Neighborhoods of Certain Classes of Analytic Functions Defined by a Generalized Differential Operator

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Abstract

Due to some familiar differential operators, we introduce here a generalized differential operator. By means of this operator, we define and investigate two new subclasses. We obtain the coefficient estimates and further study the neighborhood results.

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1 Introduction

Let \( A(n) \) denote the class of functions \( f(z) \) of the form

\[
f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k, \quad (a_k \geq 0, \ k \in \mathbb{N} \setminus \{1\}, \ n \in \mathbb{N})
\]

which are analytic in the open unit disk \( U = \{z : z \in \mathbb{C}, \ |z| < 1\} \).

To begin with our investigation, we first state some definitions given by Goodman [5] and Ruscheweyh [6]. For any \( f \in A(n) \) and \( \delta \geq 0 \), we define

\[
N_{n,\delta}(f) = \{ g \in A(n) : g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k \text{ and } \sum_{k=n+1}^{\infty} k |a_k - b_k| \leq \delta \}
\]
which is the \((n, \delta)\)-neighborhood of \(f(z)\). For \(e(z) = z\), we see that

\[
N_{n, \delta}(e) = \{ g \in A(n) : g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k \text{and} \sum_{k=n+1}^{\infty} k |b_k| \leq \delta \}.
\]

(1.2)

The subclass \(S_n^*(\gamma)\) of \(A(n)\), is the class of starlike functions of complex order \(\gamma\) satisfying

\[
\Re\{1 + \frac{1}{\gamma} [\frac{zf'(z)}{f(z)} - 1]\} > 0,
\]

\((z \in U, \ \gamma \in C \setminus \{0\})\).

The subclass \(C_n(\gamma)\) of \(A(n)\), is the class of convex functions of complex order \(\gamma\) satisfying

\[
\Re\{1 + \frac{1}{\gamma} \frac{zf''(z)}{f'(z)}\} > 0,
\]

\((z \in U, \ \gamma \in C \setminus \{0\})\).

The classes \(S_n^*(\gamma)\) and \(C_n(\gamma)\) were studied by [1]. Let \(S_n(\gamma, \lambda, \beta)\) denote the subclass of \(A(n)\) consisting of functions \(f\) which satisfy the following inequality

\[
\left| \frac{1}{\gamma} \left[ \frac{\lambda z^3 f'''(z) + (1 + 2\lambda) z^2 f''(z) + zf'(z)}{\lambda z^2 f''(z) + zf'(z)} - 1 \right] \right| < \beta,
\]

\((z \in U, \ \gamma \in C \setminus \{0\}, 0 \leq \lambda \leq 1, 0 < \beta \leq 1)\).

Let \(R_n(\gamma, \lambda, \beta)\) denote the subclass of \(A(n)\) consisting of functions \(f\) which satisfy the following inequality

\[
\left| \frac{1}{\gamma} \left[ \lambda z^2 f'''(z) + (1 + 2\lambda) zf''(z) + f'(z) - 1 \right] \right| < \beta,
\]

\((z \in U, \ \gamma \in C \setminus \{0\}, 0 \leq \lambda \leq 1, 0 < \beta \leq 1)\).

The class \(S_n(\gamma, \lambda, \beta)\) was studied by [3].

Let \(A\) be class of functions \(f\) of the form \(f(z) = z + \sum_{k=2}^{\infty} a_k z^k\) which are analytic in the open unit disk \(U\). For \(f \in A\), we now define a generalized differential operator as follows

\[
D_{\alpha, \mu}^{\sigma, \rho} f(z) = z + \sum_{k=2}^{\infty} \left[ 1 + (\alpha \mu k + \alpha - \mu)(k - 1) \right]^\sigma G(\rho, k) a_k z^k
\]
where \( G(\rho, k) = \left( \frac{k + \rho - 1}{\rho} \right) \) and \( \rho \in N_0 \).

When \( \alpha = 0 \) and \( \mu = 0 \) we get the Sălăgean differential operator \[8\], when \( \mu = 0 \) we obtain the differential operator defined by Al-Oboudi \[2\], when \( \sigma = 0 \) we obtain the Ruscheweyh operator \[7\] and when \( \rho = 0 \) we obtain the differential operator defined by Dorina and Halit \[4\]. Due to the popularity of the generalization of operators, many related work has recently been seen in the literature (see examples \[9\]-[13\])

If \( f \in A(n) \) and \( z \in U \), we obtain the power series expansion of the form

\[
D_{\alpha, \mu}^{\sigma, \rho} f(z) = z - \sum_{k=n+1}^{\infty} [1 + (\alpha \mu k + \alpha - \mu)(k - 1)]^\sigma G(\rho, k) a_k z^k. \tag{1.3}
\]

Let \( S_{\alpha, \mu}^{\sigma, \rho}(\gamma, \lambda, \beta, \alpha, \mu) \) denote the subclass of \( A(n) \) consisting of functions \( f \) which satisfy the inequality

\[
\left| \frac{1}{\gamma} \left[ \frac{\lambda(z^2 (D_{\alpha, \mu}^{\sigma, \rho} f(z))^n' + (z(D_{\alpha, \mu}^{\sigma, \rho} f(z))')'}{\gamma (z(D_{\alpha, \mu}^{\sigma, \rho} f(z))')' + (1 - \lambda)(D_{\alpha, \mu}^{\sigma, \rho} f(z))')' - 1 \right] \right| < \beta, \tag{1.4}
\]

\((z \in U, \gamma \in C \setminus \{0\}, 0 \leq \lambda \leq 1, 0 < \beta \leq 1)\).

Also, let \( R_{\alpha, \mu}^{\sigma, \rho}(\gamma, \lambda, \beta, \alpha, \mu) \) denote the subclass of \( A(n) \) consisting of \( f \) which satisfy the inequality

\[
\left| \frac{1}{\gamma} [\lambda(z^2 (D_{\alpha, \mu}^{\sigma, \rho} f(z))^n' + (z(D_{\alpha, \mu}^{\sigma, \rho} f(z))')' - 1] \right| < \beta
\]

\((z \in U, \gamma \in C \setminus \{0\}, 0 \leq \lambda \leq 1, 0 < \beta \leq 1)\).

In this paper we obtain the coefficient estimates and the consequent inclusion relationships involving neighborhoods of some analytic functions.

## 2 Coefficient estimates

In our investigation of the inclusion relations involving \((n, \delta)\)–neighborhoods, we shall require the following theorems.

**Theorem 2.1.** Let the function \( f \in A(n) \) be defined by (1.1). Then \( f \in S_{\alpha, \mu}^{\sigma, \rho}(\gamma, \lambda, \beta, \alpha, \mu) \) if and only if

\[
\sum_{k=n+1}^{\infty} k\{ (k - 1)(\lambda(k - 2) + 1) + (1 - \lambda)k + \beta |\gamma| (\lambda(k - 2) + 1) \} B_{\sigma, \rho}(\alpha, \mu, k) a_k \leq \beta |\gamma| \tag{2.5}
\]

for

\[
B_{\sigma, \rho}(\alpha, \mu, k) = [1 + (\alpha \mu k + \alpha - \mu)(k - 1)]^\sigma G(\rho, k). \tag{2.6}
\]
Proof: Let $f \in S^\sigma_n(\gamma, \lambda, \beta, \alpha, \mu)$. Then, we have

$$\Re\{ \frac{\lambda(z^2(D_{\alpha,\mu}^\sigma f(z))'' + (z(D_{\alpha,\mu}^\sigma f(z))')'}{\lambda(z(D_{\alpha,\mu}^\sigma f(z))')' + (1 - \lambda)(D_{\alpha,\mu}^\sigma f(z))'} - 1 \} > -\beta |\gamma|, \ z \in U. \quad (2.7)$$

Equivalently

$$- \sum_{k=n+1}^{\infty} k[(k-1)(\lambda(k-2) + 1) + (1 - \lambda)k] \ B_{\sigma,\rho}(\alpha, \mu, k) \ a_k z^{k-1} \ \Re\left\{ 1 - \sum_{k=n+1}^{\infty} k[\lambda(k-2) + 1] \ B_{\sigma,\rho}(\alpha, \mu, k) \ a_k z^{k-1} \right\} \geq -\beta |\gamma|, \ z \in U\quad (2.8)$$

Letting $z \to 1^-$, through the real values, the inequality (2.4) yields the desired condition (2.1).

Conversely, by applying the hypothesis (2.1) and letting $|z| = 1$, we obtain

$$\left| \frac{\lambda(z^2(D_{\alpha,\mu}^\sigma f(z))'' + (z(D_{\alpha,\mu}^\sigma f(z))')'}{\lambda(z(D_{\alpha,\mu}^\sigma f(z))')' + (1 - \lambda)(D_{\alpha,\mu}^\sigma f(z))'} - 1 \right|$$

$$= \left| - \sum_{k=n+1}^{\infty} k[(k-1)(\lambda(k-2) + 1) + (1 - \lambda)k] \ B_{\sigma,\rho}(\alpha, \mu, k) \ a_k z^{k-1} \right|$$

$$\leq \beta |\gamma| \ \left\{ 1 - \sum_{k=n+1}^{\infty} k[\lambda(k-2) + 1] \ B_{\sigma,\rho}(\alpha, \mu, k) \ a_k \right\}$$

$$\sum_{k=n+1}^{\infty} k[\lambda(k-2) + 1] \ B_{\sigma,\rho}(\alpha, \mu, k) \ a_k \ = \beta |\gamma|.$$ 

Hence, by maximum modulus theorem, we have $f \in S^\sigma_n(\gamma, \lambda, \beta, \alpha, \mu)$. Thus the proof is complete.

Similarly, we can prove the following theorem.

**Theorem 2.2.** Let the function $f \in A(n)$ be defined by (1.1). Then $f \in R^\sigma_n(\gamma, \lambda, \beta, \alpha, \mu)$ if and only if

$$\sum_{k=n+1}^{\infty} k(k-1)[\lambda(k-1) + 1] \ B_{\sigma,\rho}(\alpha, \mu, k) \ a_k.$$
3 Neighborhoods for the classes $S_{n}^{\sigma,\rho}(\gamma, \lambda, \beta, \alpha, \mu)$ and $R_{n}^{\sigma,\rho}(\gamma, \lambda, \beta, \alpha, \mu)$

Our first inclusion relations involving $(n, \delta) -$ neighborhoods for the Classes $S_{n}^{\sigma,\rho}(\gamma, \lambda, \beta, \alpha, \mu)$ and $R_{n}^{\sigma,\rho}(\gamma, \lambda, \beta, \alpha, \mu)$ given in the following theorems.

**Theorem 3.1** If

$$\delta = \frac{\beta |\gamma|}{(n+1)\{n(\lambda(n-1)+1) + (1-\lambda)(n+1) + \beta |\gamma| [\lambda(n-1)+1]\} B_{\sigma,\rho}(\alpha, \mu, n+1)},$$

where $|\gamma| < 1$, then $S_{n}^{\sigma,\rho}(\gamma, \lambda, \beta, \alpha, \mu) \subset N_{n,\delta}(e)$.

**Proof:** For $S_{n}^{\sigma,\rho}(\gamma, \lambda, \beta, \alpha, \mu)$, Theorem 2.1 immediately yields

$$(n+1)\{n(\lambda(n-1)+1) + (1-\lambda)(n+1) + \beta |\gamma| [\lambda(n-1)+1]\} B_{\sigma,\rho}(\alpha, \mu, n+1) \sum_{k=n+1}^{\infty} a_k \leq \beta |\gamma|,$$

so that

$$\sum_{k=n+1}^{\infty} a_k \leq \frac{\beta |\gamma|}{(n+1)\{n(\lambda(n-1)+1) + (1-\lambda)(n+1) + \beta |\gamma| [\lambda(n-1)+1]\}},$$

(3.9)

on the other hand, we also find from (2.1) and (3.1) that

$$(n+1) B_{\sigma,\rho}(\alpha, \mu, n+1) \sum_{k=n+1}^{\infty} k a_k \leq \beta |\gamma| + [\lambda(n+1)^2 - n(n+1)\lambda(n-1)+1] - \beta |\gamma| \{(n+1)(\lambda(n-1)+1)\} B_{\sigma,\rho}(\alpha, \mu, n+1) \sum_{k=n+1}^{\infty} a_k$$

$$\leq \frac{(n+1)^2 \beta |\gamma|}{[(n+1)\{n(\lambda(n-1)+1) + (1-\lambda)(n+1)\} + \beta |\gamma| (n+1)(\lambda(n-1)+1)] B_{\sigma,\rho}(\alpha, \mu, n+1)}.$$

Thus

$$\sum_{k=n+1}^{\infty} k a_k \leq \frac{(n+1) \beta |\gamma|}{[(n+1)\{n(\lambda(n-1)+1) + (1-\lambda)(n+1)\} + \beta |\gamma| (n+1)(\lambda(n-1)+1)]} B_{\sigma,\rho}(\alpha, \mu, n+1).$$

Thus, by the definition given by (1.2), $f \in N_{n,\delta}(e)$ which completes the proof.

Similarly, by applying Theorem 2.2 instead of Theorem 2.1. We can prove the following.
Theorem 3.2 If

\[ \delta = \frac{\beta |\gamma|}{n(n+1)(\lambda n + 1)} \]

where \(|\gamma| < 1\), then \(R_{n}^{\sigma,\rho}(\gamma, \lambda, \beta, \alpha, \mu) \subset N_{n,\delta}(e)\).

4 Neighborhoods properties for the classes \(S_{n}^{\sigma,\rho,\eta}(\gamma, \lambda, \beta, \alpha, \mu)\) and \(R_{n}^{\sigma,\rho,\eta}(\gamma, \lambda, \beta, \alpha, \mu)\)

In this section, we define the subclasses \(S_{n}^{\sigma,\rho,\eta}(\gamma, \lambda, \beta, \alpha, \mu)\) and \(R_{n}^{\sigma,\rho,\eta}(\gamma, \lambda, \beta, \alpha, \mu)\) of \(A(n)\) and neighborhoods of these classes are obtained.

A function \(f \in A(n)\) is said to be in the class \(S_{n}^{\sigma,\rho,\eta}(\gamma, \lambda, \beta, \alpha, \mu)\) if there exists a function \(h \in S_{n}^{\sigma,\rho}(\gamma, \lambda, \beta, \alpha, \mu)\) such that

\[ \left| \frac{f(z)}{h(z)} - 1 \right| < 1 - \eta, \quad (z \in U, \ 0 \leq \eta < 1) \quad (4.10) \]

also a function \(f \in R_{n}^{\sigma,\rho,\eta}(\gamma, \lambda, \beta, \alpha, \mu)\) if there exists a function \(h \in R_{n}^{\sigma,\rho}(\gamma, \lambda, \beta, \alpha, \mu)\) such that the inequality (4.1) holds true.

Theorem 4.1 If \(h \in S_{n}^{\sigma,\rho}(\gamma, \lambda, \beta, \alpha, \mu)\) and

\[ \eta = 1 - \frac{\delta[(n+1)\{n(\lambda(n-1)+1)+(1-\lambda)(n+1)\}+\beta|\gamma|(n+1)\lambda(n+1)\}]}{(n+1)\{(n+1)\{n(\lambda(n-1)+1)+(1-\lambda)(n+1)\}+\beta|\gamma|(n+1)\lambda(n+1)\}+\beta|\gamma|(n+1)(\lambda(n+1)\}]} \quad (4.11) \]

then \(N_{n,\delta}(h) \subset S_{n}^{\sigma,\rho,\eta}(\gamma, \lambda, \beta, \alpha, \mu)\).

Proof: Let \(f \in N_{n,\delta}(h)\). Then

\[ \sum_{k=n+1}^{\infty} k|a_{k} - b_{k}| \leq \delta, \]

which readily implies the coefficient inequality

\[ \sum_{k=n+1}^{\infty} |a_{k} - b_{k}| \leq \frac{\delta}{n+1}, \quad n \in N; \]

since \(h \in S_{n}^{\sigma,\rho}(\gamma, \lambda, \beta, \alpha, \mu)\), we have from equation (3.1)

\[ \sum_{k=n+1}^{\infty} b_{k} \leq \frac{\beta |\gamma|}{(n+1)\{n(\lambda(n-1)+1)+(1-\lambda)(n+1)\}+\beta|\gamma|(n+1)(\lambda(n+1)+1)} \quad (4.12) \]
so that

\[ \left| \frac{f(z)}{h(z)} - 1 \right| < \frac{\sum_{k=n+1}^{\infty} |a_k - b_k|}{1 - \sum_{k=n+1}^{\infty} b_k} \]

\[ \leq \frac{\delta}{n + 1} \cdot \frac{[(n + 1)\{n(\lambda(n - 1) + 1) + (1 - \lambda)(n + 1)\} + \beta |\gamma| ((n + 1)(\lambda(n - 1) + 1))]}{[(n + 1)\{n(\lambda(n - 1) + 1) + (1 - \lambda)(n + 1)\} + \beta |\gamma| ((n + 1)(\lambda(n - 1) + 1)) - \beta |\gamma|]}

= 1 - \eta \]

which completes the proof of Theorem 4.1.

Similarly, we can prove the following theorem.

**Theorem 4.2** If \( h \in R_{n}^{\sigma, \rho}(\gamma, \lambda, \beta, \alpha, \mu) \) and

\[ \eta = 1 - \frac{\delta n(n + 1)(\lambda n + 1)}{(n + 1)\{n(n + 1)(\lambda n + 1) - \beta |\gamma|\}}, \]

then \( N_{n, \delta}(h) \subset R_{n}^{\sigma, \rho, \eta}(\gamma, \lambda, \beta, \alpha, \mu) \).

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**References**


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