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Fourier-Bessel Transform for Tempered Boehmians

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Abstract

In the present paper the tempered Boehmian is introduced as an extension of tempered distribution, properties of which are studied using relation between the Fourier and the Hankel transform.

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1. INTRODUCTION

The Hankel transform is also called the Fourier-Bessel transform, relation of which with the Fourier transform is discussed in the sequel that follows. The Fourier transform of a function with respect to each of the n-independent variable is defined [6, p.76] as

$$F_{(1)}(\xi_1, x_2, \ldots, x_n) = \mathcal{F}_{(1)}[f\ (x_1, x_2, \ldots, x_n); x_1 \to \xi_1] \tag{1}$$

and the double transform is

$$F_{(2)}(\xi_1, \xi_2, x_3, \ldots, x_n) = \mathcal{F}_{(2)}[f\ (x_1, x_2, \ldots, x_n); x_1 \to \xi_1, x_2 \to \xi_2]\ , \tag{2}$$

which in succession gives

$$F_{(n)}(\xi_1, \xi_2, \ldots, \xi_n) = \mathcal{F}_{(n)}[f\ (x_1, x_2, \ldots, x_n); x_1 \to \xi_1, \ldots, x_n \to \xi_n]\ . \tag{3}$$

Considering X and ξ by n - vectors $(x_1, x_2, \ldots, x_n)$ and $(\xi_1, \xi_2, \ldots, \xi_n)$,which has the inner product $x_1\xi_1 + x_2\xi_2 + \ldots + x_n\xi_n = \mathrm{X}\cdot\xi$, the n - dimensional Fourier transform

$$F_{(n)}(\xi) = \mathcal{F}_{(n)}[f\ (\mathrm{X}); x \to \xi] \tag{4}$$

of f (X) is defined by

$$F_{(n)}(\xi) = (2\pi)^{-\frac{n}{2}} \int\limits_{E_n} f\ (\mathrm{X}) e^{i(\xi\cdot\mathrm{X})} dX \quad , \tag{5}$$

where E_n is the n-dimensional Euclidean space. The inversion formula for the Fourier transform of f (X) and the convolution of that of for f (X) and g(X) of the vector X are defined, respectively, by

$$f\ (\mathrm{X}) = (2\pi)^{-\frac{n}{2}} \int\limits_{E_n} F_{(n)}(\xi) e^{-i(\xi\cdot\mathrm{X})} d\xi \quad , \tag{6}$$

and

$$\mathcal{F}_{(n)}[f \circ g\ (\mathrm{X}); \mathrm{X} \to \xi] = F_{(n)}(\xi) G_{(n)}(\xi) \quad . \tag{7}$$

For two dimensional Fourier transform in plane polar coordinates (r, θ) in x_1x_2- plane and (ρ, φ) in $\xi_1\xi_2$-plane, we have $x_1 = r\cos\theta, x_2 = r\sin\theta, \xi_1 = \rho\cos\varphi, \xi_2 = \rho\sin\varphi$, which yields $(\mathrm{X}\cdot\xi) = r\rho\cos(\theta - \varphi)$,from which we obtain

$$\begin{aligned} \mathcal{F}_{(2)}[f(r), x_1 &\to \xi_1, x_2 \to \xi_2] \\ &= \frac{1}{2\pi} \int\limits_0^\infty r dr \int\limits_0^{2\pi} f\ (r) e^{ir\rho\cos(\theta-\varphi)} d\theta \quad . \end{aligned} \tag{8}$$

We also have [7]

$$J_0(r\rho) = \frac{1}{2\pi} \int\limits_0^{2\pi} e^{ir\rho\cos(\theta-\varphi)} d\theta \tag{9}$$

such that (8) is written as

$$\mathcal{F}_{(2)}[f(r), x_1 \to \xi_1, x_2 \to \xi_2] = \int_0^\infty r f(r) J_0(r\rho) dr \quad , \tag{10}$$

in other words,

$$\mathcal{F}_{(2)}[f(r), x_1 \to \xi_1, x_2 \to \xi_2] = \mathcal{H}_0[f(r); \rho] \quad , \tag{11}$$

$\mathcal{H}_0$ is the Hankel transform of order zero, properties of which can be proved by using relation between the Fourier transform and those given by (8) and (11), respectively.

Howell [2, Art.2.4.3]defined the transform of circularly system function and the Hankel transform by replacing (x, y) and (ω, υ) with polar equivalents in

$$F(\omega, \upsilon) = \mathcal{F}[f(x, y)]_{\omega,\upsilon} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-i(\omega x + \upsilon y)} dx dy \tag{12}$$

such that $(x, y) = (r \cos\theta, r \sin\theta)$ and $(\omega, \upsilon) = (\rho\cos\varphi, \rho \sin\varphi)$, which reduce (12) as

$$\begin{aligned} F(\rho\cos\varphi, \rho\sin\varphi) &= \int_0^\infty \int_{-\pi}^{\pi} f(r\cos\theta, r\sin\theta) \\ &\quad \times e^{-ir\rho\cos(\theta-\varphi)} r d\theta dr \quad . \end{aligned} \tag{13}$$

The two - dimensional inverse Fourier transform in polar form is

$$\begin{aligned} f(r\cos\theta, r\sin\theta) &= \frac{1}{4\pi^2} \int_0^\infty \int_{-\pi}^{\pi} F(\rho\cos\varphi, \rho\sin\varphi) \\ &\quad \times e^{ir\rho\cos(\theta-\varphi)} \rho d\varphi d\rho \quad . \end{aligned} \tag{14}$$

If $f(r\cos\theta, r\sin\theta)$ is separable with respect to r, θ, i.e.,

$$f(r\cos\theta, r\sin\theta) = f_r(r) f_\theta(\theta) \quad , \tag{15}$$

then (13) reduces to

$$F(\rho\cos\varphi, \rho\sin\varphi) = \int_0^\infty r f_r(r) K^-(r\rho, \varphi) d\varphi dr \quad , \tag{16}$$

where

$$K^{-}(z,\varphi)=\int_{-\pi}^{\pi} f_{\theta}(\theta)e^{-iz\cos(\theta-\varphi)}d\theta$$

i.e.

$$K^{-}(z,\varphi)=\int_{-\pi}^{\pi} f_{\theta}(\theta'+\varphi)=e^{-iz\cos(\theta')}d\theta'\ ,\ \ \theta'=(\theta-\varphi)\ . \tag{17}$$

Similarly, if $F(\rho\cos\varphi,\rho\sin\varphi)$ is separable with respect to ρ,φ, i.e.,

$$F(\rho\cos\varphi,\rho\sin\varphi)=F_{\rho}(\rho)F_{\varphi}(\varphi) \qquad , \tag{18}$$

then (14) gives

$$f(r\cos\theta,r\sin\theta)=\frac{1}{4\pi^2}\int_{0}^{\infty}\rho F_{\rho}(\rho)K^{+}(r\rho,\theta)d\rho\ \ , \tag{19}$$

where

$$K^{+}(z,\theta)=\int_{-\pi}^{\pi} F(\theta'+\theta)e^{iz}\cos\varphi' d\varphi' \tag{20}$$

Therefore, it follows from (16) through (19) that if either $f(x,y)$ or $F(\omega,\upsilon)$ is circularly symmetric, then

$$f(r\cos\theta,r\sin\theta)=f_{r}(r) \tag{21}$$

and

$$F(\rho\cos\varphi,\rho\sin\varphi)=F_{\rho}(\rho) \tag{22}$$

The Bessel function identity [7] is

$$2\pi J_0(z)=\int\limits_{-\pi}^{\pi}\cos(z\cos\omega)d\omega$$

and

$$K^{\pm}(r\rho,\omega)=\int_{-\pi}^{\pi} e^{\pm ir\rho}\cos\omega d\omega$$

$$=\int_{-\pi}^{\pi}\cos(r\rho\cos\omega)d\omega=2\pi J_0(r\rho)\qquad , \tag{23}$$

where J_0 is the Bessel function of first kind of order zero.

Further, due to above relations, (16) and (19) reduce to

$$F_\rho(\rho)=2\pi\int_0^{\infty} f_r(r)J_0(r\rho)rdr \tag{24}$$

and

$$f_r(r)=\frac{1}{2\pi}\int_0^{\infty} F_\rho(\rho)J_0(r\rho)\rho d\rho\qquad . \tag{25}$$

The Hankel transform of order zero for the function $g(r)$ [2] is

$$\hat{g}(\rho)=\int_0^{\infty} g(r)J_0(r\rho)rdr \tag{26}$$

by which we express (24) and (25) in terms of the Hankel transform of order zero as

$$F_\rho(\rho)=2\pi\hat{f}_r(r) \tag{27}$$

and

$$f_r(r)=\frac{1}{2\pi}\hat{F}_\rho(\rho) \tag{28}$$

This is an evidence to view the Hankel transform of order zero as two dimensional Fourier transform of circularly symmetric functions. Since the Fourier transform for the tempered distribution and tempered Boehmian is established [cf. [4, 5]], we, therefore, use relation between two-dimensional Fourier transform and the Hankel transform to establish Boehmians of slow growth (the tempered Boehmians), an illustration follows.

The distribution space of slow growth S', for the Fourier transform is defined in terms of the Parseval equation, by [8]

$$\left\langle \tilde{f}, \tilde{\varphi} \right\rangle = 2\pi \left\langle f, \varphi \right\rangle \tag{29}$$

and

$$\left\langle f, \overline{\tilde{\varphi}} \right\rangle = 2\pi \left\langle f, \bar{\varphi} \right\rangle \quad , \tag{30}$$

respectively. Similarly, using relation between two - dimensional Fourier transform and the Hankel transform, the tempered distribution space S' for the Hankel transform is given by

$$\left\langle \hat{g}(\rho), \hat{\varphi}(\rho) \right\rangle = \left\langle g(r), \varphi(r) \right\rangle \quad , \tag{31}$$

in other words,

$$\left\langle \mathcal{F}_2(f), \mathcal{F}_2(\varphi) \right\rangle = \left\langle f, \varphi \right\rangle$$

i.e.

$$\left\langle \mathcal{H}_0(f), \mathcal{H}_0(\varphi) \right\rangle = \left\langle f, \varphi \right\rangle \quad . \tag{32}$$

Tempered Boehmians : The pair of sequence (f_n, φ_n) is called a quotient of sequence, denoted by f_n/φ_n, whose numerator belongs to some set $\mathcal{G}$ and the denominator is a delta sequence such that

$$f_n * \varphi_m = f_m * \varphi_n \quad , \ \forall n, m \in \mathbb{N} \quad . \tag{33}$$

Two quotients of sequence f_n/φ_n and g_n/ψ_n are said to be equivalent if

$$f_n * \psi_n = g_n * \varphi_n \quad , \ \forall n \in \mathbb{N} \quad . \tag{34}$$

The equivalence classes are called the Boehmians, for details, see [1]. The space of Boehmians is denoted by $\mathcal{B}$, an element of which is written as

$x = f_n/\varphi_n$. Application of construction of Boehmians to function spaces with the convolution product yields various spaces of generalized functions. The spaces, so obtained, contain the standard spaces of generalized functions defined as dual spaces. For example, if $\mathcal{G} = C(\mathbb{R}^N)$ and a delta sequence defined as sequence of functions $\varphi_n \in \mathcal{D}$ such that

(i) $\int \varphi_n dx = 1 \quad , \ \forall a \in \mathbb{N}$
(ii) $\int |\varphi_n| \, dx \leq C \quad$, for some constant C and $\forall n \in \mathbb{N}$,
(iii) supp $\varphi_n(x) \to 0$, as $n \to \infty$,

then the space of Boehmian that is obtained, contains properly the space of Schwartz distributions. Similarly, this space of Boehmians also contains properly the space of tempered distributions S' , when $\mathcal{G}$ is the space of slowly increasing functions with delta sequence. The Hankel transform of tempered Boehmian form a proper subspace of Schwartz distribution $\mathcal{D}'$. Boehmian space have two types of convergence, namely, the δ- and Δ- convergences, which are stated as:

(i) A sequence of Boehmians (x_n) in the Boehmian space $\mathcal{B}$ is said to be δ- convergent to a Boehmian x in $\mathcal{B}$, which is denoted by $x_n \overset{\delta}{\to} x$ if there exists a delta sequence (δ_n) such that $(x_n * \delta_n), (x * \delta_n) \in \mathcal{G}, \forall n \in \mathbb{N}$ and $(x_n * \delta_k) \to (x * \delta_k)$ as $n \to \infty$ in $\mathcal{G}$, $\forall k \in \mathbb{N}$.

(ii) A sequence of Boehmians (x_n) in $\mathcal{B}$ is said to be Δ- convergent to a Boehmian x in $\mathcal{B}$, denoted by $x_n \overset{\delta}{\to} x$ if there exists a delta sequence $(\delta_n) \in \Delta$ such that $(x_n - x) * \delta_n \in \mathcal{G}, \forall n \in \mathbb{N}$ and $(x_n - x) * \delta_n \to 0$ as $n \to \infty$ in $\mathcal{G}$.

For details of the properties and convergence of Boehmians one can refer to [3]. We have employed following notations and definitions.

A complex valued infinitely differentiable function f , defined on $\mathbb{R}^N$, is called rapidly decreasing, if

$$\sup_{|\alpha| \leq m} \sup_{x \in R^N} \left(1 + x_1^2 + x_2^2 + \cdots + x_N^2\right)^m |D^\alpha f(x)| < \infty \quad ,$$

for every non-negative integer m. Here $|\alpha| = |\alpha_1| + \cdots + |\alpha_N|$, and

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}} \quad .$$

The space of all rapidly decreasing functions on $\mathbb{R}^N$ is denoted by S. The delta sequence, i.e., sequence of real valued functions $\varphi_1, \varphi_2, \ldots \in S$, is such that

(i) $\int \varphi_n dx = 1 \quad , \ \forall a \in \mathbb{N}$
(ii) $\int |\varphi_n| \, dx \leq C \quad$, for some constant C and $\forall n \in \mathbb{N}$,

(iii) $\lim_{n\to\infty} \int_{\|x\|\geq\epsilon} \|x\|^k |\varphi_n| \, dx = 0$, for every $k \in \mathbb{N}, \epsilon > 0$.

If $\varphi \in S$ and $\int \varphi = 1$, then the sequence of functions φ_n is a delta sequence.

A complex-valued function f on $\mathbb{R}^N$ is called slowly increasing if there exists a polynomial p on $\mathbb{R}^N$ such that $f(x)/p(x)$ is bounded. The space of all increasing continuous functions on $\mathbb{R}^N$ is denoted by $\mathcal{I}$. If $f_n \in \mathcal{I}$, $\{\varphi_n\}$ is a delta sequence under usual notion, then the space of equivalence classes of quotients of sequence will be denoted by $\mathcal{B}_{\mathcal{I}}$, elements of which will be called tempered Boehmians.

For $F = [f_n/\varphi_n] \in \mathcal{B}_{\mathcal{I}}$, define $D^\alpha F = [(f_n * D^\alpha \varphi_n)/(\varphi_n * \varphi_n)]$. If F is a Boehmian corresponding to differentiable function, then $D^\alpha F \in \mathcal{B}_{\mathcal{I}}$.

If $F = [f_n/\varphi_n] \in \mathcal{B}_{\mathcal{I}}$ and $f_n \in S$, for all $n \in \mathbb{N}$, then F is called a rapidly decreasing Boehmian. The space of all rapidly decreasing Boehmian is denoted by $\mathcal{B}_S$. If $F = [f_n/\varphi_n] \in \mathcal{B}_{\mathcal{I}}$ and $G = [g_n/\psi_n] \in \mathcal{B}_S$,then the convolution is

$$F * G = [(f_n * g_n)/(\varphi_n * \psi_n)] \in \mathcal{B}_{\mathcal{I}} \quad .$$

The convolution quotient is denoted by f/φ and $\frac{f}{\varphi}$ denotes a usual quotient. Let $f \in \mathcal{I}$. Then the Hankel transformation of f, denoted as $\mathcal{H}_0(f)$, is defined for distribution spaces (given as in Equation (31)) of slowly increasing function f in the following forms

$$\langle \mathcal{H}_0(f), \varphi \rangle = \langle f, \mathcal{H}_0(\varphi) \rangle \quad , \quad \varphi \in S$$

and

$$\mathcal{H}_0(f) = F_\rho(\rho) = 2\pi \int_0^\infty f_r(r) J_0(r\rho) r dr \quad .$$

2. HANKEL TRANSFORM FOR TEMPERED BOEHMIAN

Theorem 1 *: If $[f_n/\varphi_n] \in \mathcal{B}_{\mathcal{I}}$, then the sequence $\mathcal{H}_0\{f_n\}$, $n = 1, 2, \ldots, \infty$ converges in $\mathcal{D}'$. Moreover, if $[f_n/\varphi_n] = [g_n/\psi_n]$,then $\mathcal{H}_0\{f_n\}$ and $\mathcal{H}_0\{g_n\}$ converges to the same limit for the Hankel transformation of tempered Boehmians.*

Proof. Let $\varphi \in \mathcal{D}$ (testing function space) and $k \in \mathbb{N}$ be such that $\mathcal{H}_0\varphi_k > 0$ on the support of φ. Then we write

$$\langle \mathcal{H}_0\{ f_n\}, \varphi_n \rangle = \left\langle \mathcal{H}_0\{f_n\}, \varphi \cdot \frac{\mathcal{H}_0\{\varphi_k\}}{\mathcal{H}_0\{\varphi_k\}} \right\rangle$$

$$= \left\langle \mathcal{H}_0\{f_n\} \cdot \mathcal{H}_0\{\varphi_k\}, \frac{\varphi}{\mathcal{H}_0\{\varphi_k\}} \right\rangle$$

$$= \left\langle \mathcal{H}_0\{f_k\} \cdot \mathcal{H}_0\{\varphi_n\}, \frac{\varphi}{\mathcal{H}_0\{\varphi_k\}} \right\rangle$$

$$= \left\langle \mathcal{H}_0\{f_k\}, \frac{\varphi \cdot \mathcal{H}_0\{\varphi_n\}}{\mathcal{H}_0\{\varphi_k\}} \right\rangle \quad .$$

Since the sequence $\left\{\frac{\varphi \cdot \mathcal{H}_0 \varphi_n}{\mathcal{H}_0 \varphi_n}\right\}$ converges to $\frac{\varphi}{\mathcal{H}_0 \varphi_k}$ in $\mathcal{D}$. Indeed, the sequence $\{\mathcal{H}_0(f_n), \varphi\}$ converges .This proves that the sequence $\mathcal{H}_0\{f_n\}$ converges in $\mathcal{D}'$ (dual of space $\mathcal{D}$). Now, assume that $[f_n/\varphi_n] = [g_n/\psi_n] \in \mathcal{B}_{\mathcal{I}}$. Let us define

$$h_n = \begin{cases} f_{\frac{n+1}{2}} * \psi_{\frac{n+1}{2}} & \text{, if } n \text{ is odd} \\ \\ g_{\frac{n}{2}} * \varphi_{\frac{n}{2}} & \text{, if } n \text{ is even} \end{cases} \quad .$$

and

$$\delta_n = \begin{cases} \varphi_{\frac{n+1}{2}} * \psi_{\frac{n+1}{2}} & \text{, if } n \text{ is odd} \\ \\ \varphi_{\frac{n}{2}} * \psi_{\frac{n}{2}} & \text{, if } n \text{ is even} \end{cases} \quad .$$

Then $[h_n/\delta_n] = [f_n/\varphi_n] = [g_n/\psi_n]$ and the sequence $\mathcal{H}_0\{h_n\}$ converges in $\mathcal{D}'$. Moreover, $\lim\limits_{n\to\infty} \mathcal{H}_0\{h_{2n-1}\}(\varphi_n) = \lim\limits_{n\to\infty} \mathcal{H}_0\{f_n * \psi_n\} \cdot (\varphi) = \lim\limits_{n\to\infty} \mathcal{H}_0\{f_n \cdot \psi_n\}$ $\cdot(\varphi) = \lim\limits_{n\to\infty} \mathcal{H}_0\{f_n\}\{\mathcal{H}_0(\psi_n) \cdot (\varphi)\} = \lim\limits_{n\to\infty} \mathcal{H}_0\{f_n\}(\varphi)$. ■

Thus, sequence $\mathcal{H}_0\{f_n\}$ and $\mathcal{H}_0\{h_n\}$ have the same limit, which implies

that $\mathcal{H}_0\{h_n\}$ and $\mathcal{H}_0\{g_n\}$ will also have the same limit.This completes the proof of the theorem .

Definition 1 : Let $F = [f_n/\varphi_n] \in \mathcal{B}_{\mathcal{I}}$ and the limit of sequence $\mathcal{H}_0\{f_n\}$ is in $\mathcal{D}'$. The Hankel transform of $\mathcal{H}_0\{F\}$ of F, thus, has a mapping from $\mathcal{B}_{\mathcal{I}}$ into $\mathcal{D}'$, which is linear.

Theorem 2 *: Let $F = [f_n/\varphi_n] \in \mathcal{B}_{\mathcal{I}}$ and $G = [g_n/\psi_n] \in \mathcal{B}_S$.*
Then (i) $\mathcal{H}_0(G)$ is an infinitely differentiable function
(ii) $\mathcal{H}_0[F * G] = \mathcal{H}_0[F]\mathcal{H}_0[G]$
and (iii) $\mathcal{H}_0(F) \cdot \mathcal{H}_0(\varphi_n) = \mathcal{H}_0(f_n)$, $\forall n \in \mathbb{N}$

Proof. (i) Let $G = [g_n/\psi_n] \in \mathcal{B}_S$ and U be a bounded open subset of $\mathbb{R}^N$

(the n- dimensional Euclidean space). Then there exists $m \in \mathbb{N}$ such that $\mathcal{H}_0\{\psi_m\} > 0$ on U, and also ■

$$\mathcal{H}_0(G) = \lim_{n\to\infty} \mathcal{H}_0\{g_n\} = \lim_{n\to\infty} \frac{\mathcal{H}_0\{g_n\}\mathcal{H}_0\{\psi_m\}}{H_0\{\psi_m\}}$$

$$= \lim_{n\to\infty} \frac{\mathcal{H}_0\{g_n\}\mathcal{H}_0\{\psi_m\}}{\mathcal{H}_0\{\psi_m\}} = \frac{\mathcal{H}_0\{g_m\}}{\mathcal{H}_0\{\psi_m\}} \lim_{n\to\infty} \mathcal{H}_0\{\psi_n\}$$

$$= \frac{\mathcal{H}_0\{g_m\}}{\mathcal{H}_0\{\psi_m\}} \quad \text{on } U \quad .$$

Since $\mathcal{H}_0\{g_m\}, \mathcal{H}_0\{\psi_m\} \in S$ and $\mathcal{H}_0\{\psi_m\} > 0$ on U, thus $\mathcal{H}_0\{G\}$ is an infinitely differentiable function on U.

(ii) If $\varphi \in \mathcal{D}$, then there exists $m \in \mathbb{N}$ such that $\mathcal{H}_0\{\psi_m\} > 0$ on the support of φ. Thus, we have

$$\mathcal{H}_0\{F * G\}\{\varphi\} = \lim_{n\to\infty} \mathcal{H}_0\{f_n * g_n\}\{\varphi\}$$

$$= \lim_{n\to\infty} \mathcal{H}_0\{f_n g_n\}(\varphi) = \lim_{n\to\infty} \mathcal{H}_0\{f_n\}\{\mathcal{H}_0 g_n(\varphi)\}$$

$$= \lim_{n\to\infty} \mathcal{H}_0\{f_n\} \left\{ \mathcal{H}_0 g_n \varphi \frac{\mathcal{H}_0\psi_m}{\mathcal{H}_0\psi_m} \right\} \quad , \quad m \in \mathbb{N}$$

$$= \lim_{n\to\infty} \mathcal{H}_0\{f_n\} \left\{ \frac{\mathcal{H}_0 g_m \cdot \mathcal{H}_0\psi_n}{\mathcal{H}_0\psi_m} \cdot \varphi \right\}$$

$$= \lim \mathcal{H}_0\{f_n\} \left\{ \frac{\mathcal{H}_0 g_m}{\mathcal{H}_0\psi_m} \cdot \varphi \mathcal{H}_0(\psi_n) \right\}$$

$$= \lim_{n\to\infty} \mathcal{H}_0\{f_n\}\{\mathcal{H}_0(G)\varphi\mathcal{H}_0(\psi_n)\}$$

$$= \mathcal{H}_0\{G\} \lim_{n\to\infty} \{\mathcal{H}_0(f_n)\mathcal{H}_0(\psi_n)\}(\varphi)$$

$$= \mathcal{H}_0\{G\} \lim_{n\to\infty} \mathcal{H}_0(f_n * \psi_n)(\varphi)$$

$$= \mathcal{H}_0\{F\}\mathcal{H}_0\{G\}\{\varphi\} = \mathcal{H}_0\{F\} \cdot \mathcal{H}_0\{G\}\{\varphi\} \tag{35}$$

(iii) Let $\varphi \in \mathcal{D}$. Then

$$\{\mathcal{H}_0 F \cdot \mathcal{H}_0 \varphi_m\}\{\varphi\} = \{\mathcal{H}_0 F\}\{\mathcal{H}_0(\varphi_m)\varphi\} \quad , \quad m \in \mathbb{N}$$

$$= \lim_{n\to\infty} \{\mathcal{H}_0 f_n\}\{\mathcal{H}_0(\varphi_m)\varphi\} = \lim_{n\to\infty} \{\mathcal{H}_0(f_n) \cdot \mathcal{H}_0(\varphi_m)\}\{\varphi\}$$

$$= \lim_{n\to\infty} \{\mathcal{H}_0(f_m) \cdot \mathcal{H}_0(\varphi_n)\}\{\varphi\} = \lim_{n\to\infty} \mathcal{H}_0\{f_m\}\{\mathcal{H}_0(\varphi_n) \cdot \varphi\}$$

$$= \mathcal{H}_0\{f_m\}\{\varphi\} = \mathcal{H}_0\{F_m\}\{\varphi\}$$

The theorem is, therefore, completely proved.

Theorem 3 *: A distribution f is the Hankel transform of tempered Boehmian if and only if there exists a delta sequence $\{\delta_n\}$ such that $\{f\mathcal{H}_0(\delta_n)\}^{\vee}$ is a tempered distribution for every $n \in \mathbb{N}$.*

Proof. Let $F = [f_n/\varphi_n] \in \mathcal{B}_{\mathcal{I}}$ and $f = \mathcal{H}_0\{F\}$.Then $f\mathcal{H}_0\{\varphi_n\} = \mathcal{H}_0\{F\}\mathcal{H}_0\{\varphi_n\}$. Thus, $f\mathcal{H}_0(\varphi_n)$ is a tempered distribution. Now let $f \in \mathcal{D}'$, and (δ_n) be a delta sequence such that $f\mathcal{H}_0(\delta_n)$ is tempered distribution for every $n \in \mathbb{N}$. We define

$$F = \left[\frac{\{f\mathcal{H}_0(\delta_n)\}^{\vee} * \delta_n}{(\delta_n * \delta_n)}\right] \tag{36}$$

where $\{f\mathcal{H}_0(\delta_n)\}^{\vee}$ is the inverse Hankel transform of $\{f\mathcal{H}_0(\delta_n)\}$. Since $\{f\mathcal{H}_0(\delta_n)\}$ is a tempered distribution, therefore, $\{f\mathcal{H}_0(\delta_n)\}^{\vee}$ is also a tempered distribution. ■

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