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Somewhat b-Continuous and Somewhat b-Open Functions in Topological Spaces

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Abstract

In this paper, new classes of functions are introduced and studied by making use of b-open sets and b-closed sets. Relationship between the new classes and other classes of functions are established besides giving examples, counterexamples, properties and characterizations.

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1 Introduction

The concepts of feebly continuous functions and feebly open functions were first introduced and studied by Zdenek Frolik [2], in which it is proved that the almost continuous, feebly open image of a Baire space is a Baire space and it is observed that almost continuous function can be replaced by one-to-one and feebly continuous maps. Gentry and Hoyle [3] introduced and studied the concept of somewhat open functions which are Frolik functions with some conditions being dropped. These ideas are also closely related to the idea of weakly equivalent topologies which was first introduced by Yougslova [7].

D. Andrijevic [1] introduced and studied the concept of b-open sets in topological spaces. In this paper, using the notion of b-open sets, the concepts of somewhat b-continuous functions and somewhat b-open functions are introduced and studied. Also characterizations for somewhat b-continuity is obtained besides giving examples and counterexamples.

2 Preliminary Notes

Throughout this paper, all spaces X, Y and Z (or (X, τ), (Y, σ) and (Z, η)) are always topological spaces with no separation axioms assumed, unless otherwise explicitly stated. Let $A \subseteq X$. The closure of A and the interior of A will be denoted by $Cl(A)$ and $Int(A)$, respectively.

Definition 2.1 *A subset A of a space X is said to be:*
(1) semi open [5] if $A \subseteq Cl(Int(A))$.
(2) b-open [1] if $A \subseteq Cl(Int(A)) \cup Int(Cl(A))$.

Definition 2.2 *A function $f : X \to Y$ is said to be somewhat continuous [3] if for $U \in \sigma$ and $f^{-1}(U) \neq \phi$ there exists an open set V in X such that $V \neq \phi$ and $V \subseteq f^{-1}(U)$.*

Definition 2.3 *A function $f : X \to Y$ is said to be somewhat semi continuous [6] if for $U \in \sigma$ and $f^{-1}(U) \neq \phi$ there exists a semi open set V in X such that $V \neq \phi$ and $V \subseteq f^{-1}(U)$.*

Remark 2.4 *Every somewhat continuous function is somewhat semi continuous function [6].*

Definition 2.5 *A function $f : X \to Y$ is said to be somewhat open function [3] provided that for $U \in \tau$ and $U \neq \phi$, there exists an open set V in Y such that $V \neq \phi$ and $V \subseteq f(U)$.*

Definition 2.6 *A function $f : X \to Y$ is said to be somewhat semi open function [6] provided that for $U \in \tau$ and $U \neq \phi$, there exists a semi open set V in Y such that $V \neq \phi$ and $V \subseteq f(U)$.*

Remark 2.7 *Every somewhat open function is somewhat semi open function but the converse need not be true in general [6].*

Example 2.8 *Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a, b\}\}$, $\sigma = \{X, \phi, \{a\}\}$. Then the identity function $f : (X, \tau) \to (X, \sigma)$ is somewhat semi open function but not a somewhat open function.*

3 Somewhat b-continuous functions

Definition 3.1 *Let (X,τ) and (Y,σ) be any two topological spaces. A function $f : X \to Y$ is said to be somewhat b-continuous function if for every $U \in \sigma$ and $f^{-1}(U) \neq \phi$ there exists a b-open set V in X such that $V \neq \phi$ and $V \subset f^{-1}(U)$.*

Example 3.2 *Let $X = \{a,b\}$, $\tau = \{X,\phi\}$, $\sigma = \{X,\phi,\{a\}\}$. Now define a function $f : (X,\tau) \to (X,\sigma)$ as follows : $f(a) = b$, $f(b) = a$. Then clearly f is somewhat b-continuous function.*

Theorem 3.3 *Every somewhat semi continuous function is somewhat b-continuous function.*

Proof. Let $f : X \to Y$ be somewhat semi continuous function. Let U be any open set in Y such that $f^{-1}(U) \neq \phi$. Since f is somewhat semi continuous, there exists a semi open set V in X such that $V \neq \phi$ and $V \subseteq f^{-1}(U)$. Since every semi open set is b-open, there exists a b-open set V such that $V \neq \phi$ and $V \subseteq f^{-1}(U)$, which implies that f is somewhat b-continuous function.

Remark 3.4 *Converse of the above theorem need not be true in general which follows from the following example.*

Example 3.5 *Let $X = \{a,b\}$, $\tau = \{X,\phi\}$, $\sigma = \{X,\phi,\{a\}\}$. Define a function $f : (X,\tau) \to (X,\sigma)$ by $f(a) = a$ and $f(b) = b$. Then f is somewhat b-continuous function but not somewhat semi continuous function.*

Theorem 3.6 *Every somewhat continuous function is somewhat b-continuous function.*

Proof. Follows from Theorem 3.3 and Remark 2.4.

Remark 3.7 *Converse of the above theorem need not be true in general which follows from the following example.*

Example 3.8 *Let $X = \{a,b,c,d\}$, $\tau = \{X,\phi\}$, $\sigma = \{X,\phi,\{a\},\{b\},\{a,b\}\}$. Define a function $f : (X,\tau) \to (X,\sigma)$ by $f(a) = b, f(b) = c, f(c) = d$ and $f(d) = b$. Then clearly f is somewhat b-continuous function but not a somewhat continuous function.*

Theorem 3.9 *Let $f : (X,\tau) \to (Y,\sigma)$ and $g : (Y,\sigma) \to (Z,\eta)$ be any two functions. If f is somewhat b-continuous function and g is continuous function, then $g \circ f$ is somewhat b-continuous function.*

Proof. Let $U \in \eta$. Suppose that $g^{-1}(U) \neq \phi$. Since $U \in \eta$ and g is continuous function $g^{-1}(U) \in \sigma$. Suppose that $f^{-1}g^{-1}(U) \neq \phi$. Since by hypothesis f is somewhat b-continuous function, there exists a b-open set V in X such that $V \neq \phi$ and $V \subseteq f^{-1}(g^{-1}(U))$. But $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$, which implies that $V \subset (g \circ f)^{-1}(U)$. Therefore $g \circ f$ is somewhat b-continuous function.

Remark 3.10 *In the above Theorem 3.9, if f is continuous function and g is somewhat b-continuous function, then it is not necessarily true that $g \circ f$ is somewhat b-continuous function. The following example serves this purpose.*

Example 3.11 *Let $X = \{a, b\}$, $\tau = \sigma = \{X, \phi, \{a\}\}$ and $\eta = \{X, \phi, \{b\}\}$. Define $f : (X, \tau) \to (X, \sigma)$ by $f(a) = f(b) = a$ and define $g : (X, \sigma) \to (X, \eta)$ by $g(a) = b$ and $g(b) = a$. Then clearly f is continuous function and g is somewhat b-continuous function but $g \circ f$ is not a somewhat b-continuous function.*

Definition 3.12 *Let M be a subset of a topological space (X, τ). Then M is said to be b-dense in X if there is no proper b-closed set C in X such that $M \subset C \subset X$*

Theorem 3.13 *Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then the following are equivalent:*
(i) f is somewhat b-continuous function.
(ii) If C is a closed subset of Y such that $f^{-1}(C) \neq X$, then there is a proper b-closed subset D of X such that $D \supset f^{-1}(C)$.
(iii) If M is a b-dense subset of X then $f(M)$ is a dense subset of Y.

Proof. $(i) \Rightarrow (ii)$: Let C be a closed subset of Y such that $f^{-1}(C) \neq X$. Then $Y - C$ is an open set in Y such that $f^{-1}(Y - C) = X - f^{-1}(C) \neq \phi$. By hypothesis (i) there exists a b-open set V in X such that $V \neq \phi$ and $V \subset f^{-1}(Y - C) = X - f^{-1}(C)$. This means that $X - V \supset f^{-1}(C)$ and $X - V = D$ is a b-closed set in X. This proves (ii).
$(ii) \Rightarrow (iii)$: Let M be a b-dense set in X. We have to show that $f(M)$ is dense in Y. Suppose not, then there exists a proper closed set C in Y such that $f(M) \subset C \subset Y$. Clearly $f^{-1}(C) \neq X$. Hence by (ii) there exists a proper b-closed set D such that $M \subset f^{-1}(C) \subset D \subset X$. This contradicts the fact that M is b-dense in X.
$(iii) \Rightarrow (ii)$: Suppose that (ii) is not true. This means there exists a closed set C in Y such that $f^{-1}(C) \neq X$. But there is no proper b-closed set D in X such that $f^{-1}(C) \subseteq D$. This means that $f^{-1}(C)$ is b-dense in X. But by (iii) $f(f^{-1}(C)) = C$ must be dense in Y, which is contradiction to the choice of C.
$(ii) \Rightarrow (i)$: Let $U \in \sigma$ and $f^{-1}(U) \neq \phi$. Then $Y - U$ is closed and $f^{-1}(Y - U) = X - f^{-1}(U) \neq \phi$. By hypothesis of (ii) there exists a proper b-closed set D such that $D \supset f^{-1}(Y - U)$. This implies that $X - D \subset f^{-1}(U)$ and $X - D$ is b-open and $X - D \neq \phi$.

Theorem 3.14 *Let (X, τ) and (Y, σ) be any two topological spaces, A be an open set in X and $f : (A, \tau/A) \to (Y, \sigma)$ be somewhat b-continuous function such that $f(A)$ is dense in Y. Then any extension F of f is somewhat b-continuous function.*

Proof. Let U be any open set in (Y, σ) such that $F^{-1}(U) \neq \phi$. Since $f(A) \subset Y$ is dense in Y and $U \cap f(A) \neq \phi$ it follows that $F^{-1}(U) \cap A \neq \phi$. That is $f^{-1}(U) \cap A \neq \phi$. Hence by hypothesis on f, there exists a b-open set V in A such that $V \neq \phi$ and $V \subset f^{-1}(U) \subset F^{-1}(U)$ which implies F is somewhat b-continuous function.

Theorem 3.15 *Let (X, τ) and (Y, σ) be any two topological spaces, $X = A \cup B$ where A and B are open subsets of X and $f : (X, \tau) \to (Y, \sigma)$ be a function such that f/A and f/B are somewhat b-continuous functions. Then f is somewhat b-continuous function.*

Proof. Let U be any open set in (Y, σ) such that $f^{-1}(U) \neq \phi$. Then $(f/A)^{-1}(U) \neq \phi$ or $(f/B)^{-1}(U) \neq \phi$ or both $(f/A)^{-1}(U) \neq \phi$ and $(f/B)^{-1}(U) \neq \phi$.

Case 1. Suppose $(f/A)^{-1}(U) \neq \phi$

Since f/A is somewhat b-continuous, there exists a b-open set V in A such that $V \neq \phi$ and $V \subset (f/A)^{-1}(U) \subseteq f^{-1}(U)$. Since V is b-open in A and A is open in X, V is b-open in X. Thus f is somewhat b-continuous function.

Case 2. Suppose $(f/B)^{-1}(U) \neq \phi$

Since f/B is somewhat b-continuous function, there exists a b-open set V in B such that $V \neq \phi$ and $V \subset (f/B)^{-1}(U) \subseteq f^{-1}(U)$. Since V is b-open in B and B is open in X, V is b-open in X. Thus f is somewhat b-continuous function.

Case 3. Suppose $(f/A)^{-1}(U) \neq \phi$ and $(f/B)^{-1}(U) \neq \phi$

This follows from both the cases 1 and 2. Thus f is somewhat b-continuous function.

Definition 3.16 *A topological space X is said to be b-separable if there exists a countable subset B of X which is b-dense in X.*

Theorem 3.17 *If f is somewhat b-continuous function from X onto Y and if X is b-separable, then Y is separable.*

Proof. Let $f : X \to Y$ be somewhat b-continuous function such that X is b-separable. Then by definition there exists a countable subset B of X which is b-dense in X. Then by Theorem 3.13, $f(B)$ is dense in Y. Since B is countable $f(B)$ is also countable which is dense in Y, which indicates that Y is separable.

4 b-Weakly equivalent topologies

Definition 4.1 *If X is a set and τ and σ are topologies for X, then τ is said to be weakly equivalent to σ [3] provided if $U \in \tau$ and $U \neq \phi$, then there is an open set V in (X, σ) such that $V \neq \phi$ and $V \subset U$ and if $U \in \sigma$ and $U \neq \phi$, then there is an open set V in (X, τ) such that $V \neq \phi$ and $V \subset U$.*

Definition 4.2 *If X is a set and τ and σ are topologies for X, then τ is said to be b-weakly equivalent to σ provided if $U \in \tau$ and $U \neq \phi$, then there is a b-open set V in (X, σ) such that $V \neq \phi$ and $V \subset U$ and if $U \in \sigma$ and $U \neq \phi$ then there is a b-open set V in (X, τ) such that $V \neq \phi$ and $V \subset U$.*

Theorem 4.3 *Let $f : (X, \tau) \to (Y, \sigma)$ be somewhat continuous function and let τ^* be a topology for X, which is b-weakly equivalent to τ then the function $f : (X, \tau^*) \to (Y, \sigma)$ is somewhat b-continuous function.*

Proof. Let U be any open set in (Y, σ) such that $f^{-1}(U) \neq \phi$. Since by hypothesis $f : (X, \tau) \to (Y, \sigma)$ is somewhat continuous by definition there exists an open set O in (X, τ) such that $O \neq \phi$ and $O \subset f^{-1}(U)$.

Since O is an open set in (X, τ) such that $O \neq \phi$ and since by hypothesis τ is b-weakly equivalent to τ^* by definition there exists a b-open set V in (X, τ^*) such that $V \neq \phi$ and $V \subset O \subset f^{-1}(U)$. So $O \subset f^{-1}(U)$

Thus for any open set U in (Y, σ) such that $f^{-1}(U) \neq \phi$ there exist a b-open set V in (X, τ^*) such that $V \subset f^{-1}(U)$. So $f : (X, \tau^*) \to (Y, \sigma)$ is somewhat b-continuous function.

Theorem 4.4 *Let $f : (X, \tau) \to (Y, \sigma)$ be somewhat b-continuous function and let σ^* be a topology for Y which is weakly equivalent to σ. Then $f : (X, \tau) \to (Y, \sigma^*)$ is somewhat b-continuous function.*

Proof. Let U be an open set in (Y, σ^*) such that $f^{-1}(U) \neq \phi$ which implies $U \neq \phi$. Since σ and σ^* are weakly equivalent there exists an open set W in (Y, σ) such that $W \neq \phi$ and $W \subset U$.

Now, W is an open set such that $W \neq \phi$, which implies $f^{-1}(W) \neq \phi$. Now by hypothesis $f : (X, \tau) \to (Y, \sigma)$ is somewhat b-continuous function. Therefore there exists a b-open set V in X, such that $V \subset f^{-1}(W)$. Now $W \subset U$ implies $f^{-1}(W) \subset f^{-1}(U)$. So we have $V \subset f^{-1}(U)$, which implies that $f : (X, \tau) \to (Y, \sigma^*)$ is somewhat b-continuous function.

5 Somewhat b-open functions

Definition 5.1 *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be somewhat b-open function provided that for $U \in \tau$ and $U \neq \phi$ there exists a b-open set V in Y such that $V \neq \phi$ and $V \subseteq f(U)$.*

Example 5.2 *Let* $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a, b\}, \{c\}\}$ *and* $\sigma = \{X, \phi\}$. *Define a function* $f : (X, \tau) \to (X, \sigma)$ *by*

$$f(a) = b, \;\; f(b) = c, \;\; f(c) = a$$

Then clearly f *is somewhat b-open function.*

Theorem 5.3 *Every somewhat semi-open function is somewhat b-open function.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a somewhat semi-open function. Let $U \in \tau$ and $U \neq \phi$. Since f is somewhat semi-open there exists a semi-open set V in Y such that $V \neq \phi$ and $V \subset f(U)$. But every semi-open set is b-open. Therefore there exists a b-open set V in Y such that $V \neq \phi$ and $V \subset f(U)$, which implies that f is somewhat b-open function.

Remark 5.4 *Converse of the above theorem need not be true in general, which follows from the following example.*

Example 5.5 *Let* $X = \{a, b, c, d\}$. *Let* $\tau = \{X, \phi, \{a, b\}\}$ *and* $\sigma = \{X, \phi\}$. *Define a function* $f : (X, \tau) \to (X, \sigma)$ *by*

$$f(a) = c, \;\; f(b) = a, \;\; f(c) = c$$

Then clearly f *is somewhat b-open function which is not a somewhat semi-open function.*

Theorem 5.6 *Every somewhat open function is somewhat b-open function.*

Proof. Let $f : (X, \tau) \to (X, \sigma)$ be somewhat open function. Let $U \in \tau$ and $U \neq \phi$. Since f is somewhat open function, there exists an open set $V \in \sigma$ such that $V \neq \phi$ and $V \subseteq f(U)$. But every open set is b-open. So there exists a b-open set $V \in \sigma$ such that $V \neq \phi$ and $V \subseteq f(U)$. Thus f is somewhat b-open function.

Remark 5.7 *Converse of the above theorem need not be true in general, which follows from the following example.*

Example 5.8 *Let* $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a\}, \{b\} \{a, b\}\}$ *and* $\sigma = \{X, \phi\}$. *Define a function* $f : (X, \tau) \to (X, \sigma)$ *as follows*

$$f(a) = b, \;\; f(b) = c, \;\; f(d) = a, \;\; f(c) = a.$$

Then clearly f *is somewhat b-open function, but not a somewhat open function.*

Theorem 5.9 *If $f : (X, \tau) \to (Y, \sigma)$ is an open map and $g : (Y, \sigma) \to (Z, \eta)$ is somewhat b-open map then $g \circ f : (X, \tau) \to (Z, \eta)$ is somewhat b-open map.*

Proof. Let $U \in \tau$. Suppose that $U \neq \phi$. Since f is an open map $f(U)$ is open and $f(U) \neq \phi$. Thus $f(U) \in \sigma$ and $f(U) \neq \phi$.

Since g is somewhat b-open map and $f(U) \in \sigma$ such that $f(U) \neq \phi$ there exists a b-open set $V \in \eta$, $V \subset g(f(U))$, which implies $g \circ f$ is somewhat b-open function.

Theorem 5.10 *If $f : (X, \tau) \to (Y, \sigma)$ is a one-one and onto mapping, then the following conditions are equivalent.*
(i) f is somewhat b-open map.
(ii) If C is a closed subset of X such that $f(C) \neq Y$, then there is a b-closed subset D of Y such that $D \neq Y$ and $D \supset f(C)$

Proof. $(i) \Rightarrow (ii)$:
Let C be any closed subset of X such that $f(C) \neq Y$.Then $X - C$ is open in X and $X - C \neq \phi$. Since f is somewhat b-open, there exists a b-open set $V \neq \phi$ in Y such that $V \subset f(X - C)$. Put $D = Y - V$. Clearly D is b-closed in Y and we claim that $D \neq Y$. For if $D = Y$, then $V = \phi$ which is a contradiction. Since $V \subset f(X - C)$, $D = Y - V \supset Y - [f(X - C)] = f(C)$.
$(ii) \Rightarrow (i)$**:**

Let U be any non-empty open set in X. Put $C = X - U$. Then C is a closed subset of X and $f(X - U) = f(C) = Y - f(U)$ implies $f(C) \neq \phi$. Therefore, by(ii) there is a b-closed subset D of Y such that $D \neq Y$ and $f(C) \subset D$. Put $V = X - D$. Clearly V is a b-open set and $V \neq \phi$. Further, $V = X - D \subset Y - f(C) = Y - [Y - f(U)] = f(U)$.

Theorem 5.11 *Let $f : (X, \tau) \to (Y, \sigma)$ be somewhat b-open function and A be any open subset of X. Then $f/A : (A, \tau/A) \to (Y, \sigma)$ is also somewhat b-open function.*

Proof. Let $U \in \tau/A$ such that $U \neq \phi$. Since U is open in A and A is open in (X, τ), U is open in (X, τ) and since by hypothesis $f : (X, \tau) \to (Y, \sigma)$ is somewhat b-open function, there exists a b-open set V in Y, such that $V \subset f(U)$.

Thus, for any open set U in $(A, \tau/A)$ with $U \neq \phi$, there exists a b-open set V in Y such that $V \subset f(U)$ which implies f/A is somewhat b-open function.

Theorem 5.12 *Let (X, τ) and (Y, σ) be any two topological spaces and $X = A \cup B$ where A and B are open subsets of X and $f : (X, \tau) \to (Y, \sigma)$ be a* *function such that f/A and f/B are somewhat b-open, then f is also somewhat b-open function.*

Proof. Let U be any open subset of (X, τ) such that $U \neq \phi$. Since $X = A \cup B$, either $A \cap U \neq \phi$ or $B \cap U \neq \phi$ or both $A \cap U \neq \phi$ and $B \cap U \neq \phi$. Since U is open in (X, τ) U is open in both $(A, \tau/A)$ and $(B, \tau/B)$.
Case (i): Suppose that $U \cap A \neq \phi$ where $U \cap A$ is open in τ/A. Since by hypothesis f/A is somewhat b-open function, there exists a b-open set $V \in (Y, \sigma)$ such that $V \subset f(U \cap A) \subset f(U)$, which implies f is somewhat b-open function.
Case (ii): Suppose that $U \cap B \neq \phi$, where $U \cap B$ is open in $(B, \sigma/B)$. Since by hypothesis f/B is somewhat b-open function, there exists a b-open set V in (Y, σ) such that $V \subset f(U \cap B) \subset f(U)$, which implies that f is also somewhat b-open function.
Case (iii): Suppose that both $U \cap B \neq \phi$ and $U \cap A \neq \phi$. Then obviously f is somewhat b-open function from the case (i) and case (ii). Thus f is somewhat b-open function.

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