Int. Journal of Math. Analysis, Vol. 4, 2010, no. 48, 2367 - 2372

Fixed Point Theorems for Weak Contractions in Cone Metric Spaces

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Abstract

We prove fixed point theorems for weak contractions in a non-normal cone metric space.

Keywords: Fixed point, weak contraction, cone metric space

1. Introduction

Let (E, τ) be a topological vector space and P a subset of E, P is called a cone if

1. P is closed, non-empty and $P \neq \{0\}$,
2. $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b,
3. $P \cap (-P) = \{0\}$.

For a given cone $P \subseteq E$, a partial ordering $\leq$ with respect to P is defined by $x \leq y$ if and only if $y - x \in P, x < y$ if $x \leq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in int\ P$, $int\ P$ denotes the the interior of P. If E is a normed space, then the cone P is called normal (with respect to this norm) if there is a number $M > 0$ such that for all $x, y \in E, 0 \leq x \leq y$ implies $\| x \| \leq M \| y \|$. The least positive integer satisfying this norm inequality is called the normal constant of P [2]. Of course, there are non-normal cones [4].

Definition 1.1 ([2]).

Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}_{n=1}^{\infty}$ a sequence in X. Then

1. $\{x_n\}_{n=1}^{\infty}$ converges to x whenever for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $\lim_{n \longrightarrow \infty} x_n = x$ or $x_n \longrightarrow x$.
2. $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence whenever for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.
3. (X, d) is a complete cone metric space if every Cauchy sequence is convergent.

Recently, the following results were obtained.

Theorem 1.1 ([2]). Let (X, d) be a complete cone metric space and P be a normal cone with normal constant K. Suppose the mapping $T : X \to X$ satisfies the contractive condition

$$d(Tx, Ty) \leq kd(x, y), \quad \text{for all } x, y \in X,$$

where $k \in [0, 1)$ is a constant. Then T has a unique fixed point in X. For any $x \in X$, iterative sequence $\{T^n x\}_{n=1}^{\infty}$ converges to the fixed point.

Theorem 1.2 ([4]). Let (X, d) be a complete cone metric space and the mapping $T : X \to X$ satisfy the contractive condition

$$d(Tx, Ty) \leq kd(x, y), \quad \text{for all } x, y \in X,$$

where $k \in [0, 1)$ is a constant. Then T has a unique fixed point in X. For each $x \in X$, the iterative sequence $\{T^n x\}_{n=1}^{\infty}$ converges to the fixed point.

Theorem 1.3 ([4]). Let (X, d) be a complete cone metric space and the mapping $T : X \to X$ satisfy the contractive condition

$$d(Tx, Ty) \leq k(d(Tx, x) + d(Ty, y)), \tag{1}$$

for all $x, y \in X$,where $k \in [0, \frac{1}{2})$ is a constant. Then T has a unique fixed point in X. For each $x \in X$, the iterative sequence $\{T^n x\}_{n=1}^{\infty}$ converges to the fixed point.

Theorem 1.4 ([4]). Let (X, d) be a complete cone metric space and the mapping $T : X \to X$ satisfy the contractive condition

$$d(Tx, Ty) \leq k(d(Tx, y) + d(x, Ty)), \tag{2}$$

for all $x, y \in X$,where $k \in [0, \frac{1}{2})$ is a constant. Then T has a unique fixed point in X. For each $x \in X$, the iterative sequence $\{T^n x\}_{n=1}^{\infty}$ converges to the fixed point.

For other related results see [5] and [6].

In this paper we use the definition of the weak contraction mappings due to Berinde [1] on cone metric spaces and prove some fixed point theorems of weak contractions. It is worth mentioning that the class of weak contractions includes the classes of Kannan mappings [3] and Zamfirescu mappings [7].

Main Results

Definition 2.1 Let (X, d) be a complete cone metric space. A map $T :$ $X \to X$ is called a weak contraction if there exists a constant $a \in (0, 1)$ and some $b \geq 0$ such that

$$d(Tx, Ty) \leq ad(x, y) + bd(y, Tx) \text{ for all } x, y \in X. \tag{3}$$

Theorem 2.1 Let (X, d) be a complete cone metric space and the mapping $T : X \to X$ a weak contraction. Then T has a fixed point in X.

Proof. For each $x_0 \in X$ and $n \geq 1$, let $x_1 = Tx_0$, and $x_{n+1} = Tx_n = T^{n+1}x_0$. Then

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \leq ad(x_{n-1}, x_n) + bd(x_n, Tx_{n-1}) = ad(x_{n-1}, x_n) \\ &\leq a^2 d(x_{n-2}, x_{n-1}) \leq ... \leq a^n d(x_0, x_1). \end{aligned}$$

So for $n > m$,

$$d(x_m, x_n) \leq (a^m + a^{m+1} + ... + a^{n-1})d(x_0, x_1) \leq \frac{a^m}{1-a} d(x_0, x_1).$$

Let $0 \ll c$ be given. Choose a natural number N such that $\frac{a^m}{1-a} d(x_0, x_1) \ll c$ for every $m \geq N$. Thus

$$d(x_m, x_n) \leq \frac{a^m}{1-a} d(x_0, x_1) \ll c \text{ for every } n > m \geq N.$$

Therefore the sequence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in (X, d). Since (X, d) is complete, there exists $z \in X$ such that $x_n \longrightarrow z$.Choose a natural number N_1 such that

$$d(x_n, z) \leq \frac{c}{2(1+b)} \text{ for every } n \geq N_1.$$

Hence for $n \geq N_1$ we have

$$\begin{aligned} d(z, Tz) &\leq d(z, x_{n+1}) + d(x_{n+1}, Tz) \\ &= d(z, x_{n+1}) + d(Tx_n, Tz) \\ &\leq d(z, x_{n+1}) + ad(x_n, z) + bd(z, Tx_n) \\ &= (1+b)d(z, x_{n+1}) + ad(x_n, z) \\ &\leq (1+b)d(z, x_{n+1}) + d(x_n, z) \\ &\leq (1+b)[d(z, x_{n+1}) + d(x_n, z)] \\ &\ll (1+b)[\frac{c}{2(1+b)} + \frac{c}{2(1+b)}] = c \text{ for every } n \geq N_1. \end{aligned}$$

Thus

$$d(z, Tz) \ll \frac{c}{m} \text{ for all } m \geq 1.$$

So $\frac{c}{m} - d(z, Tz) \in P$ for all $m \geq 1$. Since $\frac{c}{m} \to 0($ as $m \to \infty)$, and P is closed, $-d(z, Tz) \in P$. But $d(z, Tz) \in P$. Therefore $d(z, Tz) = 0$ and so $Tz = z$.

We now show that Theorem 1.3 and Theorem 1.4 are corollaries of our results.

Corollary 2.1. Let (X, d) be a complete cone metric space. Any mapping $T : X \to X$ satisfying the contractive condition (1) is a weak contraction and so has a fixed point.

Proof. We follow [1]. By (1), we have,

$$\begin{aligned} d(Tx, Ty) &\leq k(d(x, Tx) + d(y, Ty)) \\ &\leq k\{[d(x, y) + d(y, Tx)] + [d(y, Tx) + d(Tx, Ty)]\}, \end{aligned}$$

which implies,

$$(1-k)d(Tx,Ty) \leq kd(x,y) + 2kd(y,Tx),$$

and which yields,

$$d(Tx,Ty) \leq \frac{k}{1-k}d(x,y) + \frac{2k}{1-k}d(y,Tx)$$

for all $x, y \in X$. Since $0 < k < \frac{1}{2}$, (3) holds with $a = \dfrac{k}{1-k}$, and $b = \dfrac{2k}{1-k}$.

Corollary 2.2. Let (X, d) be a complete cone metric space. Any mapping $T : X \to X$ satisfying the contractive condition (2) is a weak contraction and so has a fixed point.

Proof. We follow [1]. Using $d(x,Ty) \leq d(x,y) + d(y,Tx) + d(Tx,Ty)$, by (2) we get

$$d(Tx,Ty) \leq k[d(x,y) + d(y,Tx) + d(Tx,Ty) + d(y,Tx)]$$

$$(1-k)d(Tx,Ty) \leq kd(x,y) + 2kd(y,Tx)$$

$$d(Tx,Ty) \leq \frac{k}{1-k}d(x,y) + \frac{2k}{1-k}d(y,Tx)$$

which is (3) with $a = \dfrac{k}{1-k}$, and $b = \dfrac{2k}{1-k} \geq 0$.

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Received: July, 2010