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Neighborhoods of Certain Classes of Analytic Functions Defined by a Generalized Differential Operator

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Abstract

Due to some familiar differential operators, we introduce here a generalized differential operator. By means of this operator, we define and investigate two new subclasses. We obtain the coefficient estimates and further study the neighborhood results.

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1 Introduction

Let $A(n)$ denote the class of functions $f(z)$ of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k, \ (\ a_k \geq 0, \, k \in N \backslash \{1\}, \, n \in N) \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : \ z \in C, \, |z| < 1\}$.

To begin with our investigation, we first state some defintions given by Goodman [5] and Ruscheweyh [6]. For any $f \in A(n)$ and $\delta \geq 0$, we define

$$N_{n,\delta}(f) = \{ \, g \in A(n) : \ g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k and \quad \sum_{k=n+1}^{\infty} k \, |a_k - b_k| \leq \delta \}$$

which is the $(n,\delta)-$ neighborhood of $f(z)$. For $e(z)=z$, we see that

$$N_{n,\delta}(e)=\{\,g\in A(n):\,g(z)=z-\sum_{k=n+1}^{\infty}b_kz^k and \sum_{k=n+1}^{\infty}k\,|b_k|\le\delta\}. \tag{1.2}$$

The subclass $S_n^*(\gamma)$ of $A(n)$, is the class of starlike functions of complex order γ satisfying

$$\Re\{1+\frac{1}{\gamma}[\frac{z\,f'(z)}{f(z)}-1]\}>0,$$

$$(z\in U,\ \ \gamma\in C\backslash\{0\}).$$

The subclass $C_n(\gamma)$ of $A(n)$, is the class of convex functions of complex order γ satisfying

$$\Re\{1+\frac{1}{\gamma}\frac{z\,f''(z)}{f'(z)}\}>0,$$

$$(z\in U,\ \ \gamma\in C\backslash\{0\}).$$

The classes $S_n^*(\gamma)$ and $C_n(\gamma)$ were studied by [1]. Let $S_n(\gamma,\lambda,\beta)$ denote the subclass of $A(n)$ consisting of functions f which satisfy the following inequality

$$\left|\frac{1}{\gamma}[\frac{\lambda z^3f'''(z)+(1+2\lambda)z^2f''(z)+zf'(z)}{\lambda z^2f''(z)+z\,f'(z)}-1]\right|<\beta,$$

$$(z\in U,\ \ \gamma\in C\backslash\{0\},\,0\le\lambda\le1,\,0<\beta\le1).$$

Let $R_n(\gamma,\lambda,\beta)$ denote the subclass of $A(n)$ consisting of functions f which satisfy the following inequality

$$\left|\frac{1}{\gamma}[\lambda z^2f'''(z)+(1+2\lambda)zf''(z)+f'(z)-1]\right|<\beta,$$

$$(z\in U,\ \ \gamma\in C\backslash\{0\},\,0\le\lambda\le1,\,0<\beta\le1).$$

The class $S_n(\gamma,\lambda,\beta)$ was studied by [3].

Let A be class of functions f of the form $f(z)=z+\sum\limits_{k=2}^{\infty}a_kz^k$ which are analytic in the open unit disk U. For $f\in A$, we now define a generalized differential operator as follows

$$D_{\alpha,\mu}^{\sigma,\rho}f(z)=z+\sum_{k=2}^{\infty}[1+(\alpha\mu k+\alpha-\mu)(k-1)]^{\sigma}\,G(\rho,k)\,a_kz^k$$

where $G(\rho,k) = \binom{k+\rho-1}{\rho}$ and $\rho \in N_0$.
When $\alpha = 0$ and $\mu = 0$ we get the Sãlãgean differential operator [8], when $\mu = 0$ we obtain the differential operator defined by Al-Oboudi [2], when $\sigma = 0$ we obtain the Ruscheweyh operator [7] and when $\rho = 0$ we obtain the differential operator defined by Dorina and Halit [4]. Due to the popularity of the generalization of operators, many related work has recently been seen in the literature (see examples [9]-[13]).

If $f \in A(n)$ and $z \in U$, we obtain the power series expansion of the form

$$D^{\sigma,\rho}_{\alpha,\mu} f(z) = z - \sum_{k=n+1}^{\infty} [1 + (\alpha\mu k + \alpha - \mu)(k-1)]^{\sigma}\, G(\rho,k)\, a_k z^k. \tag{1.3}$$

Let $S^{\sigma,\rho}_n(\gamma,\lambda,\beta,\alpha,\mu)$ denote the subclass of $A(n)$ consisting of functions f which satisfy the inequality

$$\left| \frac{1}{\gamma} [\frac{\lambda(z^2(D^{\sigma,\rho}_{\alpha,\mu} f(z))'')' + (z(D^{\sigma,\rho}_{\alpha,\mu} f(z))')'}{\lambda(z(D^{\sigma,\rho}_{\alpha,\mu} f(z))')' + (1-\lambda)\,(D^{\sigma,\rho}_{\alpha,\mu} f(z))'} - 1] \right| < \beta, \tag{1.4}$$

$$(z \in U,\;\; \gamma \in C\backslash\{0\},\; 0 \le \lambda \le 1,\; 0 < \beta \le 1).$$

Also, let $R^{\sigma,\rho}_n(\gamma,\lambda,\beta,\alpha,\mu)$ denote the subclass of $A(n)$ consisting of f which satisfy the inequality

$$\left| \frac{1}{\gamma} [\lambda(z^2(D^{\sigma,\rho}_{\alpha,\mu} f(z))'')' + (z(D^{\sigma,\rho}_{\alpha,\mu} f(z))')' - 1] \right| < \beta$$

$$(z \in U,\;\; \gamma \in C\backslash\{0\},\; 0 \le \lambda \le 1,\; 0 < \beta \le 1).$$

In this paper we obtain the coefficient estimates and the consequent inclusion relationships involving neighborhoods of some analytic functions.

2 Coefficient estimates

In our investigation of the inclusion relations involving $(n,\delta)-$ neighborhoods, we shall require the following theorems.

Theorem 2.1 . *Let the function $f \in A(n)$ be defined by (1.1). Then $f \in S^{\sigma,\rho}_n(\gamma,\lambda,\beta,\alpha,\mu)$ if and only if*

$$\sum_{k=n+1}^{\infty} k\{(k-1)(\lambda(k-2)+1) + (1-\lambda)k + \beta\,|\gamma|\,(\lambda(k-2)+1)\}\; B_{\sigma,\rho}(\alpha,\mu,k)\, a_k \le \beta\,|\gamma| \tag{2.5}$$

for

$$B_{\sigma,\rho}(\alpha,\mu,k) = [1 + (\alpha\mu k + \alpha - \mu)(k-1)]^{\sigma}\, G(\rho,k). \tag{2.6}$$

Proof: Let $f \in S_n^{\sigma,\rho}(\gamma,\lambda,\beta,\alpha,\mu)$. Then, we have

$$\Re\{\frac{\lambda(z^2(D^{\sigma,\rho}_{\alpha,\mu}f(z))'')' + (z(D^{\sigma,\rho}_{\alpha,\mu}f(z))')'}{\lambda(z(D^{\sigma,\rho}_{\alpha,\mu}f(z))')' + (1-\lambda)\,(D^{\sigma,\rho}_{\alpha,\mu}f(z))'} - 1\} > -\beta\,|\gamma|\,,\ z\in U. \tag{2.7}$$

Equivalently

$$\Re\{\frac{-\sum_{k=n+1}^{\infty} k[(k-1)(\lambda(k-2)+1)+(1-\lambda)k]\ B_{\sigma,\rho}(\alpha,\mu,k)\,a_k z^{k-1}}{1-\sum_{k=n+1}^{\infty} k[\lambda(k-2)+1]\,B_{\sigma,\rho}(\alpha,\mu,k)\,a_k z^{k-1}}\} \geq -\beta\,|\gamma|\,,\ z\in U \tag{2.8}$$

Letting $z \to 1^-$, through the real values, the inequality (2.4) yields the desired condition (2.1).

Conversely, by applying the hypothesis (2.1) and letting $|z| = 1$, we obtain

$$\left|\frac{\lambda(z^2(D^{\sigma,\rho}_{\alpha,\mu}f(z))'')' + (z(D^{\sigma,\rho}_{\alpha,\mu}f(z))')'}{\lambda(z(D^{\sigma,\rho}_{\alpha,\mu}f(z))')' + (1-\lambda)\,(D^{\sigma,\rho}_{\alpha,\mu}f(z))'} - 1\right|$$

$$= \left|\frac{-\sum_{k=n+1}^{\infty} k[(k-1)(\lambda(k-2)+1)+(1-\lambda)k]\ B_{\sigma,\rho}(\alpha,\mu,k)\,a_k z^{k-1}}{1-\sum_{k=n+1}^{\infty} k[\lambda(k-2)+1]\ B_{\sigma,\rho}(\alpha,\mu,k)\,a_k z^{k-1}}\right|$$

$$\leq \frac{\beta\,|\gamma|\,\{1-\sum_{k=n+1}^{\infty} k[\lambda(k-2)+1]\ B_{\sigma,\rho}(\alpha,\mu,k)\,a_k\}}{\sum_{k=n+1}^{\infty} k[\lambda(k-2)+1]\ B_{\sigma,\rho}(\alpha,\mu,k)\,a_k}$$

$$= \beta\,|\gamma|\,.$$

Hence, by maximum modulus theorem, we have $f \in S_n^{\sigma,\rho}(\gamma,\lambda,\beta,\alpha,\mu)$. Thus the proof is complete.

Similarly, we can prove the following theorem.

Theorem 2.2 . *Let the function $f \in A(n)$ be defined by (1.1). Then $f \in R_n^{\sigma,\rho}(\gamma,\lambda,\beta,\alpha,\mu)$ if and only if*

$$\sum_{k=n+1}^{\infty} k(k-1)[\lambda(k-1)+1]\ B_{\sigma,\rho}(\alpha,\mu,k)\,a_k.$$

3 Neighborhoods for the classes $S_n^{\sigma,\rho}(\gamma,\lambda,\beta,\alpha,\mu)$ and $R_n^{\sigma,\rho}(\gamma,\lambda,\beta,\alpha,\mu)$

Our first inclusion relations involving $(n,\delta)-$ neighborhoods for the Classes $S_n^{\sigma,\rho}(\gamma,\lambda,\beta,\alpha,\mu)$ and $R_n^{\sigma,\rho}(\gamma,\lambda,\beta,\alpha,\mu)$ given in the following theorems.

Theorem 3.1 *If*

$$\delta = \frac{\beta\,|\gamma|}{(n+1)\{n(\lambda(n-1)+1)+(1-\lambda)(n+1)+\beta\,|\gamma|\,[\lambda(n-1)+1]\,\}B_{\sigma,\rho}(\alpha,\mu,n+1)},$$

where $|\gamma|<1$*, then* $S_n^{\sigma,\rho}(\gamma,\lambda,\beta,\alpha,\mu)\subset N_{n,\delta}(e)$*.*

Proof: For $S_n^{\sigma,\rho}(\gamma,\lambda,\beta,\alpha,\mu)$, Theorem 2.1 immediately yields

$$(n+1)[n(\lambda(n-1)+1)+(1-\lambda)(n+1)+$$

$$\beta\,|\gamma|\,(\lambda(n-1)+1)]\ B_{\sigma,\rho}(\alpha,\mu,n+1)\sum_{k=n+1}^{\infty}a_k \le\ \beta\,|\gamma|\,,$$

so that

$$\sum_{k=n+1}^{\infty}a_k \le$$

$$\frac{\beta\,|\gamma|}{(n+1)[n(\lambda(n-1)+1)+(1-\lambda)(n+1)+\beta\,|\gamma|\,(\lambda(n-1)+1)]}, \tag{3.9}$$

on the other hand, we also find from (2.1) and (3.1) that

$$(n+1)\,B_{\sigma,\rho}(\alpha,\mu,n+1)\sum_{k=n+1}^{\infty}k\,a_k \le\ \beta\,|\gamma|+[\lambda(n+1)^2-n(n+1)(\lambda(n-1)+1)$$
$$-\beta\,|\gamma|\,\{(n+1)(\lambda(n-1)+1\}]B_{\sigma,\rho}(\alpha,\mu,n+1)\sum_{k=n+1}^{\infty}a_k$$

$$\le \frac{(n+1)^2\beta\,|\gamma|}{[(n+1)\{n(\lambda(n-1)+1)+(1-\lambda)(n+1)\}+\beta\,|\gamma|\,(n+1)(\lambda(n-1)+1)]\ B_{\sigma,\rho}(\alpha,\mu,n+1)}.$$

Thus

$$\sum_{k=n+1}^{\infty}k\,a_k\ \le \frac{(n+1)\beta\,|\gamma|}{[(n+1)\{n(\lambda(n-1)+1)+(1-\lambda)(n+1)\}+\beta\,|\gamma|\,(n+1)(\lambda(n-1)+1)]}$$

Thus, by the definition given by (1.2), $f\in N_{n,\delta}(e)$ which completes the proof.

Similarly, by applying Theorem 2.2 instead of Theorem 2.1. We can prove the following.

Theorem 3.2 *If*

$$\delta = \frac{\beta|\gamma|}{n(n+1)(\lambda n+1)}$$

where $|\gamma| < 1$, *then* $R_n^{\sigma,\rho}(\gamma,\lambda,\beta,\alpha,\mu) \subset N_{n,\delta}(e)$.

4 Neighborhoods properties for the classes $S_n^{\sigma,\rho,\eta}(\gamma,\lambda,\beta,\alpha,\mu)$ and $R_n^{\sigma,\rho,\eta}(\gamma,\lambda,\beta,\alpha,\mu)$

In this section, we define the subclasses $S_n^{\sigma,\rho,\eta}(\gamma,\lambda,\beta,\alpha,\mu)$ and $R_n^{\sigma,\rho,\eta}(\gamma,\lambda,\beta,\alpha,\mu)$ of $A(n)$ and neighborhoods of these classes are obtained.

A function $f \in A(n)$ is said to be in the class $S_n^{\sigma,\rho,\eta}(\gamma,\lambda,\beta,\alpha,\mu)$ if there exists a function $h \in S_n^{\sigma,\rho}(\gamma,\lambda,\beta,\alpha,\mu)$ such that

$$\left|\frac{f(z)}{h(z)} - 1\right| < 1-\eta, \quad (z \in U,\ 0 \le \eta < 1) \tag{4.10}$$

also a function $f \in R_n^{\sigma,\rho,\eta}(\gamma,\lambda,\beta,\alpha,\mu)$ if there exists a function$h \in R_n^{\sigma,\rho}(\gamma,\lambda,\beta,\alpha,\mu)$ such that the inequality (4.1) holds true.

Theorem 4.1 *If* $h \in S_n^{\sigma,\rho}(\gamma,\lambda,\beta,\alpha,\mu)$ *and*

$$\eta = 1-$$

$$\frac{\delta[(n+1)\{n(\lambda(n-1)+1)+(1-\lambda)(n+1)\}+\beta|\gamma|(n+1)(\lambda(n-1)+1)]}{(n+1)\{[(n+1)\{n(\lambda(n-1)+1)+(1-\lambda)(n+1)\}+\beta|\gamma|(n+1)(\lambda(n-1)+1)]-\beta|\gamma|\}} \tag{4.11}$$

then $N_{n,\delta}(h) \subset S_n^{\sigma,\rho,\eta}(\gamma,\lambda,\beta,\alpha,\mu)$.

Proof: Let $f \in N_{n,\delta}(h)$. Then

$$\sum_{k=n+1}^{\infty} k|a_k - b_k| \le \delta,$$

which readily implies the coefficient inequality

$$\sum_{k=n+1}^{\infty} |a_k - b_k| \le \frac{\delta}{n+1}, \quad n \in N,$$

since $h \in S_n^{\sigma,\rho}(\gamma,\lambda,\beta,\alpha,\mu)$, we have from equation (3.1)

$$\sum_{k=n+1}^{\infty} b_k \le \frac{\beta|\gamma|}{[(n+1)\{n(\lambda(n-1)+1)+(1-\lambda)(n+1)\}+\beta|\gamma|(n+1)(\lambda(n-1)+1)]}, \tag{4.12}$$

so that

$$\left|\frac{f(z)}{h(z)} - 1\right| < \frac{\sum\limits_{k=n+1}^{\infty} |a_k - b_k|}{1 - \sum\limits_{k=n+1}^{\infty} b_k}$$

$$\leq \frac{\delta}{n+1} \frac{[(n+1)\{n(\lambda(n-1)+1) + (1-\lambda)(n+1)\} + \beta |\gamma| ((n+1)(\lambda(n-1)+1)]}{[(n+1)\{n(\lambda(n-1)+1) + (1-\lambda)(n+1)\} + \beta |\gamma| (n+1)(\lambda(n-1)+1)] - \beta |\gamma|}$$

$$= 1 - \eta$$

which completes the proof of Theorem 4.1.

Similarly, we can prove the following theorem.

Theorem 4.2 *If* $h \in R_n^{\sigma,\rho}(\gamma, \lambda, \beta, \alpha, \mu)$ *and*

$$\eta = 1 - \frac{\delta n(n+1)(\lambda n + 1)}{(n+1)\{n(n+1)(\lambda n+1) - \beta |\gamma|\}},$$

then $N_{n,\delta}(h) \subset R_n^{\sigma,\rho,\eta}(\gamma, \lambda, \beta, \alpha, \mu)$.

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