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Harmonic Functions in Rectangular Domains. Classical Solutions Revisited

Michael Doschoris

Department of Chemical Engineering, Division of Applied Mathematics,
University of Patras and
Institute of Chemical Engineering and High Temperature Chemical Processes
ICE/HT - FORTH, GR-26504, Patras, Greece
mdoscho@chemeng.upatras.gr

Abstract

Novel integral formulae for harmonic functions in rectangular domains are presented. This representations are analytic in the complex C−plane, displaying strong decay as the complex variable tends to infinity and are therefore suitable for numerical computations and asymptotic analysis of the solution. The analysis is based on the new method introduced by Fokas and his collaborators, yielding novel formulae even for simple problems that can be solved by the method of the classical transforms. This is achieved by implementing the global relation, which is an integral relation connecting the boundary values of the solution with the normal derivative of the solution on the boundary, in appropriate subdomains of the fundamental domain. Furthermore, a changing-type boundary value problem is solved.

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1 Introduction

In most cases, a given, well-posed, boundary-value problem can be solved by means of separation of variables, if there exist a coordinate system that fits the boundary of the fundamental domain and at the same time it separates the partial differential equation (PDE). Furthermore, separation of variables leads to the solution of PDE's by a transform pair, to which the "prototype" is the Fourier transform. However, for complicated problems the classical transform method fails. For example, there do not exist proper transforms for solving many boundary-value problems for elliptic equations of second order and in simple domains.

Within the last decade, A.S. Fokas proposed a general method, known as the generalized transform method, for solving boundary-value problems for two-dimensional linear and integrable nonlinear PDE's and it is presented in [4]. An equation in two dimensions (x_1, x_2) is called integrable if it can be expressed as the condition that a certain differential 1-form $W(x_1, x_2; k)$, $k \in \mathrm{C}$, is closed, e.g. linear PDE's with constant coefficients. This approach can be seen as a generalization of the separation of variables method, but more effectively (an overview is provided in [1,2]). The generalized transform method does not depend on the geometry of the domain at hand, but on the linearity of the PDE and constructs the solutions without the need of using eigenfunction expansions, arriving at separable solutions without actually assuming separation [3]. The novelty of the Fokas method lies in the fact that it is not based on the existence of a "classical" transform pair and therefore is applicable even in the case where classical transforms do not exist. Moreover, if a given BVP can be solved by a classical transform, the new method provides an alternative approach deriving this transform and also yields two kind of novel integral representations in the complex $\mathrm{C}-$plane, one of which is useful for solving changing type BVP's and the other, involving a strong decay as the complex variable involved tends to ∞, is suitable for numerical computations and for the study of the asymptotic properties of these solutions [5]. The key feature of this methodology lies in the successful manipulation of the so-called global relation, a formula connecting the solution of the BVP with its derivatives on the boundary. In order to obtain the global relation, the given PDE in connection with the formal adjoint associated with the PDE, is re-formulated as a divergence form. Employing then Green's second identity yields immediately the global relation. This procedure, as a "side effect", also implies a Lax pair formulation. The elimination of the unknown boundary values is possible by using the global relation and its invariant forms, which are introduced via the separation "constant" in domains which are separable.

The article is organized in a number of sections as follows. In section 2 the problem is formulated. In the sequence, in order to fix notation and terminology, the classical transform is given, which is then rederived in section 5, by means of the analysis of the global relation. In sections 6-8, the main results of the article are presented, where alternative formulae for the solution in terms of an integral instead of a series are derived. This is realized by algebraic manipulation of the global relation in appropriate subdomains. The integral representations presented are analytic in the complex $k-$plane, with strong decay as $k \to \infty$, and therefore suitable for numerical computations and asymptotic analysis of the solution. Moreover, the machinery introduced is utilized in section 9 to solve a changing-type boundary value problem (such as Dirichlet data on part of the boundary and Neumann data on the complementary part). In the latter case, one must combine the new method with the Riemann-Hilbert formulation.

2 Formulation of the Problem

The two dimensional Laplace equation in Cartesian coordinates, namely

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)q(x,y)=0,\quad (x,y)\in\Omega\,, \tag{1}$$

in the interior Ω of a square defined by

$$\Omega=\Big\{-L\le x\le L,\ -L\le y\le L\Big\} \tag{2}$$

and depicted in Figure 1, where $q(x,y)$ is a real valued function, is investigated.

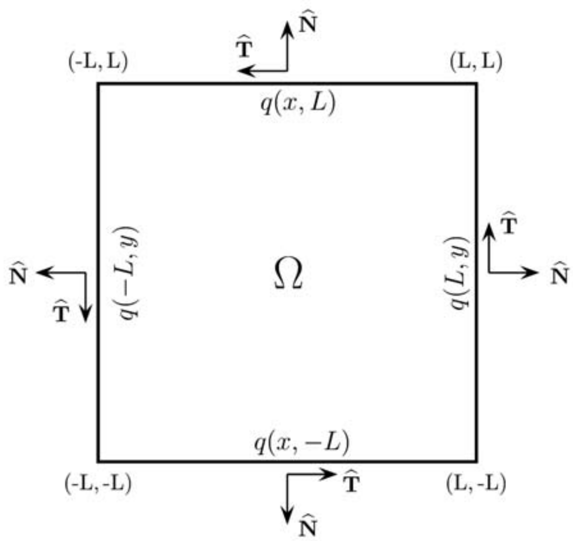

Figure 1: The domain $\Omega=\Big\{-L\le x\le L,\ -L\le y\le L\Big\}$

We analyze the general Dirichlet problem

$$\begin{aligned}&q(L,y)=f_D^{(1)}(y),\ q(x,-L)=f_D^{(2)}(x),\\&q(-L,y)=f_D^{(3)}(y),\ q(x,L)=f_D^{(4)}(x)\end{aligned} \tag{3}$$

which, after a suitable parametrization, becomes

$$q^{(j)}(s)=f_D^{(j)}(s),\quad s\in[-L,L],\quad j=1,2,3,4\,, \tag{4}$$

where (j) corresponds to the $j-$th side of the square.
We assume that the functions $f_D^{(j)}$ are smooth and compatible at the corners of the square. The general Neumann problem can be treated in the same manner, where, furthermore, the Neumann data have to satisfy the compatibility condition

$$\oint_{\partial\Omega}\left(-\frac{\partial q}{\partial y}\,\mathrm{d}x+\frac{\partial q}{\partial x}\,\mathrm{d}y\right)=0\,,$$

and $\partial\Omega$ is the boundary of the domain.
Throughout the analysis presented, emanating from the linearity of the Laplacian operator, the fact that the solution $q(x,y)$ can be written as a linear combination of "partial solutions" $q_j(x,y)$, corresponding to specific subproblems, namely particular boundary conditions, is applied.

3 The Classical Transform

When we apply the classical transform we assume the solution expanded in a series of eigenfunctions of one of the variables, with the coefficient depending upon the other variable. Separation of variables relies upon the completeness of the eigenfunctions corresponding to one of the variables. The solution will depend on functions which enter into the boundary conditions, and since the spatial domain Ω is rectangular, the relative eigenfunctions are trigonometric. Furthermore, every function can be written uniquely as the sum of an even and an odd function, or in terms of a Fourier expansion, every function, satisfying Dirichlet's conditions, which enters into the boundary conditions can be written as

$$f(s)\sim\sum_n\left[\alpha_n\,\sin\left(\tfrac{n\pi}{L}s\right)+\beta_n\,\cos\left(\left(n+\tfrac{1}{2}\right)\tfrac{\pi}{L}s\right)\right],\quad s\in[-L,L] \tag{5}$$

where the set $\mathcal{S}=\{1\}\cup\{\sin\frac{n\pi}{L}s,\,n\in\mathrm{Z}-\{0\}\}\cup\{\cos\left(n+\frac{1}{2}\right)\frac{\pi}{L}s,\,n\in\mathrm{Z}\}$ form a complete orthogonal basis of $L_2[-L,L]$.

Proposition 3.1 *Let the real valued function $q(x,y)$ satisfy the Laplace equation (1) in the domain Ω defined in (2), with boundary conditions (4), where the given functions $f_D^{(j)}(s)$, $j=1,2,3,4$ have sufficient smoothness and are continuous at the vertices. Then the classical representation for the solution* *is given by*

$$q(x,y) = \sum_{n=1}^{\infty} \left[a_n \sinh\left(\tfrac{n\pi}{L}(x+L)\right) + c_n \sinh\left(\tfrac{n\pi}{L}(x-L)\right) \right] \sin\left(\tfrac{n\pi}{L} y\right)$$

$$+ \sum_{n=0}^{\infty} \left[b_n \sinh\left(\left(n+\tfrac{1}{2}\right) \tfrac{\pi}{L}(x+L)\right) + d_n \sinh\left(\left(n+\tfrac{1}{2}\right) \tfrac{\pi}{L}(x-L)\right) \right]$$

$$\times \cos\left(\left(n+\tfrac{1}{2}\right) \tfrac{\pi}{L} y\right) + \sum_{n=1}^{\infty} \left[e_n \sinh\left(\tfrac{n\pi}{L}(y-L)\right) + g_n \sinh\left(\tfrac{n\pi}{L}(y+L)\right) \right]$$

$$\times \sin\left(\tfrac{n\pi}{L} x\right) + \sum_{n=0}^{\infty} \left[f_n \sinh\left(\left(n+\tfrac{1}{2}\right) \tfrac{\pi}{L}(y-L)\right) + h_n \sinh\left(\left(n+\tfrac{1}{2}\right) \tfrac{\pi}{L}(y+L)\right) \right]$$

$$\times \cos\left(\left(n+\tfrac{1}{2}\right) \tfrac{\pi}{L} x\right), \tag{6}$$

where, by introducing a intrinsic coordinate system $(\hat{\mathbf{T}}, \hat{\mathbf{N}})$ *on each side of the square, the Fourier coefficients* a_n, b_n, c_n, d_n, e_n, f_n, g_n *and* h_n *can be expressed as follows*

$$a_n = \frac{1}{L\sinh(2n\pi)} \int_{-L}^{L} f_D^{(1)}(s) \sin\left(\tfrac{n\pi}{L} s\right)\, \mathrm{d}s \tag{7}$$

$$b_n = \frac{1}{L\sinh(2n+1)\pi} \int_{-L}^{L} f_D^{(1)}(s) \cos\left(\left(n+\tfrac{1}{2}\right) \tfrac{\pi}{L} s\right) \mathrm{d}s \tag{8}$$

$$c_n = \frac{1}{L\sinh(2n\pi)} \int_{-L}^{L} f_D^{(3)}(-s) \sin\left(\tfrac{n\pi}{L} s\right)\, \mathrm{d}s \tag{9}$$

$$d_n = -\frac{1}{L\sinh(2n+1)\pi} \int_{-L}^{L} f_D^{(3)}(-s) \cos\left(\left(n+\tfrac{1}{2}\right) \tfrac{\pi}{L} s\right) \mathrm{d}s \tag{10}$$

$$e_n = -\frac{1}{L\sinh(2n\pi)} \int_{-L}^{L} f_D^{(2)}(s) \sin\left(\tfrac{n\pi}{L} s\right)\, \mathrm{d}s \tag{11}$$

$$f_n = -\frac{1}{L\sinh(2n+1)\pi} \int_{-L}^{L} f_D^{(2)}(s) \cos\left(\left(n+\tfrac{1}{2}\right) \tfrac{\pi}{L} s\right) \mathrm{d}s \tag{12}$$

$$g_n = -\frac{1}{L\sinh(2n\pi)} \int_{-L}^{L} f_D^{(4)}(-s) \sin\left(\tfrac{n\pi}{L} s\right)\, \mathrm{d}s \tag{13}$$

$$h_n = \frac{1}{L\sinh(2n+1)\pi} \int_{-L}^{L} f_D^{(4)}(-s) \cos\left(\left(n+\tfrac{1}{2}\right) \tfrac{\pi}{L} s\right) \mathrm{d}s\,. \tag{14}$$

4 Analysis of the Global Relation

Let $q(x,y)$ and $\overline{q}(x,y)$ satisfy the Laplace equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) q(x,y) = 0\,, \tag{15}$$

and the formal adjoint of the Laplace equation

$$\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big)\, \overline{q}(x,y) = 0\,. \tag{16}$$

Multiplying (15) by $\overline{q}(x,y)$ and (16) by $q(x,y)$ and subtracting them, we obtain the divergence form

$$\frac{\partial}{\partial x}\left(\overline{q}\,\frac{\partial q}{\partial x} - q\,\frac{\partial \overline{q}}{\partial x}\right) + \frac{\partial}{\partial y}\left(\overline{q}\,\frac{\partial q}{\partial y} - q\,\frac{\partial \overline{q}}{\partial y}\right) = 0\,. \tag{17}$$

Equation (17) holds true everywhere in R^2 and applying Green's theorem to a closed subdomain of R^2, yields

$$\int\limits_C \left[\left(\overline{q}\,\frac{\partial q}{\partial x} - q\,\frac{\partial \overline{q}}{\partial x}\right)\mathrm{d}y + \left(q\,\frac{\partial \overline{q}}{\partial y} - \overline{q}\,\frac{\partial q}{\partial y}\right)\mathrm{d}x\right] = 0\,, \tag{18}$$

where C is the boundary of the subdomain.
Equation (18) provides the *global relation*, since it relates the boundary values of the solution with the values of the normal derivatives of the solution on the boundary.
Letting $\overline{q}(x,y;k) = \overline{X}(x;k)\overline{Y}(y;k)$ where k is the complex separation constant, it follows that $\overline{X}(x;k)$ and $\overline{Y}(y;k)$ satisfy the ODE's

$$\left.\begin{array}{rcl} \overline{X}'' + k^2\,\overline{X} & = & 0 \\ \overline{Y}'' - k^2\,\overline{Y} & = & 0 \end{array}\right\}, \quad k \in \mathrm{C}\,,$$

where the prime denotes differentiation with respect to the argument.
Solving above equations yields $\overline{q}(x,y) = e^{\pm ikx}\,e^{\sigma ky}$, where $\sigma = \pm 1$. Then, equations (17) and (18) become

$$\frac{\partial}{\partial x}\left(e^{\pm ikx}\,e^{\sigma ky}\left(\pm ikq - \frac{\partial q}{\partial x}\right)\right) + \frac{\partial}{\partial y}\left(e^{\pm ikx}\,e^{\sigma ky}\left(\sigma kq - \frac{\partial q}{\partial y}\right)\right) = 0 \tag{19}$$

and

$$\int\limits_C e^{\pm ikx}\,e^{\sigma ky}\left[\left(\pm ikq - \frac{\partial q}{\partial x}\right)\mathrm{d}y - \left(\sigma kq - \frac{\partial q}{\partial y}\right)\mathrm{d}x\right] = 0, \tag{20}$$

respectively. Equations (19) imply two items. First, applying Green's theorem we obtain *immediately* the global relation, and second it yields a Lax pair formulation.
Indeed, if q(x,y) is the solution of the Laplace equation in a closed subdomain $\Omega \subset \mathrm{R}^2$, then (19) implies the existences of a function $\Xi(x,y;k)$, such that

$$\left.\begin{array}{rcl} \frac{\partial}{\partial y}\,\Xi & = & e^{\pm ikx}\,e^{\sigma ky}\left(\pm\, ikq - \frac{\partial q}{\partial x}\right) \\ \frac{\partial}{\partial x}\,\Xi & = & -e^{\pm ikx}\,e^{\sigma ky}\left(\sigma kq - \frac{\partial q}{\partial y}\right) \end{array}\right\}, \quad k \in \mathrm{C}\,.$$

The assumption $\Xi(x,y;k) = e^{\pm ikx}\, e^{\sigma ky}\, \mu(x,y;k)$ where $\mu(x,y;k)$ an auxiliary function, leads right away to the Lax pairs

$$\begin{aligned} \left(\frac{\partial}{\partial y} + \sigma k\right)\mu &= \pm ikq - \frac{\partial q}{\partial x}\,, \\ \left(\frac{\partial}{\partial x} \pm ik\right)\mu &= \frac{\partial q}{\partial y} - \sigma kq\,. \end{aligned}$$

Furthermore, (19) implies that if the differential form

$$W(x,y;k) = e^{\pm ikx}\, e^{\sigma ky}\left\{\left(\pm ikq - \frac{\partial q}{\partial x}\right)\mathrm{d}y - \left(\sigma kq - \frac{\partial q}{\partial y}\right)\mathrm{d}x\right\}$$

is closed, viz

$$\mathrm{d}W(x,y;k) = e^{\pm ikx}\, e^{\sigma ky}\left(\frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial y^2}\right)\mathrm{d}x\wedge\mathrm{d}y = 0\,,$$

then Stoke's theorem

$$\int\limits_{\partial\Omega} W = \iint\limits_{\Omega} \mathrm{d}W\,,$$

provides (20).

5 The Classical Representation

To rederive the classical transform (6), apply the global relation (20) in the subdomains Ω_1 and Ω_2 defined by

$$\Omega_1 = \Big\{-L \le \eta \le x,\ |y| \le L\Big\}, \quad \Omega_2 = \Big\{x \le \eta \le L,\ |y| \le L\Big\}$$

and depicted in Figure 2, with the following boundary conditions

$$\left.\begin{array}{cc} q(L,y) = f_D^{(1)}(y), & q(-L,y) = f_D^{(3)}(y) \\ q(\eta,L) = q(\eta,-L) = 0, & \partial_y\, q(\eta,L) = \partial_y\, q(\eta,-L) = 0 \end{array}\right\}. \tag{21}$$

Thus we derive the following equations

$$\begin{aligned} &\int_{-L}^{L} e^{\sigma ky}\Big(\pm ik\, q_1(x,y) - \partial_x\, q_1(x,y)\Big)\,\mathrm{d}y \\ &= e^{\mp ik(x+L)}\int_{-L}^{L} e^{\sigma ky}\Big(\pm ik\, q(-L,y) - \partial_x\, q(-L,y)\Big)\,\mathrm{d}y\,, \end{aligned} \tag{22}$$

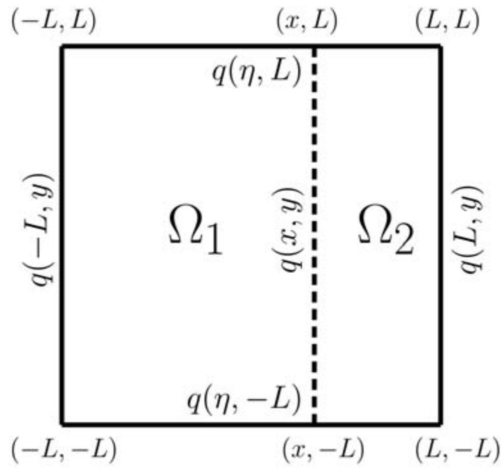

Figure 2: The subdomains $\Omega_1 \subset \Omega$ and $\Omega_2 \subset \Omega$ defined as $\Omega_1 = \{-L \leq \eta \leq x,\, |y| \leq L\}$, $\Omega_2 = \{x \leq \eta \leq L,\, |y| \leq L\}$, respectively.

$$\begin{aligned}&\int_{-L}^{L} e^{\sigma k y}\Big(\pm ikq_1(x,y) - \partial_x\, q_1(x,y)\Big)\,\mathrm{d}y \\ &= e^{\mp ik(x-L)} \int_{-L}^{L} e^{\sigma k y}\Big(\pm ik\, q(L,y) - \partial_x\, q(L,y)\Big)\,\mathrm{d}y\,, \end{aligned} \tag{23}$$

where $q_1(x,y)$ the solution corresponding to the specific boundary conditions (21).

To eliminate the unknown function $\partial_x q_1(x,y)$, subtract equations $(22)^+$ and $(23)^-$

$$\begin{aligned}\int_{-L}^{L} e^{\sigma k y} q_1(x,y)\,\mathrm{d}y &= \frac{1}{2ik}\Bigg[e^{ik(x-L)} \int_{-L}^{L} e^{\sigma k y}\Big(ik\, q(L,y) + \partial_x\; q(L,y)\Big)\,\mathrm{d}y \\ &+ e^{-ik(x+L)} \int_{-L}^{L} e^{\sigma k y}\Big(ikq(-L,y) - \partial_x\, q(-L,y)\Big)\,\mathrm{d}y\Bigg],\; k \in \mathrm{C} - \{0\}\,. \end{aligned} \tag{24}$$

Using boundary conditions (21) and denoting

$$\mathfrak{D}^{(j)}(\sigma k) = \int_{-L}^{L} e^{\sigma k y} f_{D}^{(j)}(y)\mathrm{d}y, \qquad \mathfrak{N}^{(j)}(\sigma k) = \int_{-L}^{L} e^{\sigma k y} f_{N}^{(j)}(y)\mathrm{d}y, \quad j = 1, 3\,, \tag{25}$$

where the unknown Neumann boundary values are defined as

$$\frac{\partial q}{\partial n}\Big|_{x = x_{\max},\, x_{\min}} = f_{N}^{(j)}(y), \quad j = 1, 3$$

and $\hat{\mathbf{n}}$ is the outgoing normal to the boundary, equation (24) rewrites

$$\int_{-L}^{L} e^{\sigma k y} q_1(x,y)\,\mathrm{d}y = \frac{1}{2ik}\left[e^{ik(x-L)}\Big(ik\,\mathfrak{D}^{(1)}(\sigma k) + \mathfrak{N}^{(1)}(\sigma k)\Big) + e^{-ik(x+L)}\Big(ik\,\mathfrak{D}^{(3)}(\sigma k) + \mathfrak{N}^{(3)}(\sigma k)\Big)\right], \quad k \in \mathrm{C} - \{0\}\,. \tag{26}$$

In order to compute the two unknowns $\mathfrak{N}^{(1)}(\sigma k)$ and $\mathfrak{N}^{(3)}(\sigma k)$, apply the global relation (20) in the domain Ω depicted in Figure 1, with boundary conditions (21) to derive the Dirichlet-to-Neumann correspondence,

$$e^{\pm ikL}\Big(\pm\, ik\,\mathfrak{D}^{(1)}(\sigma k) - \mathfrak{N}^{(1)}(\sigma k)\Big) - e^{\mp ikL}\Big(\pm\, ik\,\mathfrak{D}^{(3)}(\sigma k) + \mathfrak{N}^{(3)}(\sigma k)\Big) = 0, \quad k \in \mathrm{C}\,.$$

Solving the above system with respect to the unknown Neumann data and substituting the resulting expressions into (26) we obtain

$$\int_{-L}^{L} e^{\sigma k y} q_1(x,y)\,\mathrm{d}y = \frac{1}{e^{i2kL} - e^{-i2kL}}\left[\Big(e^{ik(x+L)} - e^{-ik(x+L)}\Big)\mathfrak{D}^{(1)}(\sigma k) - \Big(e^{ik(x-L)} - e^{-ik(x-L)}\Big)\mathfrak{D}^{(3)}(\sigma k)\right], \quad k \in \mathrm{C} - \left\{\frac{n\pi}{2L}\right\},\ n \in \mathrm{Z}\,. \tag{27}$$

Replacing $\sigma = 1$ and $\sigma = -1$ in the above equation respectively, and performing simple algebraic manipulations of the resulting two equations, we derive the relations

$$\int_{-L}^{L} \begin{matrix}\cosh\\ \sinh\end{matrix}\,(ky)\, q_1(x,y)\,\mathrm{d}y = \frac{1}{\sin(2kL)}\left[\sin\Big(k(x+L)\Big)\int_{-L}^{L} \begin{matrix}\cosh\\ \sinh\end{matrix}\,(ky)\, f_D^{(1)}(y)\,\mathrm{d}y - \sin\Big(k(x-L)\Big)\int_{-L}^{L} \begin{matrix}\cosh\\ \sinh\end{matrix}\,(ky)\, f_D^{(3)}(y)\,\mathrm{d}y\right], \quad k \in \mathrm{C} - \left\{\frac{n\pi}{2L}\right\},\ n \in \mathrm{Z}\,. \tag{28}$$

Evaluating equations (28) at $k = i(n+\frac{1}{2})\frac{\pi}{L}$ and at $k = i\frac{n\pi}{L}$, yields the cosine and sine Fourier transform of $q_1(x,y)$, respectively. The inversion formulae then gives

$$q_1^c(x,y) = \sum_{n=0}^{\infty}\left[b_n \sinh\Big(\big(n+\tfrac{1}{2}\big)\tfrac{\pi}{L}(x+L)\Big) + d_n \sinh\Big(\big(n+\tfrac{1}{2}\big)\tfrac{\pi}{L}(x-L)\Big)\right] \times \cos\Big(\big(n+\tfrac{1}{2}\big)\tfrac{\pi}{L}y\Big)\,, \tag{29}$$

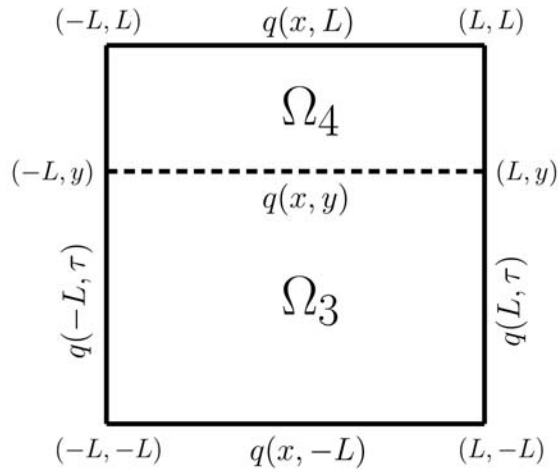

Figure 3: The subdomains Ω_3 and Ω_4 defined as $\Omega_3 = \{|x| \leq L \ -L \leq \tau \leq y\}$ and $\Omega_4 = \{|x| \leq L,\ y \leq \tau \leq L\}$, respectively.

and

$$q_1^s(x,y) = \sum_{n=1}^{\infty} \left[a_n \sinh\left(\tfrac{n\pi}{L}(x+L)\right) + c_n \sinh\left(\tfrac{n\pi}{L}(x-L)\right) \right] \sin\left(\tfrac{n\pi}{L} y\right), \quad (30)$$

where the Fourier constants a_n, b_n, c_n, d_n are given by equations (7)-(10). Analogous, applying the global relation $(20)^-$ in the subdomains

$$\Omega_3 = \Big\{|x| \leq L,\ -L \leq \tau \leq y\Big\}$$

and

$$\Omega_4 = \Big\{|x| \leq L,\ y \leq \tau \leq L\Big\},$$

depicted in Figure 3, with the following boundary conditions

$$\left.\begin{array}{cc} q(x,-L) = f_D^{(2)}(x), & q(x,L) = f_D^{(4)}(x) \\ q(L,\tau) = q(-L,\tau) = 0, & \partial_x\, q(L,\tau) = \partial_x\, q(-L,\tau) = 0 \end{array}\right\}. \quad (31)$$

we find the following equations

$$\int_{-L}^{L} e^{-ikx}\Big(\sigma k\, q_2(x,y) - \partial_y\, q_2(x,y)\Big)\mathrm{d}x = \\ e^{-\sigma k(y+L)} \int_{-L}^{L} e^{-ikx}\Big(\sigma k\, q(x,-L) - \partial_y\, q(x,-L)\Big)\mathrm{d}x\,, \quad (32)$$

$$\int_{-L}^{L} e^{-ikx}\Big(\sigma k\, q_2(x,y) - \partial_y\, q_2(x,y)\Big)\mathrm{d}x = \\ e^{-\sigma k(y-L)} \int_{-L}^{L} e^{-ikx}\Big(\sigma k\, q(x,L) - \partial_y\, q(x,L)\Big)\mathrm{d}x\,, \quad (33)$$

where $q_2(x,y)$ is the solution corresponding to boundary conditions (31). In order to eliminate the unknown function $\partial_y q_2(x,y)$, subtract (32) evaluated for $\sigma=1$ and (33) evaluated for $\sigma=-1$

$$\int_{-L}^{L} e^{-ikx}\, q_2(x,y)\,\mathrm{d}x = \frac{1}{2k}\Big[e^{-k(y+L)}\int_{-L}^{L} e^{-ikx}\Big(\sigma k\, q(x,-L) - \partial_y\, q(x,-L)\Big)\mathrm{d}x$$
$$+\, e^{k(y-L)}\int_{-L}^{L} e^{-ikx}\Big(\sigma k\, q(x,L) - \partial_y\, q(x,L)\Big)\mathrm{d}x, \quad k\in \mathrm{C}-\{0\}\,. \tag{34}$$

Using boundary conditions (31) and denoting

$$\mathfrak{D}^{(j)}(-ik) = \int_{-L}^{L} e^{-ikx} f_{D}^{(j)}(x)\mathrm{d}x, \quad \mathfrak{N}^{(j)}(-ik) = \int_{-L}^{L} e^{-ikx} f_{N}^{(j)}(x)\mathrm{d}x, \quad j=2,4\,. \tag{35}$$

where the unknown Neumann boundary values are defined as

$$\frac{\partial q}{\partial n}\Big|_{y=y_{\min},\, y_{\max}} = f_N^{(j)}(x), \quad j=2,4$$

and $\hat{\mathbf{n}}$ is the outgoing normal to the boundary, equation (34) rewrites

$$\int_{-L}^{L} e^{-ikx} q_2(x,y)\mathrm{d}x = \frac{1}{2k}\Bigg[e^{-k(y+L)}\Big(k\mathfrak{D}^{(2)}(-ik) + \mathfrak{N}^{(2)}(-ik)\Big)$$
$$+\, e^{k(y-L)}\Big(k\mathfrak{D}^{(4)}(-ik) + \mathfrak{N}^{(4)}(-ik)\Big)\Bigg], \quad k\in \mathrm{C}-\{0\}\,. \tag{36}$$

To compute the unknowns $\mathfrak{N}^{(2)}(-ik)$ and $\mathfrak{N}^{(4)}(-ik)$, apply $(20)^{-}$ in Ω with boundary conditions (31) to obtain

$$e^{-\sigma kL}\Big(\sigma k\mathfrak{D}^{(2)}(-ik) + \mathfrak{N}^{(2)}(-ik)\Big) - e^{\sigma kL}\Big(\sigma k\mathfrak{D}^{(4)}(-ik) - \mathfrak{N}^{(4)}(-ik)\Big) = 0, \tag{37}$$

Replacing $\sigma=1$ and $\sigma=-1$ in (37) respectively, we obtain two equations with unknowns the Fourier transforms of the Neumann data $\mathfrak{N}^{(j)}(-ik)$, $j=2,4$. Solving this system and substituting into (36) yields

$$\int_{-L}^{L} e^{-ikx}\, q_2(x,y)\mathrm{d}x = \frac{1}{e^{2kL}-e^{-2kL}}\Bigg[-\big(e^{k(y-L)} - e^{-k(y-L)}\big)\,\mathfrak{D}^{(2)}(-ik)$$
$$+\big(e^{k(y+L)} - e^{-k(y+L)}\big)\,\mathfrak{D}^{(4)}(-ik)\Bigg], \quad k\in \mathrm{C}-\left\{i\frac{n\pi}{2L}\right\}, \quad n\in \mathrm{Z}\,. \tag{38}$$

Simple algebraic manipulations of the latter equation together with (38) with k replaced by $-k$, leads to

$$\int_{-L}^{L} \begin{matrix}\cos\\ \sin\end{matrix}\,(kx)\, q_2(x,y)\,\mathrm{d}x = \frac{1}{\sinh(2kL)}\Bigg[-\sinh\Big(k(y-L)\Big)\int_{-L}^{L} \begin{matrix}\cos\\ \sin\end{matrix}\,(kx)\, f_{\scriptscriptstyle D}^{(2)}(x)\,\mathrm{d}x$$
$$+\sinh\Big(k(y+L)\Big)\int_{-L}^{L} \begin{matrix}\cos\\ \sin\end{matrix}\,(kx)\, f_{\scriptscriptstyle D}^{(4)}(x)\,\mathrm{d}x\Bigg], \quad k\in \mathrm{C}-\Big\{i\frac{n\pi}{2L}\Big\}, \quad n\in \mathrm{Z}\,. \tag{39}$$

Evaluating equations (39) at $k=(n+\frac{1}{2})\frac{\pi}{L}$ and at $k=\frac{n\pi}{L}$ yields the cosine and sine Fourier transform of $q_2(x,y)$ respectively. The inversion formulae then implies,

$$q_2^c(x,y) = \sum_{n=0}^{\infty}\Bigg[f_n \sinh\Big(\big(n+\tfrac{1}{2}\big)\tfrac{\pi}{L}(y-L)\Big) + h_n \sinh\Big(\big(n+\tfrac{1}{2}\big)\tfrac{\pi}{L}(y+L)\Big)\Bigg]$$
$$\times\cos\Big(\big(n+\tfrac{1}{2}\big)\tfrac{\pi}{L}x\Big)\,, \tag{40}$$

and

$$q_2^s(x,y) = \sum_{n=1}^{\infty}\Bigg[e_n \sinh\Big(\tfrac{n\pi}{L}(y-L)\Big) + g_n \sinh\Big(\tfrac{n\pi}{L}(y+L)\Big)\Bigg]\sin\big(\tfrac{n\pi}{L}x\big)\;, \tag{41}$$

where the Fourier constants e_n, f_n, g_n, h_n are given by equations (11)-(14). Adding equations (29),(30),(40) and (41) yields the classical transform (6).

6 Novel Integral Formulae

Proposition 6.1 *Let $q(x,y)$ satisfy the Laplace equation* (1) *in the interior Ω of the square defined by*

$$\Omega = \Big\{|x|\le L,\ |y|\le L\Big\},$$

and with boundary conditions specified in (3)*. Then $q(x,y)$ admits the following integral representation*

$$\begin{aligned} q(x,y) = &-\frac{1}{2i\pi}\int_{\mathcal{L}} e^{(y+L)k}\,\frac{\sin k(x-L)\,\mathcal{N}^{(3)}(k) - \sin k(x+L)\,\mathcal{N}^{(1)}(k)}{\sin 2kL}\,\mathrm{d}k \\ &+\frac{1}{2i\pi}\int_{\mathcal{R}} e^{(y-L)k}\,\frac{\sin k(x+L)\,\mathcal{M}^{(1)}(k) - \sin k(x-L)\,\mathcal{M}^{(3)}(k)}{\sin 2kL}\,\mathrm{d}k \\ &-\frac{1}{2\pi}\int_{\mathcal{U}} e^{i(x+L)k}\,\frac{\sinh k(y+L)\,\mathcal{N}^{(4)}(k) - \sinh k(y-L)\,\mathcal{N}^{(2)}(k)}{\sinh 2kL}\,\mathrm{d}k \\ &+\frac{1}{2\pi}\int_{\mathcal{D}} e^{i(x-L)k}\,\frac{\sinh k(y+L)\,\mathcal{M}^{(4)}(k) - \sinh k(y-L)\,\mathcal{M}^{(2)}(k)}{\sinh 2kL}\,\mathrm{d}k\,, \end{aligned} \tag{42}$$

where the functions $\mathcal{N}^{(j)}(k), \mathcal{M}^{(j)}(k), j = 1, 2, 3, 4$ *are defined as*

$$\mathcal{N}^{(j)}(k) = \sum_n (-1)^n \left(\alpha_n^{(j)} \frac{\frac{n\pi}{L}}{k^2 + \frac{n^2\pi^2}{L^2}} + \beta_n^{(j)} \frac{(n+\frac{1}{2})\frac{\pi}{L}}{k^2 + (n+\frac{1}{2})^2\frac{\pi^2}{L^2}} \right), \tag{43}$$

$$\mathcal{M}^{(j)}(k) = \sum_n (-1)^n \left(\beta_n^{(j)} \frac{(n+\frac{1}{2})\frac{\pi}{L}}{k^2 + (n+\frac{1}{2})^2\frac{\pi^2}{L^2}} - \alpha_n^{(j)} \frac{\frac{n\pi}{L}}{k^2 + \frac{n^2\pi^2}{L^2}} \right), \tag{44}$$

for every $k \in \mathrm{C} - \left\{ \pm i\frac{n\pi}{L}, \pm i\frac{(n+\frac{1}{2})\pi}{L} \right\}$, *if* $j = 1, 3$, *and*

$$\mathcal{N}^{(j)}(k) = \sum_n (-1)^n \left(\alpha_n^{(j)} \frac{\frac{n\pi}{L}}{k^2 - \frac{n^2\pi^2}{L^2}} + \beta_n^{(j)} \frac{(n+\frac{1}{2})\frac{\pi}{L}}{k^2 - (n+\frac{1}{2})^2\frac{\pi^2}{L^2}} \right), \tag{45}$$

$$\mathcal{M}^{(j)}(k) = \sum_n (-1)^n \left(\alpha_n^{(j)} \frac{\frac{n\pi}{L}}{k^2 - \frac{n^2\pi^2}{L^2}} - \beta_n^{(j)} \frac{(n+\frac{1}{2})\frac{\pi}{L}}{k^2 - (n+\frac{1}{2})^2\frac{\pi^2}{L^2}} \right), \tag{46}$$

for every $k \in \mathrm{C} - \left\{ \pm \frac{n\pi}{L}, \pm \frac{(n+\frac{1}{2})\pi}{L} \right\}$, *if* $j = 2, 4$. *The Fourier coefficients* $\alpha_n^{(j)}$ *and* $\beta_n^{(j)}$ *correlate with the coefficients* (7)-(14) *as* $\alpha_n^{(1)} = \sinh 2n\pi\, a_n$, $\alpha_n^{(2)} = -\sinh 2n\pi\, e_n$, $\alpha_n^{(3)} = -\sinh 2n\pi\, c_n$, $\alpha_n^{(4)} = \sinh 2n\pi\, g_n$ *and* $\beta_n^{(1)} = \sinh(2n+1)\pi\, b_n$, $\beta_n^{(2)} = -\sinh(2n+1)\pi\, f_n$, $\beta_n^{(3)} = -\sinh(2n+1)\pi\, d_n$, $\beta_n^{(4)} = \sinh(2n+1)\pi\, h_n$.
The contours $\mathcal{L}$, $\mathcal{R}$, $\mathcal{U}$ *and* $\mathcal{D}$ *are obtained by deformation processes described in the sequence and depicted in Figure 4.*

Equation (24), with σ replaced by -1, can be thought as the bilateral Laplace transform of $q_1(x,y)$, provided that the function $q_1(x,y)$ is such that the integral is convergent for some values of k. The inversion formula then implies

$$\begin{aligned} q_1(x,y) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \frac{e^{ky}}{2ik} \Bigg[& e^{ik(x-L)} \int_{-L}^{L} e^{\sigma ky} \Big(ik\, q(L,y) + \partial_x\, q(L,y) \Big)\, \mathrm{d}y \\ & + e^{-ik(x+L)} \int_{-L}^{L} e^{\sigma ky} \Big(ikq(-L,y) - \partial_x\, q(-L,y) \Big)\, \mathrm{d}y \Bigg] \mathrm{d}k, \end{aligned} \tag{47}$$

a formula useful for changing-type boundary value problems, as we will see in section 9.

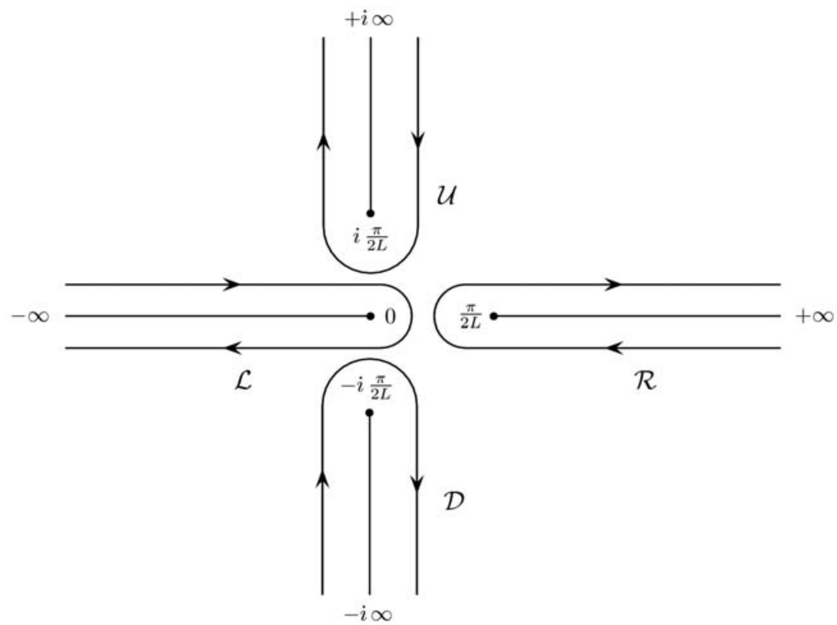

Figure 4: The contours $\mathcal{L}$, $\mathcal{R}$, $\mathcal{U}$, $\mathcal{D}$.

But since we are primarily concerned with Dirichlet data prescribed on the boundary, the inversion of (27) implies

$$q_1(x,y) = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} \frac{e^{ky}}{e^{i2kL} - e^{-i\,2kL}} \Bigg[\Big(e^{i\,k(x+L)} - e^{-i\,k(x+L)} \Big) \mathfrak{D}^{(1)}(-k) - \Big(e^{i\,k(x-L)} - e^{-i\,k(x-L)} \Big) \mathfrak{D}^{(3)}(-k) \Bigg] \mathrm{d}k\,, \tag{48}$$

where the Dirichlet transforms $\mathfrak{D}^{(j)}$ are given by equations (25).
Expanding the Dirichlet data $f^{j}_{_D}$ in a series of the form (5) yields $\mathfrak{D}^{(j)}(-k) = e^{kL}\,\mathcal{N}^{(j)}(k) + e^{-kL}\,\mathcal{M}^{(j)}(-k)$, where we note that

$$\mathcal{N}^{(j)}(k),\, \mathcal{M}^{(j)}(k) \to 0, \qquad \text{as} \quad k \to \infty\,.$$

Plugging the latter expression into eq. (48) we find

$$\begin{aligned} q_1(x,y) = {} & \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} e^{(y+L)k}\, \frac{\sin k(x+L)\,\mathcal{N}^{(1)}(k) - \sin k(x-L)\,\mathcal{N}^{(3)}(k)}{\sin 2kL}\, \mathrm{d}k \\ & + \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} e^{(y-L)k}\, \frac{\sin k(x+L)\,\mathcal{M}^{(1)}(k) - \sin k(x-L)\,\mathcal{M}^{(3)}(k)}{\sin 2kL}\, \mathrm{d}k\,. \end{aligned} \tag{49}$$

The Laplace transform of the function $q_1(x,y)$ displays a rapid decay as k approaches large values. Indeed, as $k \to \infty$ the denominator $e^{i\,2kL} - e^{-i\,2kL}$ is dominated by $e^{-i\,2kL}$ for $\operatorname{Im} k < 0$ and by $-e^{i\,2kL}$ for $\operatorname{Im} k > 0$. On the other hand, the nominator e^{ky} is bounded in the left ($\operatorname{Re} k \leq 0$) complex k-plane if $y \in [0, L]$ and in the right ($\operatorname{Re} k > 0$) complex k-plane if $y \in [-L, 0]$. Hence as $k \to \infty$,

$$\frac{e^{i\,k(x+L)} - e^{-i\,k(x+L)}}{e^{i\,2kL} - e^{-i\,2kL}} \sim \begin{cases} e^{i\,k(x-L)} - e^{-i\,k(x+3L)} & , \operatorname{Im} k < 0 \\ -e^{i\,k(x+3L)} + e^{-i\,k(x-L)} & , \operatorname{Im} k > 0 \end{cases} , \quad k \to \infty \,,$$

$$\frac{e^{i\,k(x-L)} - e^{-i\,k(x-L)}}{e^{i\,2kL} - e^{-i\,2kL}} \sim \begin{cases} e^{i\,k(x-3L)} - e^{-i\,k(x+L)} & , \operatorname{Im} k < 0 \\ -e^{i\,k(x+L)} + e^{-i\,k(x-3L)} & , \operatorname{Im} k > 0 \end{cases} , \quad k \to \infty \,.$$

Furthermore, the exponentials $e^{(y+L)k}$ and $e^{(y-L)k}$ are bounded in the left (Re $<$ 0) or the right (Re $>$ 0) complex k-plane, respectively.
The aforementioned analysis implies that the Bromwich contour in (49) can be replaced either by the contour $\mathcal{L}$ or by the contour $\mathcal{R}$, depicted in Figure 5. Equation (49) then becomes

$$\begin{aligned} q_1(x,y) = &-\frac{1}{2i\pi}\int_{\mathcal{L}} e^{(y+L)k}\,\frac{\sin k(x-L)\,\mathcal{N}^{(3)}(k) - \sin k(x+L)\,\mathcal{N}^{(1)}(k)}{\sin 2kL}\,\mathrm{d}k \\ &+\frac{1}{2i\pi}\int_{\mathcal{R}} e^{(y-L)k}\,\frac{\sin k(x+L)\,\mathcal{M}^{(1)}(k) - \sin k(x-L)\,\mathcal{M}^{(3)}(k)}{\sin 2kL}\,\mathrm{d}k\,. \end{aligned} \tag{50}$$

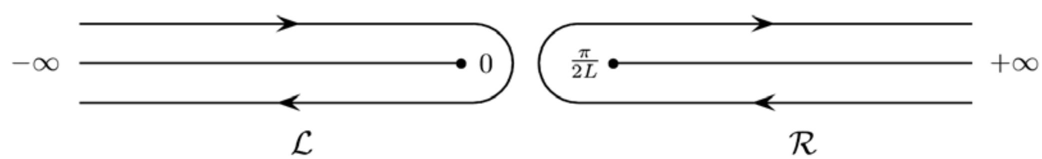

Figure 5: The contours $\mathcal{L}$ and $\mathcal{R}$.

The contour $\mathcal{L}$ begins and ends in the left ($\operatorname{Re} k < 0$) complex $k-$plane, such that $\operatorname{Re} k$ tends to $-\infty$ at each end, a technique known as Talbot's method [9]. In Talbot's method the initial contour is deformed to the region of the complex $k-$plane in which the factor $e^{f(k)}$ reduces in magnitude as much as possible. Analogous, the contour $\mathcal{R}$ begins and ends in the right ($\operatorname{Re} k > 0$) complex $k-$plane, such that $\operatorname{Re} k \to \infty$ at each end.
Similarly, equation (34) can be seen as the Fourier transform of $q_2(x, y)$. Thus,

the inversion formula implies

$$q_2(x,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{ikx}}{2k}\Bigg[e^{-k(y+L)}\int_{-L}^{L} e^{-ikx}\Big(\sigma k\, q(x,-L) - \partial_y\, q(x,-L)\Big)\mathrm{d}x$$
$$+\, e^{k(y-L)}\int_{-L}^{L} e^{-i\,kx}\Big(\sigma k\, q(x,L) - \partial_y\, q(x,L)\Big)\mathrm{d}x\Bigg]\mathrm{d}k\,, \tag{51}$$

is a relation which will prove valuable for changing-type boundary value problems.
For Dirichlet data the inversion of (38) yields

$$q_2(x,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{ikx}}{e^{2kL}-e^{-2kL}}\Bigg[-\left(e^{k(y-L)} - e^{-k(y-L)}\right)\mathfrak{D}^{(2)}(-ik)$$
$$+\left(e^{k(y+L)} - e^{-k(y+L)}\right)\mathfrak{D}^{(4)}(-ik)\Bigg]\mathrm{d}k\,.$$

Applying the previous analysis, the above equations yields

$$q_2(x,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i(x+L)k}\,\frac{\sinh k(y-L)\,\mathcal{N}^{(2)}(k) - \sinh k(y+L)\,\mathcal{N}^{(4)}(k)}{\sinh 2kL}\,\mathrm{d}k$$
$$+\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i(x-L)k}\,\frac{\sinh k(y+L)\,\mathcal{M}^{(4)}(k) - \sinh k(y-L)\,\mathcal{M}^{(2)}(k)}{\sinh 2kL}\,\mathrm{d}k\,. \tag{52}$$

From (52) it is evident that the Fourier transform of the function $q_2(x,y)$ displays a rapid decay as k approaches large values. Indeed, as $k\to\infty$ the denominator $e^{2kL}-e^{-2kL}$ is dominated by e^{-2kl} for $\mathrm{Re}\,k>0$ and by $-e^{2kl}$ for $\mathrm{Re}\,k<0$. The nominator $e^{i\,kx}$ on the other hand is bounded in the lower ($\mathrm{Im}\,k<0$) complex k-plane for every $x\in[-L,0]$ and in the upper ($\mathrm{Im}\,k>0$) complex k-plane for every $x\in[0,L]$. Hence as $k\to\infty$,

$$\frac{e^{k(y-L)}-e^{-k(y-L)}}{e^{2kL}-e^{-2kL}} \sim \begin{cases} -e^{k(y+L)} + e^{-k(y-3L)} & ,\mathrm{Re}\,k<0 \\ e^{k(y-3L)} - e^{-k(y+L)} & ,\mathrm{Re}\,k>0 \end{cases}, \quad k\to\infty\,,$$

$$\frac{e^{k(y+L)}-e^{-k(y+L)}}{e^{2kL}-e^{-2kL}} \sim \begin{cases} -e^{k(y+3L)} + e^{-k(y-L)} & ,\mathrm{Re}\,k<0 \\ e^{k(y-L)} - e^{-k(y+3L)} & ,\mathrm{Re}\,k>0 \end{cases}, \quad k\to\infty\,.$$

More over, the exponentials $e^{i(x+L)k}$ and $e^{i(x-L)k}$ are bounded in the upper ($\mathrm{Im}>0$) or the lower ($\mathrm{Im}<0$) complex k-plane, respectively.
Thus, the line with endpoints $-\infty$ and $+\infty$ present in (52), can be replaced

by either the contour $\mathcal{U}$ or by the contour $\mathcal{D}$ depicted in Figure 6. Hence, (52) can be rewritten as

$$q_2(x,y) = -\frac{1}{2\pi}\int_{\mathcal{U}} e^{i\,(x+L)k}\,\frac{\sinh k(y+L)\,\mathcal{N}^{(4)}(k) - \sinh k(y-L)\,\mathcal{N}^{(2)}(k)}{\sinh 2kL}\,\mathrm{d}k$$
$$+\frac{1}{2\pi}\int_{\mathcal{D}} e^{i\,(x-L)k}\,\frac{\sinh k(y+L)\,\mathcal{M}^{(4)}(k) - \sinh k(y-L)\,\mathcal{M}^{(2)}(k)}{\sinh 2kL}\,\mathrm{d}k\,. \tag{53}$$

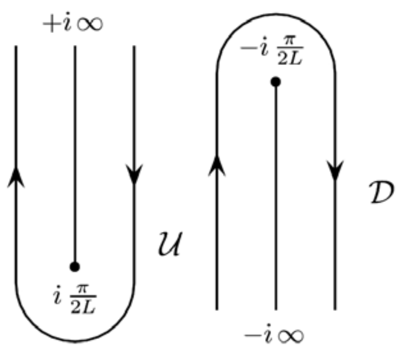

Figure 6: The contours $\mathcal{U}$ and $\mathcal{D}$ respectively.

Adding equations (50) and (53) yields (42).

7 A Novel Integral Representation

Proposition 7.1 *Suppose that there exist a function $q(x,y)$ with sufficient smoothness all the way to the boundary, satisfying the Laplace equation* (1) *in the interior of the square Ω defined by*

$$\Omega = \Big\{|x| \le L,\ |y| \le L\Big\},$$

with Dirichlet boundary conditions prescribed by equations (3)*. Then the solution $q(x,y)$ admits the following integral representation*

$$\begin{aligned} q(x,y) = {} & \frac{i}{2\pi}\int_{-\infty}^{+\infty} e^{ik(x-L)}\,\Big(\mathcal{J}(y;k)\,f_{D}^{(1)}(\tau)\Big)\,\mathrm{d}k \\ & + \frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty} e^{k(y+L)}\,\Big(\mathcal{I}(x;k)\,f_{D}^{(2)}(\eta)\Big)\,\mathrm{d}k \\ & - \frac{i}{2\pi}\int_{-\infty}^{+\infty} e^{i\,k(x+L)}\,\Big(\mathcal{J}(y;k)\,f_{D}^{(3)}(\tau)\Big)\,\mathrm{d}k \\ & - \frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty} e^{k(y-L)}\,\Big(\mathcal{I}(x;k)\,f_{D}^{(4)}(\eta)\Big)\,\mathrm{d}k, \quad k \in \mathrm{C}\,, \end{aligned} \tag{54}$$

where the integral operators $\mathcal{I}(x;k)$ *and* $\mathcal{J}(y;k)$ *are defined as*

$$\mathcal{I}(x;k) = \int_{-L}^{x} d\eta \, e^{ik(\eta - x)} + \int_{x}^{L} d\eta \, e^{-ik(\eta - x)}, \quad k \in \mathrm{C}\,, \tag{55}$$

and

$$\mathcal{J}(y;k) = \int_{-L}^{y} d\tau \, e^{k(\tau - y)} + \int_{y}^{L} d\tau \, e^{-k(\tau - y)}, \quad k \in \mathrm{C}\,, \tag{56}$$

respectively.

Employing the global relation (20) in the subdomains Ω_3 and Ω_4 depicted in Figure 3, with boundary conditions

$$\left. \begin{array}{ll} q(L,\tau) = f_{D}^{(1)}(\tau), & q(x,-L) = q(-L,\tau) = q(x,L) = 0 \\ & \partial_y \, q(x,-L) = \partial_x \, q(-L,\tau) = \partial_y \, q(x,L) = 0 \end{array} \right\}\,, \tag{57}$$

we derive the following equations

$$\begin{aligned} &\int_{-L}^{L} e^{\pm ikx} \Big(\sigma k \, q_1(x,y) - \partial_y \; q_1(x,y) \Big) \mathrm{d}x = \\ &- e^{\pm ikL} \int_{-L}^{y} e^{\sigma k(\tau - y)} \Big(\pm ik f_{D}^{(1)}(\tau) - f_{N}^{(1)}(\tau) \Big) d\tau \,, \end{aligned} \tag{58}$$

$$\begin{aligned} &\int_{-L}^{L} e^{\pm ikx} \Big(\sigma k \, q_1(x,y) - \partial_y \; q_1(x,y) \Big) \mathrm{d}x = \\ &e^{\pm ikL} \int_{y}^{L} e^{\sigma k(\tau - y)} \Big(\pm ik f_{D}^{(1)}(\tau) - f_{N}^{(1)}(\tau) \Big) d\tau \,, \end{aligned} \tag{59}$$

where the solution $q_1(x,y)$ corresponds to the boundary conditions (57). Replace in the former $\sigma = 1$ and in the latter $\sigma = -1$. Subtracting the resulting equations, not only eliminates the unknown function $\partial_y \, q_1(x,y)$, but also provides the Fourier transform for the solution $q_1(x,y)$,

$$\int_{-L}^{L} e^{\pm ikx} \; q_1(x,y) \, \mathrm{d}x = -\frac{e^{\pm ikL}}{2k} \, \mathcal{J}(y;k) \Big(\pm ik f_{D}^{(1)}(\tau) - f_{N}^{(1)}(\tau) \Big), \quad k \in \mathrm{C} - \{0\}, \tag{60}$$

where the integral operator $\mathcal{J}(y;k)$ is defined by eq. (56).
The inverse of $(60)^{-}$ gives

$$q_1(x,y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ik(x-L)} \frac{1}{2k} \mathcal{J}(y;k) \Big(ik f_{D}^{(1)}(\tau) + f_{N}^{(1)}(\tau) \Big) \mathrm{d}k \,. \tag{61}$$

Eliminating the unknown Neumann boundary data $f_N^{(1)}(\tau)$ in (61), with the aid of $(60)^+$, we find

$$q_1(x,y) = \frac{i}{2\pi}\int_{-\infty}^{+\infty} e^{ik(x-L)}\Big(\mathcal{J}(y;k) f_D^{(1)}(\tau)\Big)\,\mathrm{d}k + \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ik(x-L)}\left\{\int_{-L}^{L} e^{ik(x-L)} q_1(x,y)\mathrm{d}x\right\}\mathrm{d}k\,. \tag{62}$$

As k tends to infinity, both $e^{k(\tau-y)}$ and $e^{-k(\tau-y)}$ tend to zero since $\tau - y \leq 0$ for $\tau \in [-L, y]$ and $\tau - y \geq 0$ for $\tau \in [y, L]$, respectively. Thus, the integral operator $\mathcal{J}(y;k)$ is bounded as a function of k in the right ($\mathrm{Re}\,k \geq 0$) complex k-plane. Furthermore, since $x - L \leq 0$, the exponential $e^{ik(x-L)}$ is bounded in the lower ($\mathrm{Im}\,k \leq 0$) complex k-plane.

Assuming the change of the order of integration being permitted, the second integral appearing on the right-hand side of eq. (62) takes the form

$$\int_{-\infty}^{+\infty} e^{ik2(x-L)}\,\mathrm{d}k\,. \tag{63}$$

By deforming the line with endpoints $-\infty$ and $+\infty$ into a contour that begins and ends in the lower ($\mathrm{Im}\,k \leq 0$) complex k-plane, such that $\mathrm{Im}\,k \to -\infty$ at each end, the integral (63) yields a zero contribution since $e^{ik(x-L)}$ is *analytic* and *bounded* in $\mathrm{Im}\,k \leq 0$.

Hence, (62) becomes

$$q_1(x,y) = \frac{i}{2\pi}\int_{-\infty}^{+\infty} e^{ik(x-L)}\Big(\mathcal{J}(y;k) f_D^{(1)}(\tau)\Big)\,\mathrm{d}k\,. \tag{64}$$

Repeating the above procedure in the subdomains Ω_3 and Ω_4 with boundary conditions

$$\begin{aligned} &q(-L,\tau) = f_D^{(3)}(\tau), \quad q(L,\tau) = q(x,-L) = q(x,L) = 0\\ &\partial_x\, q(L,\tau) = \partial_y\, q(x,-L) = \partial_y\, q(x,L) = 0\,, \end{aligned}$$

we derive the relation

$$q_3(x,y) = -\frac{i}{2\pi}\int_{-\infty}^{+\infty} e^{ik(x+L)}\Big(\mathcal{J}(y;k) f_D^{(3)}(\tau)\Big)\,\mathrm{d}k\,, \tag{65}$$

where the solution $q_3(x,y)$ corresponds to the specific boundary conditions described above.

Similar, by applying the global relation (20) in the subdomains Ω_1 and Ω_2, depicted in Figure 2, with boundary conditions

$$\left.\begin{aligned} q(\eta,-L) = f_D^{(2)}(\eta), \qquad &q(L,y) = q(-L,y) = q(\eta,L) = 0\\ &\partial_x\, q(L,y) = \partial_x\, q(-L,y) = \partial_y\, q(\eta,L) = 0 \end{aligned}\right\}, \tag{66}$$

we derive the following equations

$$\int_{-L}^{L} e^{\sigma k y}\Big(\pm ik\, q_2(x,y) - \partial_x\ q_2(x,y)\Big)\mathrm{d}y = \\ e^{-\sigma k L}\int_{-L}^{x} e^{\pm ik(\eta - x)}\Big(\sigma k f_{D}^{(2)}(\eta) + f_{N}^{(2)}(\eta)\Big)d\eta\,, \tag{67}$$

$$\int_{-L}^{L} e^{\sigma k y}\Big(\pm ik\, q_2(x,y) - \partial_x\ q_2(x,y)\Big)\mathrm{d}y = \\ -\,e^{-\sigma k L}\int_{x}^{L} e^{\pm ik(\eta - x)}\Big(\sigma k f_{D}^{(2)}(\eta) + f_{N}^{(2)}(\eta)\Big)d\eta\,, \tag{68}$$

where the solution $q_2(x,y)$ corresponds to the boundary conditions (66). The unknown function $\partial_x\, q_2(x,y)$, is eliminated by adding equations $(67)^+$ and $(68)^-$

$$\int_{-L}^{L} e^{\sigma k y}\, q_2(x,y)\,\mathrm{d}y = \frac{e^{-\sigma k L}}{2ik}\,\mathcal{I}(x;k)\Big(\sigma\, k\, f_{D}^{(2)}(\eta) + f_{N}^{(2)}(\eta)\Big), \quad k \in \mathrm{C} - \{0\}\,, \tag{69}$$

where the integral operator $\mathcal{I}(x;k)$ is defined by eq. (55).
Evaluate equation (69) for $\sigma = -1$ to retrieve the bilateral Laplace transform for the solution $q_2(x,y)$, provided that $q_2(x,y)$ is such that the integral is convergent for some values of k. Then inversion implies the representation

$$q_2(x,y) = -\frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} e^{k(y+L)}\frac{1}{2ik}\,\mathcal{I}(x;k)\Big(k f_{D}^{(2)}(\eta) - f_{N}^{(2)}(\eta)\Big)\,\mathrm{d}k\,. \tag{70}$$

The unknown Neumann boundary values $f_{N}^{(2)}(\eta)$ are eliminated with the aid of (69) evaluated at $\sigma = 1$.
Eq. (70) then becomes

$$q_2(x,y) = \frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty} e^{k(y+L)}\,\Big(\mathcal{I}(x;k) f_{D}^{(2)}(\eta)\Big)\,\mathrm{d}k \\ + \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} e^{k(y+L)}\bigg\{\int_{-L}^{L} e^{k(y+L)} q_2(x,y)\mathrm{d}y\bigg\}\,\mathrm{d}k\,. \tag{71}$$

The exponentials appearing in equation (55) are bounded in the lower ($\mathrm{Im}\, k \le 0$) complex k-plane. Hence, as $k \to \infty$, the integral operator $\mathcal{I}(x;k)$ is bounded as a function of k in the lower ($\mathrm{Im}\, k \le 0$) complex k-plane. Moreover, as $k \to \infty$ the exponential $e^{k(y+L)}$ tends to zero in the left ($\mathrm{Re}\, k \le 0$) complex k-plane.

Interchanging the order of integration in the second integral appearing on the right-hand side of eq. (71) we find

$$\int_{c-i\infty}^{c+i\infty} e^{2k(y+L)}\,\mathrm{d}k\,. \tag{72}$$

By deforming the Bromwich line into a contour that begins and ends in the left ($\operatorname{Re} k \leq 0$) complex k-plane, such that $\operatorname{Re} k \to -\infty$ at both ends, the integral (72) yields a zero contribution.
Hence, (71) yields

$$q_2(x,y) = \frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty} e^{k(y+L)}\Big(\mathcal{I}(x;k)f_D^{(2)}(\tau)\Big)\,\mathrm{d}k\,, \tag{73}$$

An analysis similar to the one described previously, applied in the subdomains Ω_1 and Ω_2, with boundary conditions

$$\left.\begin{array}{ll} q(\eta,L) = f_D^{(4)}(\eta), & q(L,y) = q(-L,y) = q(\eta,-L) = 0 \\ & \partial_x\, q(L,y) = \partial_x\, q(-L,y) = \partial_y\, q(\eta,-L) = 0 \end{array}\right\}, \tag{74}$$

reveals that

$$q_4(x,y) = -\frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty} e^{k(y-L)}\Big(\mathcal{I}(x;k)f_D^{(4)}(\eta)\Big)\,\mathrm{d}k\,, \tag{75}$$

where $q_4(x,y)$ is the solution corresponding to the boundary conditions (74). Finally, adding equations (64), (65), (73) and (75) we obtain (54).

8 Changing-type Boundary Value Problems

The Dirichlet-to-Neumann correspondence, i.e. the global relation implemented at the boundary of the fundamental domain, can be used for the analysis of problems with changing-type boundary conditions. For example, consider the following problem

$$q(L,y) = f_D^{(1)}(y), \quad y\in[-L,0], \quad \partial_x\, q(L,y) = f_N^{(1)}(y), \quad y\in[0,L]\,, \tag{76}$$

$$q(x,-L) = f_D^{(2)}(x), \quad x\in[-L,0], \quad -\partial_y\, q(x,-L) = f_N^{(2)}(x), \quad x\in[0,L]\,, \tag{77}$$

$$q(-L,y) = f_D^{(3)}(y), \quad y\in[-L,0], \quad -\partial_x\, q(-L,y) = f_N^{(3)}(y), \quad y\in[0,L]\,, \tag{78}$$

$$q(x,L) = f_D^{(4)}(x), \quad x\in[-L,0], \quad \partial_y\, q(x,L) = f_N^{(4)}(x), \quad x\in[0,L]\,, \tag{79}$$

where we assume that the functions $f_D^{(j)}$ and $f_N^{(j)}$ are smooth and continuous at the corners of the square and also at the points $(0, L)$, $(0, -L)$, $(L, 0)$ and $(-L, 0)$.

It is a well known fact that, due to the linearity of the Laplacian operator, the solution $q(x, y)$ can be written as a linear combination of "partial solutions" which correspond to specific boundary conditions. Therefore, implementing the global relation $(20)^+$, with σ replaced by -1, in the domain Ω depicted in Figure 1, we obtain the following relation

$$\int_{-L}^{L} e^{-ky}\Big(ik\, q_1(L, y) - \partial_x\, q_1(L, y)\Big)\,\mathrm{d}y = 0\,, \tag{80}$$

where $q_1(x, y)$ is a "partial solution" corresponding to given boundary conditions prescribed on side 1 of the square and zero boundary conditions on the remaining sides.

Splitting the above integral into one part valid in the interval $-L \leq y \leq 0$ and a second part valid in the remaining interval and using boundary conditions (76) we find

$$\begin{aligned} ik \int_{0}^{L} e^{-ky}\, q_1(L, y)\,\mathrm{d}y - \int_{-L}^{0} e^{-ky}\,\partial_x\, q_1(L, y)\,\mathrm{d}y = \\ \int_{0}^{L} e^{-ky}\, f_N^{(1)}(y)\,\mathrm{d}y - ik \int_{-L}^{0} e^{-ky}\, f_D^{(1)}(y)\,\mathrm{d}y\,. \end{aligned} \tag{81}$$

Introducing the variable $z = e^{-kL}$, eq. (81) becomes the Riemann-Hilbert problem

$$\Phi_1^+(z) - \Phi_1^-(z) = \varphi_1(z), \quad z \in \mathrm{C}\,, \tag{82}$$

where

$$\Phi_1^+(z) = ik \int_{0}^{L} e^{-ky}\, q_1(L, y)\,\mathrm{d}y, \qquad \Phi_1^-(z) = \int_{-L}^{0} e^{-ky}\,\partial_x\, q_1(L, y)\,\mathrm{d}y\,, \tag{83}$$

and $\varphi_1(z)$ is the known function

$$\varphi_1(z) = \int_{0}^{L} e^{-ky}\, f_N^{(1)}(y)\,\mathrm{d}y - ik \int_{-L}^{0} e^{-ky}\, f_D^{(1)}(y)\,\mathrm{d}y\,. \tag{84}$$

Note that $\Phi_1^+(z)$ is analytic as z tends to zero, where else $\Phi_1^-(z)$ is analytic as $z \to \infty$. Moreover, $\Phi_1^-(z) \to 0$ as $z \to \infty$.

Employing the global relation $(20)^-$, with σ replaced by 1, in the domain Ω depicted in Figure 1, for the "partial solution" $q_2(x, y)$ corresponding to given boundary conditions prescribed on side 2 of the square and zero boundary conditions on the remaining sides, we obtain

$$\int_{-L}^{L} e^{-ikx}\Big(k\, q_2(x, -L) - \partial_y\, q_2(x, -L)\Big)\,\mathrm{d}x = 0\,. \tag{85}$$

Splitting the above integral into two parts and using boundary conditions (77) we find

$$k\int_0^L e^{-ikx}\, q_2(x,-L)\,\mathrm{d}x - \int_{-L}^0 e^{-ikx}\,\partial_y\, q_2(x,-L)\,\mathrm{d}x =$$
$$-\int_0^L e^{-ikx}\, f_{\scriptscriptstyle N}^{(2)}(x)\,\mathrm{d}x - k\int_{-L}^0 e^{-ikx}\, f_{\scriptscriptstyle D}^{(2)}(x)\,\mathrm{d}x\,. \tag{86}$$

Introducing the variable $z' = e^{-ikL}$, eq. (86) becomes the Riemann-Hilbert problem

$$\Phi_2^+(z') - \Phi_2^-(z') = \varphi_2(z'), \quad z' \in \mathrm{C}\,, \tag{87}$$

where

$$\Phi_2^+(z') = k\int_0^L e^{-ikx}\, q_2(x,-L)\,\mathrm{d}x, \qquad \Phi_2^-(z') = \int_{-L}^0 e^{-ikx}\,\partial_y\, q_2(x,-L)\,\mathrm{d}x\,, \tag{88}$$

and $\varphi_2(z')$ is the known function

$$\varphi_2(z') = -\int_0^L e^{-ikx}\, f_{\scriptscriptstyle N}^{(2)}(x)\,\mathrm{d}x - k\int_{-L}^0 e^{-ikx}\, f_{\scriptscriptstyle D}^{(2)}(x)\,\mathrm{d}x\,. \tag{89}$$

Note that $\Phi_2^+(z')$ is analytic as z' tends to zero, where else $\Phi_2^-(z')$ is analytic as $z' \to \infty$. Moreover, $\Phi_2^-(z') \to 0$ as $z' \to \infty$.
Repeating the above procedures for the sides 3 and 4, one is led to the Riemann-Hilbert problems

$$\Phi_3^+(z) - \Phi_3^-(z) = \varphi_3(z), \quad z = e^{-kL}\,, \tag{90}$$

and

$$\Phi_4^+(z') - \Phi_4^-(z') = \varphi_4(z'), \quad z' = e^{-ikL}\,, \tag{91}$$

where

$$\Phi_3^+(z) = ik\int_0^L e^{-ky}\, q_3(L,y)\,\mathrm{d}y, \quad \Phi_3^-(z) = \int_{-L}^0 e^{-ky}\,\partial_x\, q_3(L,y)\,\mathrm{d}y\,, \tag{92}$$

$$\Phi_4^+(z') = k\int_0^L e^{-ikx}\, q_4(x,-L)\,\mathrm{d}x, \quad \Phi_4^-(z') = \int_{-L}^0 e^{-ikx}\,\partial_y\, q_4(x,-L)\,\mathrm{d}x\,, \tag{93}$$

and $\varphi_3(z)$, $\varphi_4(z')$ are the known functions

$$\varphi_3(z) = -\int_0^L e^{-ky}\, f_{\scriptscriptstyle N}^{(3)}(y)\,\mathrm{d}y - ik\int_{-L}^0 e^{-ky}\, f_{\scriptscriptstyle D}^{(3)}(y)\,\mathrm{d}y\,, \tag{94}$$

$$\varphi_4(z') = \int_0^L e^{-ikx}\, f_{\scriptscriptstyle N}^{(4)}(x)\,\mathrm{d}x - k\int_{-L}^0 e^{-ikx}\, f_{\scriptscriptstyle D}^{(4)}(x)\,\mathrm{d}x\,. \tag{95}$$

The *scalar* Riemann-Hilbert problems (82), (87), (90) and (91) can be solved in closed form (see [7] and specially Appendix 2 of the reference given, since the boundary of the fundamental domain Ω is a *piecewise* smooth contour). The solution $q(x, y)$ is given by adding equations (47) and (51). Splitting the integrals on the right-hand side of the resulting equation into two parts and given boundary conditions (76)-(79), the unknown boundary conditions are obtained by solving the Riemann-Hilbert problems derived in this section, and hence the solution $q(x, y)$ is completely determined.

9 Existence of the Integral transforms and the Inversion formulae

The aforementioned operations are justified introducing the functional space $L_1(\mathrm{R})$ for every function $q : \mathrm{R} \to \mathrm{C}$ exhibiting exponential growth, i.e. equiped with the property

$$|q(\mathbf{x})| \leq C\, e^{\sigma\, x}.$$

Then [8,6],

Theorem 9.1 (Existence of the Bilateral Laplace Transform) *Let* $q \in L_1(\epsilon, E)$, $-\infty < \epsilon < E < +\infty$, *belonging to both* $L_1(\mathrm{R}; e^{-\sigma_1 x_i})$ *and* $L_1(\mathrm{R}; e^{-\sigma_2 x_i})$. *Then the bilateral Laplace transform* $Q(x_2; k) = \mathcal{BL}\{q(x_1, x_2); k\}$ *exist and the integral*

$$Q(x_2; k) = \int_{-\infty}^{\infty} e^{-k\, x_1}\, q(x_1, x_2)\, \mathrm{d}x_1$$

is absolutely and uniformly convergent in the strip $\sigma_1 < c < \sigma_2$

Theorem 9.2 (Inversion formula) *Let* $q(x_1, x_2), e^{-k\, x_i}\, q(x_1, x_2) \in C[\epsilon, E] \cap L_1(\mathrm{R})$, $\sigma_1 < c = Re\, k < \sigma_2$. *Then the following inversion formula for the bilateral Laplace transformation*

$$q(x_1, x_2) = \frac{1}{2\pi i} \lim_{R \to \infty} \int_{c-i\, R}^{c+i\, R} e^{k\, x_1}\, Q(x_2; k)\, \mathrm{d}k\,,$$

is valid for every inerval $[\epsilon, E] \subset \mathrm{R}$.

Similar conclusions, due do the connection with the (bilateral) Laplace transform, are valid for the Fourier transform.

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