}_{n,\alpha}\right\| \ \leq\ & \frac{Q}{2}\left\|x^{h,\delta}_{n-1,\alpha} - x^{\delta}_{n-1,\alpha}\right\| + \frac{C_h}{\alpha}\left\|x^{h,\delta}_{n,\alpha} - x^{h,\delta}_{n-1,\alpha}\right\| \\
\leq\ & \left(\frac{Q}{2}\right)^n \left\|x^{h,\delta}_{0,\alpha} - x^{\delta}_{0,\alpha}\right\| + \left(\frac{Q}{2}\right)^{n-1}\frac{C_h}{\alpha}\eta_h + \left(\frac{Q}{2}\right)^{n-2}\frac{C_h}{\alpha}r_h\eta_h \\
& + \cdots + \left(\frac{Q}{2}\right)\frac{C_h}{\alpha}r_h^{n-2}\eta_h + \frac{C_h}{\alpha}r_h^{n-1}\eta_h \\
\leq\ & \left(\frac{Q}{2}\right)^n b_h + \left(\frac{Q}{2}\right)^{n-1}\frac{C_h}{\alpha}\eta_h + \left(\frac{Q}{2}\right)^{n-2}\frac{C_h}{\alpha}r_h\eta_h \\
& + \cdots + \left(\frac{Q}{2}\right)\frac{C_h}{\alpha}r_h^{n-2}\eta_h + \frac{C_h}{\alpha}r_h^{n-1}\eta_h
\end{aligned}
$$

$$\begin{aligned} &\leq \left(\frac{Q}{2}\right)^n b_h + \frac{C_h}{\alpha}[\left(\frac{Q}{2}\right)^{n-1} + \left(\frac{Q}{2}\right)^{n-2} r_h \\ &\quad + \cdots + \left(\frac{Q}{2}\right) r_h^{n-2} + r_h^{n-1}]\eta_h \\ &\leq \left(\frac{Q}{2}\right)^n b_h + \frac{C_h\eta_h}{(\frac{Q}{2} - r_h)} \frac{\left(\frac{Q}{2}\right)^n}{\alpha}. \end{aligned}$$

This completes the proof.

4 Error Bounds Under Source Conditions

In view of Theorem 2.3 and Theorem 3.6; to obtain an error estimate for $\|x_{n,\alpha}^{h,\delta} - \hat{x}\|$, it is enough to obtain an error estimate for $\|x_\alpha^\delta - \hat{x}\|$. It is known (cf. [13], Proposition 3.1) that

$$\|x_\alpha^\delta - x_\alpha\| \leq \frac{\delta}{\alpha} \tag{40}$$

and (cf. [11], Theorem 3.1) that

$$\|x_\alpha - \hat{x}\| \leq (k_0 r_0 + 1)c_\varphi \varphi(\alpha). \tag{41}$$

Combining the estimates in Theorem 2.3, and Theorem 3.6, (40) and (41) we obtaining the following,

Theorem 4.1 *Let $x_{n,\alpha}^{h,\delta}$ be as in (13) and let the assumptions in Theorem 2.3,Theorem 3.6 and Lemma 3.5 be satisfied. Then we have the following;*

$$\begin{aligned} \|x_{n,\alpha}^{h,\delta} - \hat{x}\| &\leq \|x_{n,\alpha}^{h,\delta} - x_{n,\alpha}^\delta\| + \|x_{n,\alpha}^\delta - x_\alpha^\delta\| + \|x_\alpha^\delta - x_\alpha\| + \|x_\alpha - \hat{x}\| \\ &\leq \left(\frac{Q}{2}\right)^n b_h + \frac{C_h\eta_h}{(\frac{Q}{2} - r_h)} \frac{\left(\frac{Q}{2}\right)^n}{\alpha} + \frac{q^n\eta}{(1-q)} + \frac{\delta}{\alpha} + (k_0 r_0 + 1)c_\varphi\varphi(\alpha). \end{aligned}$$

Let

$$C_0 := max\{1 + b_h + \frac{C_h\eta_h}{(\frac{Q}{2} - r_h)} + \frac{\eta}{1-q}, (k_0 r_0 + 1)c_\varphi\}$$

and let

$$n_\delta := \min\{n : \left(\frac{Q}{2}\right)^n \leq \delta\}. \tag{42}$$

Note that for $0 < \alpha < 1$, $\delta \leq \delta/\alpha$. Thus by Theorem 4.1 we have the following Theorem.

Theorem 4.2 *Let $x_{n,\alpha}^{h,\delta}$ be as in (13) an let the assumptions in Theorem 2.3, Theorem 3.6 and Lemma 3.5 be satisfied. Let n_δ be as in (42). Then for $0 < \alpha < 1$ we have,*

$$\|x_{n_\delta,\alpha}^{h,\delta} - \hat{x}\| \leq C_0(\varphi(\alpha) + \frac{\delta}{\alpha}). \tag{43}$$

4.1 A Priori Choice of the Parameter

Note that the error estimate $\varphi(\alpha) + \frac{\delta}{\alpha}$ in (43) is of optimal order if $\alpha := \alpha_\delta = \alpha(\delta)$ satisfies, $\alpha_\delta \varphi(\alpha_\delta) = \delta$. Now using the function $\psi(\lambda) := \lambda\varphi^{-1}(\lambda), 0 < \lambda \leq a$ we have $\delta = \alpha_\delta \varphi(\alpha_\delta) = \psi(\varphi(\alpha_\delta))$, so that $\alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta))$. Hence by (43) we have the following.

Theorem 4.3 *Let $\psi(\lambda) := \lambda\varphi^{-1}(\lambda)$ for $0 < \lambda \leq a$, and assumptions in Theorem 4.2 holds. For $\delta > 0$, let $\alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta))$. Let n_δ be as in (42). Then*

$$\|x^{h,\delta}_{n_\delta,\alpha} - \hat{x}\| = \bigcirc(\psi^{-1}(\delta)).$$

4.2 An Adaptive Choice of the Parameter

In this subsection, we will present a parameter choice rule based on the adaptive method studied in [10, 12].

In practice, the regularization parameter α is often selected from some finite set

$$D_M(\alpha) := \{\alpha_i = \mu^i \alpha_0, i = 0, 1, \cdots, M\} \tag{44}$$

where $\mu > 1$ and M is such that $\alpha_M < 1 \leq \alpha_{M+1}$. We choose $\alpha_0 := \sqrt{\delta}$, because we expect to have an accuracy of order $\bigcirc(\sqrt{\delta})$ and from Theorem 4.3, it follows that such an accuracy cannot be guaranteed for $\alpha < \sqrt{\delta}$.

Let

$$n_M = \min\{n : \left(\frac{Q}{2}\right)^n \leq \delta\}. \tag{45}$$

and let $x_i := x^{h,\delta}_{n_M,\alpha_i}$. The parameter choice strategy that we are going to consider in this paper, we selects $\alpha = \alpha_i$ from $D_M(\alpha)$ and operates only with corresponding $x_i, \quad i = 0, 1, \cdots, M$.

Theorem 4.4 *Assume that there exists $i \in \{0, 1, 2, \cdots, M\}$ such that $\varphi(\alpha_i) \leq \frac{\delta}{\alpha_i}$. Let assumptions of Theorem 4.2 and Theorem 4.3 hold and let*

$$l := \max\{i : \varphi(\alpha_i) \leq \frac{\delta}{\alpha_i}\} < M,$$

$$k := \max\{i : \|x_i - x_j\| \leq 4C_0 \frac{\delta}{\alpha_j}, \quad \forall j = 0, 1, 2, \cdots, i\}. \tag{46}$$

Then $l \leq k$ and

$$\|\hat{x} - x_k\| \leq c\psi^{-1}(\delta)$$

where $c = 6C_0\mu$.

Proof. *To see that $l \leq k$, it is enough to show that, for each $i \in \{1, 2, \cdots, M\}$,*

$$\varphi(\alpha_i) \leq \frac{\delta}{\alpha_i} \Longrightarrow \|x_i - x_j\| \leq 4C_0 \frac{\delta}{\alpha_j}, \quad \forall j = 0, 1, \cdots, i.$$

For $j \le i$, by (43) we have

$$\begin{aligned}\|x_i - x_j\| &\le \|x_i - \hat{x}\| + \|\hat{x} - x_j\| \\ &\le C_0\varphi(\alpha_i) + C_0\frac{\delta}{\alpha_i} + C_0\varphi(\alpha_j) + C_0\frac{\delta}{\alpha_j} \\ &\le 2C_0\frac{\delta}{\alpha_i} + 2C_0\frac{\delta}{\alpha_j} \\ &\le 4C_0\frac{\delta}{\alpha_j}.\end{aligned}$$

Thus the relation $l \le k$ is proved. Next we observe that

$$\begin{aligned}\|\hat{x} - x_k\| &\le \|\hat{x} - x_l\| + \|x_l - x_k\| \\ &\le C_0\varphi(\alpha_l) + C_0\frac{\delta}{\alpha_l} + 4C_0\frac{\delta}{\alpha_l} \\ &\le 6C_0\frac{\delta}{\alpha_l}.\end{aligned}$$

Now since $\alpha_\delta \le \alpha_{l+1} \le \mu\alpha_l$, it follows that

$$\frac{\delta}{\alpha_l} \le \mu\frac{\delta}{\alpha_\delta} = \mu\varphi(\alpha_\delta) = \mu\psi^{-1}(\delta).$$

Thus

$$\begin{aligned}\|\hat{x} - x_k\| &\le 6C_0\mu\psi^{-1}(\delta) \\ &\le c\psi^{-1}(\delta)\end{aligned}$$

where $c = 6C_0\mu$.This completes the proof of the theorem.

5 Implementation of Adaptive Choice Rule

In this section we provide an algorithm for the determination of a parameter fulfilling the balancing principle (46) and also provide a starting point for the iteration (13) approximating the unique solution x_α^δ of (3) .The choice of the starting point involves the following steps:

- Choose $\alpha_0 = \sqrt{\delta}$, $\mu > 1$ and $r_h < 1$.
- Choose $x_0 \in D(F)$ such that $\|x_0 - \hat{x}\| \le \rho$ and $(1 + \frac{\gamma_{0,h}}{\alpha_0})(\frac{k_0}{2}(b_h + \rho)^2 + b_h + \rho) + \frac{\delta}{\alpha_0} \le \eta_h \le min\{C, r_0(1 - r_h)\}$.
- Choose $1 < Q < 2$ where Q is as in (33).
- Choose n_M such that $n_M = \min\{n : \left(\frac{Q}{2}\right)^n \le \delta\}$.

Finally the adaptive algorithm associated with the choice of the parameter specified in Theorem 4.4 involves the following steps:

5.1 Algorithm

- Set $i \leftarrow 0$
- Solve $x_i := x^{h,\delta}_{n_M,\alpha_i}$ by using the iteration (13).
- If $\|x_i - x_j\| > 6C_0\frac{\sqrt{\delta}}{\mu^j}, \qquad j \le i$, then take $k = i - 1$.
- Set $i = i + 1$ and return to step 2.

6 Concluding Remarks

In this paper we considered a new iterative method in the finite dimensional setting for approximately solving the nonlinear ill-posed operator equation $F(x) = y$, when the available data y^δ in place of the exact data y. It is assumed that F is Fr*é*chet differentiable in a neighborhood of some initial guess x_0 of the actual solution $\hat{x}$. The procedure involves finding the fixed point of the function

$$G_h(x) = x - (P_h F'(x) P_h + \alpha P_h)^{-1}(F(x) - y^\delta + \alpha(x - x_0)),$$

in an iterative manner in a finite dimensional subspace X_h of the Hilbert space X. Here x_0 is an initial guess and P_h is the orthogonal projection onto X_h. For choosing the regularization parameter α we made use of the adaptive method suggested by Pereversev and Schock in [12] and the stopping rule is based on a majorizing sequence.

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