Int. Journal of Math. Analysis, Vol. 4, 2010, no. 48, 2395 - 2400

Fixed Point Theorems in Vector Metric Spaces

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Abstract. In this paper we consider the idea of vector metric space introduced by Sreenivasan where the distance between two points of the space is taken to be a sequence of real numbers. In this setting, we consider the generalized contraction mapping of Krasnoselskii and uniformly locally contractive mappings of Edelstein, which are perceived to be important generalizations of Banach's fixed point theorem in metric space. Here, we show that these mappings posses unique fixed points as well.

Mathematics Subject Classification: Primary 54H25; Secondary 47H10

Keywords: vector metric space, generalized contraction, ϵ-chainable, uniformly locally contractive

1. Introduction

The idea of a vector metric space was initiated by Sreenivasan [7] from which it follows that a metric space is a vector metric space. It appears that so far little work has been done in respect of fixed point theory in such spaces. Recently in [6] it has been shown that an analogue of Banach's fixed point theorem can be obtained in a complete vector metric space. Under this situation it is reasonable to think whether some of the most useful generalizations of Banach's fixed point theorem can be extended to a vector metric space. In this paper we enquire in this direction and prove fixed point theorems for two mappings (see $[1], [2], [3], [4], and$ [5]) whose conditions are much weaker than the contraction condition.

2. Preliminaries

Let α be a vector given by a real sequence $\{a_n\}$ *i.e.* $\alpha = \{a_n\}$. Let S be the set of all such vectors. Let θ denote the zero vector $\{0\}$ and $\hat{\theta}$ stands for the unit vector $\{1\}$. The set S is partially ordered by the relation $\alpha \leq \beta$ if $a_n \leq b_n$

for all n, where $\alpha = \{a_n\}$ and $\beta = \{b_n\}$ are elements of S. Thus $\alpha \in S$ will be non-negative if $\alpha \geq \theta$. Also for any $\alpha = \{a_n\} \in S$ and any real t we define $t\alpha = \{ta_n\}$. If $\alpha = \{a_n\} \in S$ and $\beta = \{b_n\} \in S$ then $\alpha + \beta = \{a_n + b_n\}$ and $\alpha = \beta$ if $a_n = b_n$ for all n.

Further we assume that S is endowed with the Frećhet metric ρ i.e. for any $a = \{a_n\}, b = \{b_n\} \in S$

$$\rho(a, b) = \sum_{n=1}^{\infty} \frac{1}{2^n} . \frac{|a_n - b_n|}{1 + |a_n - b_n|}.$$

It is well known that S is a complete metric space with respect to the metric ρ and the convergence in S is equivalent to co-ordinate wise convergence i.e. $a^k \to a$ as $k \to \infty$ if and only if $a_n^k \to a_n$ for all n where $a^k = \{a_n^k\}$ and $a = \{a_n\}$ are members of S.

Definition 1. Let X be a non-void set and let a vector $V(x, y)$ be defined for each pair x, y of X. Then $V(x, y)$ is called a vector metric and X a vector metric space if the following conditions are satisfied

$$(i)\ V(x, y) \geq \theta \text{ and } V(x, y) = \theta \text{ iff } x = y,$$
$$(ii)\ V(x, y) = V(y, x),$$
$$(iii)\ V(x, y) \leq V(x, z) + V(z, y), \forall\ x, y, z \in X.$$

The vector metric space X (or shortly v.m.s) with the vector metric $V(x, y)$ will be denoted by (X, V).

The following Theorem from [6] will be needed.

Theorem 1. *If $V(x, y) = \{d_n(x, y)\}$ be a vector metric then each $d_n(x, y)$ is a quasi-metric function; conversely if each $d_n(x, y)$ is a quasi-metric and the relation $d_n(x, y) = 0$ for all n implies $x = y$, then $V(x, y) = \{d_n(x, y)\}$ is a vector metric.*

If (X, d) is a metric space and if $V(x, y) = \{d(x, y)\}$, then (X, V) is a vector metric space. So any metric space, is in a sense, a vector metric space. For the converse we can only say from Theorem 1 that a vector metric space is, in general, a quasi-metric space. Examples of some vector metric spaces can be found in [6] and [7].

Definition 2[6]. Let (X, V) be a v.m.s. A sequence $\{x_k\}$ in X is said to converge to an element $x \in X$, $x_k \to x$ as $k \to \infty$ if $V(x_k, x) \to \theta$ as $k \to \infty$ which is interpreted as $d_n(x_k, x) \to 0$ as $k \to \infty$ for all n, where $V(x_k, x) = \{d_n(x_k, x)\}$.

Definition 3[7]. A sequence $\{x_n\}$ in (X, V) is said to be a vector Cauchy sequence if $V(x_n, x_m) \to \theta$, whenever $m, n \to \infty$.

Definition 4[6]. A v.m.s (X, V) is said to be complete if every vector Cauchy sequence in X converges to an element in X. Otherwise X is called incomplete.

Definition 5[6]. Let (X, V) be a v.m.s. A mapping $T : X \to X$ is called a vector contraction if there exists $r,\ 0 < r < 1$ such that

$$V(Tx, Ty) \leq r.V(x, y) \text{ for all } x, y \in X.$$

In [6] it was shown that in a complete vector metric space every vector contraction has a unique fixed point.

We will also need the following definition.

Definition 6. Let $T : X \to X$ be a mapping on a v.m.s (X, V). T is said to be continuous at $x \in X$ if every sequence $\{y_n\} \in X$ converges to x, the sequence $\{Ty_n\}$ also converges to Tx, i.e. $V(Ty_n, Tx) \to \theta$ as $n \to \infty$, whenever $V(y_n, x) \to \theta$ as $n \to \infty$.

Throughout if $x \in X$ be any vector we shall denote the kth co-ordinate of x as $(x)_k$.

3. Main Results

In this section we prove our main results. We first introduce the following definition.

Definition 7. If (X, V) is a v.m.s and $T : X \to X$ be a mapping then T is called a generalized contraction mapping if

$$V(Tx, Ty) < \alpha(u, v)V(x, y)$$

where $\theta < u < v$ are such that $u \leq V(x, y) < v$ and α is a function defined on $S \times S$ with the property that $\alpha(u, v) \in [0, 1)$ for all $u, v \in S$.

Theorem 2. *If T is a generalized contraction mapping on a complete v.m.s (X, V), then there exists a unique fixed point $x_0 \in X$ of T.*

Proof. Consider the sequence of vectors $\{a_n\}$ where $a_n = V(x_n, x_{n-1})$ where $x_1 = Tx_0$ and $x_n = Tx_{n-1}$ for all n. From the properties of T it is clear that $\{a_n\}$ is a non-increasing sequence in X. Since every co-ordinate of a_n is also non-increasing, so we can find $u \in S$ such that $u = lim\ a_n$. We shall show that $u = \theta$ i.e. every co-ordinate of u is zero. If not, let the kth co-ordinate of u i.e. $(u)_k > 0$ and choose $n_0 \in N$ such that kth co-ordinate of a_{n_0} is less than $(u)_k + 1$ i.e. $(a_{n_0})_k < (u)_k + 1$. Now choose $v \in X$ such that

$$(v)_k = (u)_k + 1 \text{ and } (v)_i = (a_{n_0})_i + 1, \forall\ i\ (\neq k).$$

Then clearly $u \leq a_{n_0} < v$. Since$\{a_k\}$ is non-increasing this implies $u \leq a_k < v$, $\forall\ k \geq n_0$. Now

$$a_{n_0+1} = V(Tx_{n_0}, Tx_{n_o-1}) \leq \alpha(u, v)V(x_{n_0}, x_{n_0-1}) = \alpha(u, v)a_{n_0}$$
$$a_{n_0+2} \leq \alpha(u, v)a_{n_0+1} \leq (\alpha(u, v))^2 a_{n_0}$$

and so on. Thus we have

$$a_{n_0+m} \leq (\alpha(u, v))^m a_{n_0} \text{ for all } m \geq 1.$$

Thus

$$(a_{n_0+m})_k \leq (\alpha(u,v))^m[(u)_k + 1] \text{ for all } m \geq 1.$$

Since $\alpha(u,v) < 1$, this implies $(u)_k = \lim_{n\to\infty}(a_n)_k = 0$, a contradiction. Hence $u = \theta$.

Next we show that $\{x_n\}$ is a vector Cauchy sequence i.e. $V(x_k, x_{k+p}) \to \theta$ when $k \to \infty$ for any positive integer p. Now,

$$\begin{aligned} V(x_k, x_{k+p}) &= V(Tx_{k-1}, Tx_{k+p-1}) \\ &\leq V(Tx_{k-1}, Tx_k) + V(Tx_k, Tx_{k+1}) + ... \\ &\quad + V(Tx_{k+p-2}, Tx_{k+p-1}) \\ &= a_k + a_{k+1} + ... + a_{k+p-1} \\ &\leq (p-1)a_k \end{aligned}$$

(since $\{a_k\}$ is monotone non-increasing). Since $a_k \to \theta$ when $k \to \infty$ and as p is arbitrary but fixed, clearly every co-ordinate of $V(x_k, x_{k+p})$ tends to zero as $k \to \infty$ and therefore $\{x_n\}$ is a vector Cauchy sequence in X. As (X, V) is a complete v.m.s, $\lim_{n\to\infty} x_n$ exists in X. Let $\lim_{n\to\infty} x_n = x_0$. It can be easily shown that x_o is the unique fixed point of T. □

Next we consider another generalization of contraction mapping which is defined on some special classes of vector metric space.

For this we introduce the following definitions.

Definition 8. A complete v.m.s is called ϵ-chainable if for any two points $a, b \in X$, there exists a finite set of elements $a = x_0, x_1, ..., x_n = b$ and $\epsilon > 0$ such that

$$V(x_{i-1}, x_i) \leq \epsilon\hat{\theta}, \text{ for } i = 1, 2, ..., n.$$

Definition 9. A mapping $T : X \to X$ is called locally contractive if for every $x \in X$ there exists $\epsilon_x > 0$ and λ_x, $0 < \lambda_x < 1$, such that for all p, q in

$$\{y : V(x,y) < \epsilon_x\hat{\theta}\}$$

the relation $V(Tp, Tq) \leq \lambda_x V(p,q)$ holds.

Definition 10. A mapping $T : X \to X$ is called (ϵ, λ) uniformly locally contractive if it is locally contractive and ϵ and λ do not depend on x.

Now we are in a position to prove the following theorem.

Theorem 3. *If (X, V) is an ϵ-chainable space and $T : X \to X$ is an (ϵ, λ) uniformly locally contractive mapping then there exists a unique fixed point x_0 of T.*

Proof. Let x be an arbitrary element in X and consider the elements x and Tx. We can find an ϵ-chain

$$x = x_0, x_1, ..., x_{n-1}, x_n = Tx$$

such that $V(x_{i-1}, x_i) < \epsilon\hat{\theta}$ for all $i = 1, 2, ..., n$.

Now using property (iii) of definition 1, we have,

$$V(x,Tx) < n\epsilon\hat{\theta}$$

Now, we have

$$V(Tx_{i-1},Tx_i) < \lambda V(x_{i-1},x_i) < \lambda\epsilon\hat{\theta}$$

and by induction, we obtain for any positive integer m,

$$V(T^m x_{i-1},T^m x_i) < \lambda^m\epsilon\hat{\theta}$$

which implies

$$V(T^m x,T^{m+1}x) < \lambda^m n\epsilon\hat{\theta} \text{ using } (iii) \text{ of definition 1.}$$

Hence for any positive integer p

$$\begin{aligned} V(T^m x,T^{m+p}x) &\leq V(T^m x,T^{m+1}x) + V(T^{m+1}x,T^{m+2}x) + ... \\ &\quad + V(T^{m+p-1}x,T^{m+p}x) \\ &\leq (\lambda^m + \lambda^{m+1} + ... + \lambda^{m+p-1})n\epsilon\hat{\theta} \\ &\leq \frac{\lambda^m}{1-\lambda}n\epsilon\hat{\theta} \\ &= \{\frac{\lambda^m}{1-\lambda}n\epsilon, \frac{\lambda^m}{1-\lambda}n\epsilon, ...\}. \end{aligned}$$

By Theorem 1, the co-ordinates of $V(T^m x,T^{m+p}x)$ are quasi metrics and by above relation, each of the co-ordinates tends to zero as $m \to \infty$. Hence from definition, for each positive integer p

$$V(T^m x,T^{m+p}x) \to \theta \text{ as } m \to \infty$$

and so $\{T^m x\}$ is a vector Cauchy sequence in (X,V). Since (X,V) is complete, $\lim\limits_{n\to\infty} T^m x$ exists and we denote it by x_0. Since T is continuous we obtain

$$Tx_0 = \lim_{n\to\infty} T(T^m x) = \lim_{n\to\infty} T^{m+1}x = x_0$$

i.e. x_0 is a fixed point of T.

We now show that this is the unique fixed point of T. If possible, let y_0 be another fixed point of T.

We can find an ϵ-chain

$$x_0 = x_1, x_2, ..., x_k = y_0$$

such that $V(x_{i-1}, x_i) < \epsilon\hat{\theta}$ for all $i = 1, 2, ..., k$ and as before, we obtain

$$\begin{aligned} V(x_0, y_0) = V(Tx_0, Ty_0) &= V(T^m x_0, T^m y_0) \\ &\le \sum_{i=1}^{k} V(T^m x_{i-1}, T^m x_i) \\ &< \lambda^m k\epsilon\hat{\theta} \\ &= \{\lambda^m k\epsilon, \lambda^m k\epsilon, ...\}. \end{aligned}$$

Then since each co-ordinate of $V(Tx_0, Ty_0)$ tends to zero as $m \to \infty$, we obtain $V(x_0, y_0) = \theta$ and hence $x_0 = y_0$. This completes the proof of the theorem. □

Acknowledgement. I am thankful to Dr. Pratulananda Das, Jadavpur University, for his kind help and suggestions in the preparation of this paper.

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Received: July, 2010