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Portfolio Selection Problems Based on Fuzzy Interval Numbers Under the Minimax Rules

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Abstract

Recently, many researchers pay their attention to portfolio problems under minimax rules. Based on the uncertainty of the expected return of securities, therefore, it is of significant and interesting to study portfolio models based on fuzzy interval numbers under minimax rules(in briefly, PMFM). This PMFM-model can be regarded as a natural generalization of ordinarily portfolio model under the minimax rule. In the present paper, we study the PMFM-model and arrive at some similar and interesting conclusions. These conclusions imply that the optimal solution to PMFM is much better than that of the model without fuzzy interval numbers.

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Keywords: Fuzzy Interval Portfolio; Interval Numbers; Minimax Rules; Transaction Costs

1 Introduction

It is well known that the uncertainty of things manifests in two ways: 1. Randomness, that is the uncertainty of things that whether it happens; 2. Ambiguity, that is uncertainty of the state of thing itself.

In the random case, the people's cognition has been more complete. In 1952, Markowitz[21] proposed the mean-variance model in which the variance of the benefit measures risk and the earnings expected measures return on investment. He aims to find the best solution to the model through maximizing returns when risk is fixed, or minimizing the risk

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when the income is fixed. After that, scholars like Sharp[30], Mao[20], Knono[13, 14, 15], Young[39], Pan and Zhao[25], Swalm[32], Cai, Teo and Yang[2], Feinstein and Thapa[4] make the model better. 1999, Speranza [31] put forward mean-semi-absolute deviation portfolio selection model regarded the absolute deviation between portfolio returns of future and expected return as risk measure.

In 1950s, Samuelson [29] first proposed high-order moments. Investors then began to introduce third-order moments of the distribution of return into portfolio models, such as: Kane[11], Konno and Suzuki [14], Chunhachida, Dandapani, Hamid, and Prakash [5], etc.

In 1952, Roy[28] raised safety-first model and proposed to take minimum of the probability of a given "disaster risk level" and the portfolio return as measurement of the risk. After that, we also studied in [7, 9, 10, 37] this transformed-model with transaction costs and some risk constraints.

In 1968, Mossin[24] extended single-stage model of Markowitz to a multi-stage situation with dynamic programming method. Recently, Zhou and Li[42] in continuous time, Li and Ng[19] in the discrete time extended single-stage portfolio model of Markowitz to a multi-stage mean-variance model respectively and found efficient frontier. Another feasible way to study multi-stage portfolio selection problem is to apply sequential decision-making method. Among them, Cover[6] and others came up with a pan-portfolio selection model in order to maximize the multiple growth of terminal wealth, proposed. Other scholars Carlsson[3], Merton[22], Yoshimoto[38], Deng et al[8], Lai et al[12] and Zhang et al[44] etc have also studied the method.

Comparatively speaking, the research of investment portfolio in fuzzy situation is lagging behind. Ambiguity is the inherent characteristic of things. Zadeh[40, 41] put forward the concept of fuzzy sets. In 1970, Bellman and Zadeh[1] established a fuzzy decision model. In recent years, people started to pay attention to the ambiguity of the securities market, including the situation of securities profit fluctuating in a interval. In this case, it is urgent and significance to study portfolio problem in the setting of fuzzy.

Tanaka and Guo[33, 34] posed the central-difference model. 1998, Ramaswamy[27] studied portfolio choice based on fuzzy decision-making theory.

Leon et al[17, 18] solved the problems not feasible that may arise the portfolio selection model in use of fuzzy decision-making approach. Lai et al[16] consider the expectations of securities gains and covariance as interval numbers to see the promotion of the Markowitz model. On this basis, there are some scholars who have studied it such as Tang K-S et al[35], K. Lai et al[12], M. Boshernitzan[20], Parra, Terol and Uria[22].

From these statements, we find that there is no corresponding Minimax idea in the model of interval numbers. So we consider in this paper to apply Minimax rule to the interval number model, then construct the portfolio model based on Minimax rule with fuzzy interval numbers. That is to say, this paper wish to establish the model of investment portfolio selection in the theory of fuzzy interval numbers. Moreover, it puts forward the best solution to this model combined with the rules of Minimax and the optimization method.

The organization of this paper is as follows. In section 2, we will recall and give some necessary notations and terminologies. Section 3 is first devoted to the portfolio model without transaction costs based on Minimax rule with fuzzy interval numbers. We introduce order relation, and transform fuzzy interval model into a clear number model, then get the interval solution of the model making use of K-T condition. By comparing the optimal solution in this paper with that of the model without fuzzy interval, we can derive that the solution to the model in the setting of fuzzy is much better which means less loss for investors. Secondly, the model was extended to the portfolio model with transaction costs based on Minimax rule with fuzzy interval numbers.

2 Preliminaries

2.1 The Determination of the Security Return Interval

In the emerging stock markets, security return in the future is difficult to predict accurately. If historical data of the securities last for a relatively long period, it can not reflect the importance of the historical data for late period than the mound historical data. If the history data of the securities is not sufficient, it is difficult to make accurate estimates of statistical parameters due to the lack of data. Taking these factors into account, it is positive indeed to consider expected rate of return as an interval number, rather than a clear number. To determine the interval range of the expected rate of return, we have to consider the following three factors:

1. Factor of Arithmetic Average: Arithmetic average of historical security return is defined as factor r_a, it can be calculated by historical date of security.

2. Factor of Historical Trend in Security Return: Factor of historical trend in security return is defined as r_h which reflects changing trend in the security return. It can be represented by arithmetic average of recent historical security return or supplied by experts.

3. Predicting Factor of Future Earnings: Predicting factor of future earnings is defined as r_f, it depends on the financial reports of listed companies and judgments of experts. Due to the three factors r_a, r_h, r_f above, we can define the minimum of the three factors as lower bound for the ranging security return, while the maximum as higher bound. As a result, the ranging interval of expected security return is expressed as $[min\{r_a, r_h, r_f\}, max\{r_a, r_h, r_f\}]$.

2.2 Minimax Rules

Minimax method is to find minimum return in maximum time $[0, T]$ in constraint conditions that average rate of security return exceeds a minimum level with intent to avoid low-income, and achieve a higher rate of return under the most adverse proceeds. Supposed that there are historical dates of n kinds of securities in T periods. r_{jt}: return of security j at time t when a total investment is one; $\bar{r}_j$: average return of security j in $[0, T]$,

x_j: total investment of security j; r_{pt}: return of portfolio in t-th period, $r_{pt} = \sum\limits_{j=1}^{n} x_j r_{jt}$; $\bar{r}_p$: expected return of portfolio, $\bar{r}_p = \frac{1}{T}\sum\limits_{t=1}^{T} r_{pt}$; r_l: the lowest acceptable income of investors. The model in general is as follows:

$$\max_{w} \min_{1\leq t\leq T} \{r_{pt} = \sum_{j=1}^{n} x_j r_{jt}\}$$

$$s.t. \quad \begin{cases} \bar{r}_p = \sum\limits_{j=1}^{n} x_j \bar{r}_j = \frac{1}{T}\sum\limits_{t=1}^{T} r_{pt} \geq r_l, t = 1, 2, \cdots, T \\ \sum\limits_{j=1}^{n} x_j = 1 \\ x_j \geq 0, j = 1, 2, \cdots, n \end{cases}$$

This model is the so-called Minimax investment model by Young[39], and it shows a more conservative approach investor take to avoid the least proceeds. Minimax model does not require return to meet a normal distribution. As long as there is historical data on rates of return, we can get optimal portfolio solution through solving linear programming, avoiding difficult calculations of the $M-V$ model for solving quadratic programming problems. As is well known, in the investment activities, income is always accompanied by risk, here we use the maximum of deviation between the actual portfolio return and the average degree of portfolio return as standard of a risk measurement. That is the risk function is defined as:

$$R_M = \max_{1\leq t\leq T}\{|r_{pt} - \bar{r}_p|\} = \max_{1\leq t\leq T}\{|\sum_{j=1}^{n} x_j r_{jt} - \sum_{j=1}^{n} x_j \bar{r}_j|\}$$

Investors are also risk averse, in the hope of achieving maximum portfolio return with smallest investment risk in each period which means portfolio return balance each time without of too much fluctuations.

2.3 Interval Numbers

Interval number is a special class of fuzzy numbers, and a powerful tool to deal with uncertainty. Particularly, there is not enough data to get a valid probability density in real life, then the interval method is more applicable. The following outlines the basic theory of interval (see [35, 43]).

Definition 2.3.1 *Set $a = [\underline{a}, \bar{a}]$ as a bounded closed interval, if $\underline{a} \leq \bar{a}$, and $\underline{a}, \bar{a} \in \mathbb{R}$, then we call a=$[\underline{a}, \bar{a}]$ as an interval number, where $\underline{a}, \bar{a}$ are the lower bound and the upper bound, respectively.*

Definition 2.3.2 *Rules of interval operator:*
$a \pm b = [\underline{a} \pm \underline{b}, \bar{a} \pm \bar{b}], a \pm k = [\underline{a} \pm k, \bar{a} \pm k]$;
$ka = [k\underline{a}, k\bar{a}]$, *if* $k \geq 0$, *or* $ka = [k\bar{a}, k\underline{a}]$, *if* $k < 0$, *where* $k \in \mathbb{R}$.

Definition 2.3.3 *For arbitrary two given intervals* $a = [\underline{a}, \bar{a}], b = [\underline{b}, \bar{b}]$, *there exists order relation* $\preceq_1$ *between them:*

$$a \preceq_1 b \text{ if and only if } \underline{a} \leq \underline{b}, \frac{\underline{a}+\bar{a}}{2} \leq \frac{\underline{b}+\bar{b}}{2}.$$

Definition 2.3.4 *For arbitrary two given intervals* $a = [\underline{a}, \bar{a}], b = [\underline{b}, \bar{b}]$, *there exists order relation* $\preceq_2$ *between them:*

$$a \preceq_2 b \text{ if and only if } \bar{a} \leq \bar{b}, \frac{\underline{a}+\bar{a}}{2} \leq \frac{\underline{b}+\bar{b}}{2}.$$

3 Portfolio Selection Problems Based on Fuzzy Interval Numbers Under Minimax Rules

3.1 Minimax Rules Based on Fuzzy Interval Numbers

Suppose that the market is allowed short selling, rational investors invest in n kinds of selected risky securities $S_i(i = 1, 2, \cdots, n)$, where the expected rate of security returns is uncertain. Investors also have historical rates of return of the securities in the T periods, so we will be able to know the factor of historical trend in security return. We can derive the Corresponding scope of security return with the method described in Subsection 2.1 by analyzing the three factors: factor of arithmetic average, factor of historical trend in security return, predicting factor of future earnings. In order to study the issue of convenience, then introduce the following notation: M_0: Initial wealth, x_i: funds invested in securities S_i, and the feasible region of optimization problems as $L = \{x = (x_1, x_2, \cdots, x_n) : \sum_{i=1}^{n} = M_0\}$, r_{it}: the rate of security return in t period, $r_{ai} = \frac{1}{T} \sum_{t=1}^{T} r_{it}$: the arithmetic average of security returns. Denote $\tilde{r}_i = [\underline{r}_i, \bar{r}_i] = [\min\{r_{ai}, r_{hi}, r_{fi}\}, \max\{r_{ai}, r_{hi}, r_{fi}\}]$ by the expected rate of return security S_i, and by a fuzzy interval number.

Let $q_i = E(|r_{it} - \tilde{r}_i|)$, that is q_i stands for the absolute deviation between r_{it} and the expectation $\tilde{r}_i$, then the interval number of q_i is $q_i = [\underline{q_i}, \bar{q}_i] = [E(|r_{it} - \underline{r_i}|), E(|r_{it} - \bar{r}_i|)]$, then $q_i x_i = [\underline{q_i} x_i, \bar{q}_i x_i] = [E(|r_{it} - \underline{r_i}|)x_i, E(|r_{it} - \bar{r}_i|)x_i]$.

Let $x \in L$, and set $\tilde{l_\infty}$ risk function as $\tilde{w}_\infty = \max_{1\leq i\leq n} E(|r_{it}x_i - \tilde{r}_i x_i|) = \max_{1\leq i\leq n} q_i x_i$.

If $E(|r_{it} - \underline{r}_i|) > E(|r_{it} - \bar{r}_i|)$, then $\underline{\tilde{w}_\infty} = \max_{1\leq i\leq n} E(|r_{it}x_i - \underline{r}_i x_i|) = \max_{1\leq i\leq n} \underline{q}_i x_i$.

If $E(|r_{it} - \underline{r}_i|) < E(|r_{it} - \bar{r}_i|)$, then $\overline{\tilde{w}_\infty} = \max_{1\leq i\leq n} E(|r_{it}x_i - \bar{r}_i x_i|) = \max_{1\leq i\leq n} \bar{q}_i x_i$.

So $\tilde{w}_\infty$ also need to be considered as an interval number for the unknown specific size of $E(|r_{it} - \underline{r}_i|)$ and $E(|r_{it} - \bar{r}_i|)$, and the risk function $\tilde{w}_\infty$ can be represented as $[\max\limits_{1 \le i \le n} E(|r_{it} - \underline{r}_i|)x_i, \max\limits_{1 \le i \le n} E(|r_{it} - \bar{r}_i|)x_i]$.

3.2 Fuzzy Security Portfolio Models Without Transaction Costs

3.2.1 Establishment of Security Portfolio Models Excluding Transaction Costs

We always believe that investors wish to maximize the expected profit with the minimized risk. However, these two objectives are conflicting. As risk measure fluctuates in a range and is supposed to be transformed in the solution process. Thus, using the risk measure of $\tilde{l_\infty}$, a portfolio optimization problem will be expressed as the following bi-objective linear programming problem:

$$\begin{aligned} &\min(\max_{1 \le i \le n} q_i x_i, -\sum_{i=1}^{n} r_{it} x_i), \\ &s.t.\ x \in L \end{aligned} \tag{3.2.1}$$

3.2.2 Transaction of Security Portfolio Models Without Transaction Costs

From (3.2.1), we find that the objective function of linear programming changes in a interval, in this case, how can we deal with it? As Definition of the order relation of interval numbers is mentioned earlier, we apply it into the model in order to facilitate solution. Consider allowing short selling in this situation. By a simple transformation, (3.2.1) turns into

$$\begin{aligned} &\min(y, -\sum_{i=1}^{n} r_{it} x_i), \\ &s.t.\ q_i |x_i| \le y = [\underline{y}, \bar{y}], i = 1, 2, \cdots, n \\ &x \in L \end{aligned} \tag{3.2.2}$$

(3.2.2) is an optimization problem with interval coefficient, some explicit optimization algorithms can not directly be used. Therefore, we need to transform it into a clear number linear programming problem. We introduce order relation $\preceq_1$ in Definition 2.3.3 into the objective function of (3.2.2), then turn (3.2.2) into the following question:

$$\begin{aligned} &\min_{\preceq_1}(y, -\sum_{i=1} r_{it} x_i), \\ &s.t.\ q_i |x_i| \preceq_1 y, i = 1, 2, \cdots, n \\ &x \in L \end{aligned} \tag{3.2.3}$$

According to the order relation so we can get the following questions:

$$\min(\bar{y}, -\sum_{i=1} r_{it} x_i),$$

$$\begin{aligned}
&\min(\frac{\underline{y}+\bar{y}}{2}, -\sum_{i=1} r_{it}x_i),\\
&s.t.\ \underline{q}_i|x_i| \le \underline{y}\ ,\\
&\quad \frac{\underline{q}_i|x_i| + \bar{q}_i|x_i|}{2} \le \frac{\underline{y}+\bar{y}}{2},\\
&\quad x \in L,\ \text{and}\ i = 1, 2, \cdots, n
\end{aligned} \tag{3.2.4}$$

According to multi-objective programming theory, (3.2.4) can be translated into parametric programming problems as follows:

$$\begin{aligned}
&\min F_{\lambda_1}(x,y) = \lambda_1\bar{y} + (1-\lambda_1)(-\sum_{i=1} r_{it}x_i),\\
&\min F_{\lambda_2}(x,y) = \lambda_2\frac{\underline{y}+\bar{y}}{2} + (1-\lambda_2)(-\sum_{i=1} r_{it}x_i),\\
&s.t.\ \underline{q}_i|x_i| \le \underline{y}\ ,\\
&\quad \frac{\underline{q}_i|x_i| + \bar{q}_i|x_i|}{2} \le \frac{\underline{y}+\bar{y}}{2},\\
&\quad x \in L,\ \text{and}\ i = 1, 2, \cdots, n
\end{aligned} \tag{3.2.5}$$

By the multi-objective planning theory, we know that (3.2.5) can be settled by solving parametric programming problems as follows:

$$\begin{aligned}
\min F_\lambda(x,y) \quad = \quad & [\lambda\lambda_1 + \frac{(1-\lambda)\lambda_2}{2}]\frac{\bar{y}}{2} + (1-\lambda)\lambda_2\frac{\bar{y}}{2}\\
& + [\lambda(1-\lambda_1) + (1-\lambda_2)](-\sum_{i=1} r_{it}x_i),\\
& s.t.\ \ \underline{q}_i|x_i| \le \underline{y}\ ,\\
& \quad \frac{\underline{q}_i|x_i| + \bar{q}_i|x_i|}{2} \le \frac{\underline{y}+\bar{y}}{2},\\
& \quad \lambda_1 \in (0,1), \lambda_2 \in (0,1), \lambda \in (0,1),\\
& \quad x \in L,\ \text{and}\ i = 1, 2, \cdots, n
\end{aligned} \tag{3.2.6}$$

For simplified (3.2.6), we set $\bar{y}_i^+ = \frac{\bar{q}_i|x_i|+x_i}{2}$, $\bar{y}_i^- = \frac{\bar{q}_i|x_i|-x_i}{2}$, $\underline{y}_i^+ = \frac{\underline{q}_i|x_i|+x_i}{2}$, $\underline{y}_i^- = \frac{\underline{q}_i|x_i|-x_i}{2}$. Then we have $\bar{y}_i^+ + \bar{y}_i^- = \bar{q}_i|x_i|, \bar{y}_i^+ - \bar{y}_i^- = x_i$, $\underline{y}_i^+ + \underline{y}_i^- = \underline{q}_i|x_i|, \underline{y}_i^+ - \underline{y}_i^- = x_i$. Thus, (3.2.6) can also be turned into:

$$\begin{aligned}
\min F_\lambda(x,y) \quad = \quad & [\lambda\lambda_1 + \frac{(1-\lambda)\lambda_2}{2}]\frac{\bar{y}}{2} + (1-\lambda)\lambda_2\frac{\bar{y}}{2}\\
& + [\lambda(1-\lambda_1) + (1-\lambda_2)](-\sum_{i=1} r_{it}x_i),\\
& s.t.\ \ \underline{y}_i^+ + \underline{y}_i^- \le \underline{y}\ ,
\end{aligned} \tag{3.2.7}$$

$$\frac{\underline{y}_i^+ + \underline{y}_i^- + \bar{y}_i^+ + \bar{y}_i^-}{2} \leq \frac{\underline{y} + \bar{y}}{2},$$
$$\lambda_1 \in (0,1), \lambda_2 \in (0,1), \lambda \in (0,1),$$
$$x \in L, \text{ and } i = 1, 2, \cdots, n$$

Based on the above discussion, obviously we can solve interval programming portfolio problem by settling parametric programming problem. By adjusting the value of $\lambda_1, \lambda_2, \lambda \in (0,1)$, we can get a variety of satisfactory solution to interval linear programming problem.

3.2.3 Solutions to Security Portfolio Models Without Transaction Costs

For convenience, we study the solution to (3.2.5),it is not hard to see that (3.2.5) can be separated into two programming problems:

$$\min F_{\lambda_1}(x, y) = \lambda_1 \bar{y} + (1 - \lambda_1)(-\sum_{i=1} r_{it} x_i),$$

$$(3.2.8) \qquad (P_1) \qquad s.t. \begin{cases} \underline{y} - \underline{q}_i x_i \geq 0, x_i \geq 0, \\ \underline{y} + \underline{q}_i x_i < 0, x_i < 0, \\ \dfrac{\underline{y} + \bar{y}}{2} - \dfrac{\underline{q}_i x_i + \bar{q}_i x_i}{2} \geq 0, x_i \geq 0, \\ \dfrac{\underline{y} + \bar{y}}{2} + \dfrac{\underline{q}_i x_i + \bar{q}_i x_i}{2} < 0, x_i < 0, \\ x \in L, \ \lambda_1 \in (0,1), \quad i = 1, 2, \cdots, n \end{cases}$$

$$\min F_{\lambda_2}(x, y) = \lambda_2 \frac{\underline{y} + \bar{y}}{2} + (1 - \lambda_2)(-\sum_{i=1} r_{it} x_i),$$

$$(3.2.9) \qquad (P_2) \qquad s.t. \begin{cases} \underline{y} - \underline{q}_i x_i \geq 0, x_i \geq 0, \\ \underline{y} + \underline{q}_i x_i < 0, x_i < 0, \\ \dfrac{\underline{y} + \bar{y}}{2} - \dfrac{\underline{q}_i x_i + \bar{q}_i x_i}{2} \geq 0, x_i \geq 0, \\ \dfrac{\underline{y} + \bar{y}}{2} + \dfrac{\underline{q}_i x_i + \bar{q}_i x_i}{2} < 0, x_i < 0, \\ x \in L, \ \lambda_2 \in (0,1), \quad i = 1, 2, \cdots, n \end{cases}$$

I) Solution to (P_1): Introduce Langrange function of (3.2.8):

$$\begin{aligned} L(x, \underline{y}, \bar{y}, \lambda_0, \mu\, \nu, \alpha, \beta) &= \lambda_1 \bar{y} + (1 - \lambda_1)(-\sum_{i=1}^n r_{it} x_i) + \lambda_0(\sum_{i=1}^n x_i - M_0) - \sum_{i=1}^n \mu_i(\underline{y} - \underline{q}_i x_i) \\ &- \sum_{i=1}^n \nu_i(\underline{y} + \underline{q}_i x_i) - \sum_{i=1}^n \alpha_i(\frac{\underline{y} + \bar{y}}{2} - \frac{\underline{q}_i x_i + \bar{q}_i x_i}{2}) \end{aligned}$$

$$- \quad \sum_{i=1}^{n} \beta_i(\frac{\underline{y}+\bar{y}}{2} + \frac{\underline{q}_i x_i + \bar{q}_i x_i}{2})$$

Thus, the optimum (x, y) satisfy the $K-T$ conditions as follows:

$$\frac{\partial L}{\partial \underline{y}} = -\sum_{i=1}^{n} \mu_i - \sum_{i=1}^{n} \nu_i - \sum_{i=1}^{n} \frac{\alpha_i}{2} - \sum_{i=1}^{n} \frac{\beta_i}{2} = 0, \tag{3.2.10}$$

$$\frac{\partial L}{\partial \underline{y}} = -\lambda_1 - \sum_{i=1}^{n} \frac{\alpha_i}{2} - \sum_{i=1}^{n} \frac{\beta_i}{2} = 0, \tag{3.2.11}$$

$$\frac{\partial L}{\partial x_i} = -(1-\lambda_1) r_{it} + \lambda_0 + \mu_i \underline{q}_i - \nu_i \underline{q}_i + \alpha_i \cdot \frac{\underline{q}_i + \bar{q}_i}{2} - \beta_i \cdot \frac{\underline{q}_i + \bar{q}_i}{2} = 0, \tag{3.2.12}$$

$$\frac{\partial L}{\partial \lambda_0} = \sum_{i=1}^{n} x_i = M_0, \tag{3.2.13}$$

$$\mu_i(\underline{y} - \underline{q}_i x_i) = 0, i = 1, 2, \cdots, n, \tag{3.2.14}$$

$$\nu_i(\underline{y} + \underline{q}_i x_i) = 0, i = 1, 2, \cdots, n, \tag{3.2.15}$$

$$\alpha_i(\frac{\underline{y}+\bar{y}}{2} - \frac{\underline{q}_i x_i + \bar{q}_i x_i}{2}) = 0, i = 1, 2, \cdots, n, \tag{3.2.16}$$

$$\beta_i(\frac{\underline{y}+\bar{y}}{2} + \frac{\underline{q}_i x_i + \bar{q}_i x_i}{2}) = 0, i = 1, 2, \cdots, n. \tag{3.2.17}$$

Notation: We can see λ_1 as the risk tolerance of investors. Investors will ignore the return items when $\lambda_1 \to 1$ to minimize risk, and ignore the risk item when $\lambda_1 \to 0$ to maximize return. Thus, the risk tolerance of investors becomes smaller when λ_1 becomes bigger. It is easy to see that (3.2.8) is a convex programming problem. Therefore, $K-T$ conditions are necessary and sufficient conditions for the optimal solution. That is (x, y) will be the best solution to (3.2.8) if (x, y) is founded.

Define $G^*(\lambda_1) = \{i : \mu_i \geq o, \nu_i = 0, \alpha_i \geq 0, \beta_i = 0\}$. When $i \notin G^*(\lambda_1)$, one can set $\mu_i = 0, \nu_i \geq 0, \alpha_i = 0, \beta \geq 0$.

When $i \in G^*(\lambda_1)$, $\mu_i \geq 0$, by using (3.2.14), one arrives at $\underline{y} - \underline{q}_i x_i = 0$, so $x_i = \frac{\underline{y}}{\underline{q}_i}$.

When $i \notin G^*(\lambda_1)$, $\nu_i \geq 0$, by using (3.2.15), one arrives at $\underline{y} + \underline{q}_i x_i = 0$, so $x_i = \frac{-\underline{y}}{\underline{q}_i}$.

Thus,

$$x_i = \begin{cases} \frac{\underline{y}}{\underline{q}_i}, i \in G^*(\lambda_1) \\ \frac{-\underline{y}}{\underline{q}_i}, i \notin G^*(\lambda_1) \end{cases} \tag{3.2.18}$$

By using (3.2.13), $\sum_{i=1}^{n} x_i = M_0$, we deduce $\sum_{i \in G^*(\lambda_1)} \frac{\underline{y}}{\underline{q}_i} + \sum_{i \notin G^*(\lambda_1)} \frac{-\underline{y}}{\underline{q}_i} = M_0$. It is obvious that $\underline{y}(\sum_{i \in G^*(\lambda_1)} \frac{1}{\underline{q}_i} + \sum_{i \notin G^*(\lambda_1)} \frac{-1}{\underline{q}_i}) = M_0$. Thus, we get

$$\underline{y} = M_0(\sum_{i \in G^*(\lambda_1)} \frac{1}{\underline{q}_i} - \sum_{i \notin G^*(\lambda_1)} \frac{1}{\underline{q}_i})^{-1} \tag{3.2.19}$$

Similarly, we arrive at

$$\underline{y}+\bar{y}=M_0\Big(\sum_{i\in G^*(\lambda_1)}\frac{1}{\underline{q}_i+\bar{q}_i}-\sum_{i\notin G^*(\lambda_1)}\frac{1}{\underline{q}_i+\bar{q}_i}\Big)^{-1}$$

Therefore, by a direct computation, we know

$$\bar{y}=M_0\Big[\Big(\sum_{i\in G^*(\lambda_1)}\frac{1}{\underline{q}_i+\bar{q}_i}-\sum_{i\notin G^*(\lambda_1)}\frac{1}{\underline{q}_i+\bar{q}_i}\Big)^{-1}-\Big(\sum_{i\in G^*(\lambda_1)}\frac{1}{\underline{q}_i}-\sum_{i\notin G^*(\lambda_1)}\frac{1}{\underline{q}_i}\Big)^{-1}\Big]. \tag{3.2.20}$$

For $i\in G^*(\lambda_1)$, because of (3.2.12), we have

$$-(1-\lambda_1)r_{it}+\lambda_0+\mu_i\underline{q}_i+\alpha_i\cdot\frac{\underline{q}_i+\bar{q}_i}{2}=0, i=1,2,\cdots,n. \tag{3.2.21}$$

For $i\notin G^*(\lambda_1)$, by using (3.2.12) again, we have

$$-(1-\lambda_1)r_{it}+\lambda_0-\nu_i\underline{q}_i-\beta_i\cdot\frac{\underline{q}_i+\bar{q}_i}{2}=0, i=1,2,\cdots,n. \tag{3.2.22}$$

By (3.2.21) and (3.2.22), we get

$$\mu_i=\frac{(1-\lambda_1)r_{it}-\lambda_0-\alpha_i\cdot\frac{\underline{q}_i+\bar{q}_i}{2}}{\underline{q}_i}, i\in G^*(\lambda_1) \tag{3.2.23}$$

$$\alpha_i=\frac{2[(1-\lambda_1)r_{it}-\lambda_0-\mu_i\underline{q}_i]}{\underline{q}_i+\bar{q}_i}, i\in G^*(\lambda_1) \tag{3.2.24}$$

$$\nu_i=\frac{-(1-\lambda_1)r_{it}+\lambda_0-\beta_i\cdot\frac{\underline{q}_i+\bar{q}_i}{2}}{\underline{q}_i}, i\notin G^*(\lambda_1) \tag{3.2.25}$$

$$\beta_i=\frac{2[-(1-\lambda_1)r_{it}+\lambda_0-\nu_i\underline{q}_i]}{\underline{q}_i+\bar{q}_i}, i\notin G^*(\lambda_1) \tag{3.2.26}$$

Substituting (3.2.23), (3.2.24), (3.2.25) and (3.2.26) into (3.2.10), then we get

$$\begin{aligned}&\sum_{i\in G^*(\lambda_1)}\frac{(1-\lambda_1)r_{it}-\lambda_0-\alpha_i\cdot\frac{\underline{q}_i+\bar{q}_i}{2}}{\underline{q}_i}+\sum_{i\in G^*(\lambda_1)}\frac{2[(1-\lambda_1)r_{it}-\lambda_0-\mu_i\underline{q}_i]}{\underline{q}_i+\bar{q}_i}\\&+\sum_{i\notin G^*(\lambda_1)}\frac{-(1-\lambda_1)r_{it}+\lambda_0-\beta_i\cdot\frac{\underline{q}_i+\bar{q}_i}{2}}{\underline{q}_i}+\sum_{i\notin G^*(\lambda_1)}\frac{2[-(1-\lambda_1)r_{it}+\lambda_0-\nu_i\underline{q}_i]}{\underline{q}_i+\bar{q}_i}=0.\end{aligned}$$

Thus, one can arrive at

$$\lambda_0 \;=\; \Big[(1-\lambda_1)\Big(\sum_{i\in G^*(\lambda_1)}\frac{r_{it}}{\underline{q}_i}-\sum_{i\notin G^*(\lambda_1)}\frac{r_{it}}{\underline{q}_i}\Big) \tag{3.2.27}$$

$$
\begin{aligned}
&+\quad 2(1-\lambda_1)(\sum_{i\in G^*(\lambda_1)}\frac{r_{it}}{\underline{q}_i+\bar{q}_i}-\sum_{i\notin G^*(\lambda_1)}\frac{r_{it}}{\underline{q}_i+\bar{q}_i})-\sum_{i\in G^*(\lambda_1)}\frac{\alpha_i\cdot\frac{\underline{q}_i+\bar{q}_i}{2}}{\underline{q}_i}\\
&-\quad \sum_{i\notin G^*(\lambda_1)}\frac{\beta_i\cdot\frac{\underline{q}_i+\bar{q}_i}{2}}{\underline{q}_i}-\sum_{i\in G^*(\lambda_1)}\frac{\mu_i\underline{q}_i}{\underline{q}_i+\bar{q}_i}-\sum_{i\notin G^*(\lambda_1)}\frac{\nu_i\underline{q}_i}{\underline{q}_i+\bar{q}_i}]\\
&\times\quad (\sum_{i\in G^*(\lambda_1)}\frac{1}{\underline{q}_i}-\sum_{i\notin G^*(\lambda_1)}\frac{1}{\underline{q}_i})^{-1}(\sum_{i\in G^*(\lambda_1)}\frac{1}{\underline{q}_i+\bar{q}_1}-\sum_{i\notin G^*(\lambda_1)}\frac{1}{\underline{q}_i+\bar{q}})^{-1}.
\end{aligned}
$$

For convenience, we denote by

$$
\begin{aligned}
A &\doteq \sum_{i\in G^*(\lambda_1)}\frac{r_{it}}{\underline{q}_i}-\sum_{i\notin G^*(\lambda_1)}\frac{r_{it}}{\underline{q}_i},\\
B &\doteq \sum_{i\in G^*(\lambda_1)}\frac{r_{it}}{\underline{q}_i+\bar{q}_i}-\sum_{i\notin G^*(\lambda_1)}\frac{r_{it}}{\underline{q}_i+\bar{q}_i}),\\
C &\doteq (\sum_{i\in G^*(\lambda_1)}\frac{1}{\underline{q}_i}-\sum_{i\notin G^*(\lambda_1)}\frac{1}{\underline{q}_i})^{-1}(\sum_{i\in G^*(\lambda_1)}\frac{1}{\underline{q}_i+\bar{q}_1}-\sum_{i\notin G^*(\lambda_1)}\frac{1}{\underline{q}_i+\bar{q}})^{-1},\\
D &\doteq -\sum_{i\in G^*(\lambda_1)}\frac{\alpha_i\cdot\frac{\underline{q}_i+\bar{q}_i}{2}}{\underline{q}_i}-\sum_{i\notin G^*(\lambda_1)}\frac{\beta_i\cdot\frac{\underline{q}_i+\bar{q}_i}{2}}{\underline{q}_i}-\sum_{i\in G^*(\lambda_1)}\frac{\mu_i\underline{q}_i}{\underline{q}_i+\bar{q}_i}-\sum_{i\notin G^*(\lambda_1)}\frac{\nu_i\underline{q}_i}{\underline{q}_i+\bar{q}_i}.
\end{aligned}
$$

Then, we know

$$\mu_i=\frac{(1-\lambda_1)[r_{it}-(A+2B)C]}{\underline{q}_i}+\frac{D}{\underline{q}_i}\cdot C+\frac{\alpha_i\cdot\frac{\underline{q}_i+\bar{q}_i}{2}}{\underline{q}_i} \tag{3.2.28}$$

$$\nu_i=\frac{-(1-\lambda_1)[r_{it}-(A-2B)C]}{\underline{q}_i}+\frac{D}{\underline{q}_i}\cdot C-\frac{\beta_i\cdot\frac{\underline{q}_i+\bar{q}_i}{2}}{\underline{q}_i} \tag{3.2.29}$$

$$\alpha_i=\frac{2(1-\lambda_1)[r_{it}-(A+2B)C]}{\underline{q}_i+\bar{q}_i}-\frac{2D}{\underline{q}_i+\bar{q}_i}\cdot C+\frac{2\mu_i\underline{q}_i}{\underline{q}_i+\bar{q}_1} \tag{3.2.30}$$

$$\beta_i=\frac{-2(1-\lambda_1)[r_{it}-(A-2B)C]}{\underline{q}_i+\bar{q}_i}+\frac{2D}{\underline{q}_i+\bar{q}_i}\cdot C-\frac{2\nu_i\underline{q}_i}{\underline{q}_i+\bar{q}_i}. \tag{3.2.31}$$

Obviously, if we can determine the set of $G^*(\lambda_1)$ such that $\mu_i,\nu_i,\alpha_i,\beta_i$ being non-negative, then we can derive out the form of solutions to question P_1 in terms of y given by (3.2.18), (3.2.19), (3.2.20). In this setting, we need to consider risks as the following: In order to $\sum\limits_{i\in G^*(\lambda_1)}\frac{1}{\underline{q}_i}-\sum\limits_{i\notin G^*(\lambda_1)}\frac{1}{\underline{q}_i}>0$ and $(\sum\limits_{i\in G^*(\lambda_1)}\frac{1}{\underline{q}_i+\bar{q}_i}-\sum\limits_{i\notin G^*(\lambda_1)}\frac{1}{\underline{q}_i+\bar{q}_i})^{-1}>(\sum\limits_{i\in G^*(\lambda_1)}\frac{1}{\underline{q}_i}-\sum\limits_{i\notin G^*(\lambda_1)}\frac{1}{\underline{q}_i})^{-1}$.

Let $k_1 = \max\left\{l_1 : \sum\limits_{i=l_1+1}^{n} \frac{1}{\underline{q}_i} - \sum\limits_{i=l}^{l_1} \frac{1}{\underline{q}_i} > 0 \text{ and } (\sum\limits_{i=l_1+1}^{n} \frac{1}{\underline{q}_i+\bar{q}_i} - \sum\limits_{i=1}^{l_1} \frac{1}{\underline{q}_i+\bar{q}_i})^{-1} > (\sum\limits_{i=l_1}^{n} \frac{1}{\underline{q}_i} - \sum\limits_{i=1}^{l_1} \frac{1}{\underline{q}_i})^{-1}\right\}$. For $1 \leq l_1 \leq k_1+1$, we take $G^* = \{n, n-1, \cdots, l_1+1\}$ such that $\mu_i, \nu_i, \alpha_i, \beta_i$ being non-negative, and then we get

(1) For $l_1 = 1$, then

$$\frac{(1-\lambda_1)[r_{it} - (A+2B)]}{\underline{q}_i} + \frac{\underline{q}_i + \bar{q}_i}{2\underline{q}_i} \geq 0.$$

$$\frac{1}{\lambda_1 - 1} \geq \frac{r_{it} - \sum\limits_{i=2}^{n} \frac{r_{it}-r_1}{\underline{q}_i} + 2\sum\limits_{i=2}^{n} \frac{r_{it}-r_1}{\underline{q}_i+\bar{q}_i}}{\frac{\underline{q}_i+\bar{q}_i}{2}}$$

(2) For $2 \leq l_1 \leq k+1$, then

$$\begin{aligned} & \frac{r_{it} - (\sum\limits_{i=l_1+1}^{n} \frac{r_{it}-r_{l_1}}{\underline{q}_i} + \sum\limits_{i=1}^{l_1} \frac{r_{l_1}-r_{it}}{\underline{q}_i}) + 2(\sum\limits_{i=l_1+1}^{n} \frac{r_{it}-r_{l_1}}{\underline{q}_i+\bar{q}_i} + \sum\limits_{i=1}^{l_1} \frac{r_{l_1}-r_{it}}{\underline{q}_i+\bar{q}_i})}{\frac{\underline{q}_i+\bar{q}_i}{2}} \\ \leq \;& \frac{1}{\lambda_1 - 1} \leq \frac{r_{it} - (\sum\limits_{i=l_1}^{n} \frac{r_{it}-r_{l_1-1}}{\underline{q}_i} + \sum\limits_{i=1}^{l_1-1} \frac{r_{l_1-1}-r_{it}}{\underline{q}_i}) + 2(\sum\limits_{i=l_1}^{n} \frac{r_{it}-r_{l_1-1}}{\underline{q}_i+\bar{q}_i} + \sum\limits_{i=1}^{l_1-1} \frac{r_{l_1-1}-r_{it}}{\underline{q}_i+\bar{q}_i})}{\frac{\underline{q}_i+\bar{q}_i}{2}}. \end{aligned}$$

Thus, the solution to P_1 is as follows:

(i) When $\frac{1}{\lambda_1-1} \geq \frac{r_{it} - \sum\limits_{i=2}^{n} \frac{r_{it}-r_1}{\underline{q}_i} + 2\sum\limits_{i=2}^{n} \frac{r_{it}-r_1}{\underline{q}_i+\bar{q}_i}}{\frac{\underline{q}_i+\bar{q}_i}{2}}$, then there holds

$$\underline{y} = \frac{M_0}{\sum\limits_{i=1}^{n} \frac{1}{\underline{q}_i}}, x_i = \frac{\underline{y}}{\underline{q}_i}, 1 \leq i \leq n; \tag{3.2.32}$$

(ii) When

$$\begin{aligned} & \frac{r_{it} - (\sum\limits_{i=l_1+1}^{n} \frac{r_{it}-r_{l_1}}{\underline{q}_i} + \sum\limits_{i=1}^{l_1} \frac{r_{l_1}-r_{it}}{\underline{q}_i}) + 2(\sum\limits_{i=l_1+1}^{n} \frac{r_{it}-r_{l_1}}{\underline{q}_i+\bar{q}_i} + \sum\limits_{i=1}^{l_1} \frac{r_{l_1}-r_{it}}{\underline{q}_i+\bar{q}_i})}{\frac{\underline{q}_i+\bar{q}_i}{2}} \\ \leq \;& \frac{1}{\lambda_1 - 1} \leq \frac{r_{it} - (\sum\limits_{i=l_1}^{n} \frac{r_{it}-r_{l_1-1}}{\underline{q}_i} + \sum\limits_{i=1}^{l_1-1} \frac{r_{l_1-1}-r_{it}}{\underline{q}_i}) + 2(\sum\limits_{i=l_1}^{n} \frac{r_{it}-r_{l_1-1}}{\underline{q}_i+\bar{q}_i} + \sum\limits_{i=1}^{l_1-1} \frac{r_{l_1-1}-r_{it}}{\underline{q}_i+\bar{q}_i})}{\frac{\underline{q}_i+\bar{q}_i}{2}}. \end{aligned}$$

Then, it holds

$$\underline{y} = M_0(\sum_{i=l_1+1}^{n} \frac{1}{\underline{q}_i} - \sum_{i=1}^{l_1-1} \frac{1}{\underline{q}_i})^{-1}.$$

$$\bar{y} = M_0[(\sum_{i=l_1+1}^{n} \frac{1}{\underline{q}_i + \bar{q}_i} - \sum_{i=1}^{l_1-1} \frac{1}{\underline{q}_i + \bar{q}_i})^{-1} - (\sum_{i=l_1+1}^{n} \frac{1}{\underline{q}_i} - \sum_{i=1}^{l_1-1} \frac{1}{\underline{q}_i})^{-1}] \tag{3.2.33}$$

$$x_i = \begin{cases} \frac{M_0}{\underline{q}_i}(\sum\limits_{i=l_1+1}^{n} \frac{1}{\underline{q}_i} - \sum\limits_{i=1}^{l_1-1} \frac{1}{\underline{q}_i})^{-1}, 1 \le i \le l_1; \\ -\frac{M_0}{\underline{q}_i}(\sum\limits_{i=l_1+1}^{n} \frac{1}{\underline{q}_i} - \sum\limits_{i=1}^{l_1-1} \frac{1}{\underline{q}_i})^{-1}, l_1 + 1 \le i \le n. \end{cases}$$

II) Solution to P_2: Introduce Langrange functions of (3.2.9):

$$\begin{aligned} L(x, \underline{y}, \bar{y}, \lambda_0, \omega, \varphi, \gamma, \eta) &= \lambda_2 \cdot \frac{\underline{y} + \bar{y}}{2} + (1 - \lambda_2)(-\sum_{i=1}^{n} r_{it} x_i) + \lambda_0(\sum_{i=1}^{n} x_i - M_0) \\ &- \sum_{i=1}^{n} \omega_i(\underline{y} - \underline{q}_i x_i) - \sum_{i=1}^{n} \varphi_i(\underline{y} + \underline{q} x_i) \\ &- \sum_{i=1}^{n} \gamma_i(\frac{\underline{y} + \bar{y}}{2} - \frac{\underline{q}_i x_i + \bar{q}_i x_i}{2}) - \sum_{i=i}^{n} \eta_i(\frac{\underline{y} + \bar{y}}{2} + \frac{\underline{q}_i x_i + \bar{q}_i x_i}{2}) \end{aligned}$$

Thus, the optimum (x, y) satisfy the $K - T$ conditions:

$$\frac{\partial L}{\partial \underline{y}} = \frac{\lambda_2}{2} - \sum_{i=1}^{n} \omega_i - \sum_{i=1}^{n} \varphi_i - \sum_{i=1}^{n} \frac{\gamma_i}{2} - \sum_{i=1}^{n} \frac{\eta_i}{2} = 0, \tag{3.2.34}$$

$$\frac{\partial L}{\partial \bar{y}} = \frac{\lambda_2}{2} - \sum_{i=1}^{n} \frac{\gamma_i}{2} - \sum_{i=1}^{n} \frac{\eta_i}{2} = 0, \tag{3.2.35}$$

$$\frac{\partial L}{\partial x_i} = -(1 - \lambda_2) r_{it} + \lambda_0 + \omega_i \underline{q}_i - \varphi_i \underline{q}_i + \gamma_i \cdot \frac{\underline{q}_i + \bar{q}_i}{2} - \eta_i \cdot \frac{\underline{q}_i + \bar{q}_i}{2} = 0, \tag{3.2.36}$$

$$\frac{\partial L}{\partial \lambda_0} = \sum_{i=1}^{n} x_i = M_0, \tag{3.2.37}$$

$$\omega_i(\underline{y} - \underline{q}_i x_i) = 0, i = 1, 2, \cdots, n, \tag{3.2.38}$$

$$\varphi_i(\underline{y} + \underline{q}_i x_i) = 0, i = 1, 2, \cdots, n, \tag{3.2.39}$$

$$\gamma_i(\frac{\underline{y} + \bar{y}}{2} - \frac{\underline{q}_i x_i + \bar{q}_i x_i}{2}) = 0, i = 1, 2, \cdots, n, \tag{3.2.40}$$

$$\eta_i(\frac{\underline{y} + \bar{y}}{2} + \frac{\underline{q}_i x_i + \bar{q}_i x_i}{2}) = 0, i = 1, 2, \cdots, n. \tag{3.2.41}$$

It is easy to see that (3.2.9) is a convex programming problem. Therefore, $K-T$ conditions are necessary and sufficient conditions for the optimal solution. That is (x, y) will be the best solution to (3.2.9) if (x, y) is founded. Similar to P_1, we can define $H^*(\lambda_2)$ as follows.

Define $H^*(\lambda_2) = \{i : \omega_i \geq 0, \varphi_i = 0, \gamma_i \geq 0, \eta_i = 0\}$. When $i \notin H^*(\lambda_2)$, set $\omega_i = 0, \varphi_i \geq 0, \gamma_i = 0, \eta \geq 0$. When $i \in H^*(\lambda_2)$, $\gamma_i \geq 0$, by suing (3.2.40), we know $\frac{\underline{y}+\bar{y}}{2} - \frac{\underline{q}_i x_i + \bar{q}_i x_i}{2} = 0$, that is, $x_i = \frac{\underline{y}+\bar{y}}{\underline{q}_i+\bar{q}_i}$; When $i \notin H^*(\lambda_2)$, $\eta_i \geq 0$, by suing (3.2.41), we know $\frac{\underline{y}+\bar{y}}{2} + \frac{\underline{q}_i x_i + \bar{q}_i x_i}{2} = 0$, that is, $x_i = -\frac{\underline{y}+\bar{y}}{\underline{q}_i+\bar{q}_i}$. This implies that

$$x_i = \begin{cases} \frac{\underline{y}+\bar{y}}{\underline{q}_i+\bar{q}_i}, i \in H^*(\lambda_2), \\ -\frac{\underline{y}+\bar{y}}{\underline{q}_i+\bar{q}_i}, i \notin H^*(\lambda_2) \end{cases} \tag{3.2.42}$$

Because of (3.2.37), we can derive that

$$\underline{y} = M_0 \Big(\sum_{i \in H^*(\lambda_2)}^{n} \frac{1}{\underline{q}_i} - \sum_{i \notin H^*(\lambda_2)}^{n} \frac{1}{\underline{q}_i} \Big)^{-1} \tag{3.2.43}$$

By a direct computation, we get

$$\underline{y} + \bar{y} = M_0 \Big(\sum_{i \in H^*(\lambda_2)} \frac{1}{\underline{q}_i + \bar{q}_i} - \sum_{i \notin H^*(\lambda_2)} \frac{1}{\underline{q}_i + \bar{q}_i} \Big)^{-1}.$$

This implies that

$$\bar{y} = M_0 \Big[\Big(\sum_{i \in H^*(\lambda_2)} \frac{1}{\underline{q}_i + \bar{q}_i} - \sum_{i \notin H^*(\lambda_2)} \frac{1}{\underline{q}_i + \bar{q}_i} \Big)^{-1} - \Big(\sum_{i \in H^*(\lambda_2)} \frac{1}{\underline{q}_i} - \sum_{i \notin H^*(\lambda_2)} \frac{1}{\underline{q}_i} \Big)^{-1} \Big] \tag{3.2.44}$$

For $i \in H^*(\lambda_2)$, because of (3.2.36), one has

$$-(1-\lambda_2) r_{it} + \lambda_0 + \omega_i \underline{q}_i + \gamma_i \cdot \frac{\underline{q}_i + \bar{q}_i}{2} = 0, i = 1, 2, \cdots, n. \tag{3.2.45}$$

For $i \notin H^*(\lambda_2)$, by suing (3.2.36) again, one also has

$$-(1-\lambda_2) r_{it} + \lambda_0 - \omega_i \underline{q}_i - \eta_i \cdot \frac{\underline{q}_i + \bar{q}_i}{2} = 0, i = 1, 2, \cdots, n. \tag{3.2.46}$$

According to (3.2.45) and (3.2.46), we deduce

$$\omega_i = \frac{(1-\lambda_2) r_{it} - \lambda_0 - \gamma_i \cdot \frac{\underline{q}_i + \bar{q}_i}{2}}{\underline{q}_i}, i \in H^*(\lambda_2), \tag{3.2.47}$$

$$\gamma_i = \frac{2[(1-\lambda_2) r_{it} - \lambda_0 - \omega_i \underline{q}_i]}{\underline{q}_i + \bar{q}_i}, i \in H^*(\lambda_2), \tag{3.2.48}$$

$$\varphi_i = \frac{-(1-\lambda_2) r_{it} + \lambda_0 - \eta_i \cdot \frac{\underline{q}_i + \bar{q}_i}{2}}{\underline{q}_i}, i \notin H^*(\lambda_2), \tag{3.2.49}$$

$$\eta_i = \frac{2[-(1-\lambda_2) r_{it} + \lambda_0 - \varphi_i \underline{q}_i]}{\underline{q}_i + \bar{q}_i}, i \notin H^*(\lambda_2), \tag{3.2.50}$$

Substituting (3.2.45),(3.2.46),(3.2.47) and (3.2.48) into (3.2.33), then there holds

$$\frac{\lambda_2}{2} = \sum_{i\in H^*(\lambda_2)} \frac{(1-\lambda_2)r_{it} - \lambda_0 - \omega_i \underline{q}_i}{\underline{q}_i + \bar{q}_i} + \sum_{i\notin H^*(\lambda_2)} \frac{-(1-\lambda_2)r_{it} + \lambda_0 - \varphi_i \underline{q}_i}{\underline{q}_i + \bar{q}_i}. \tag{3.2.51}$$

For convenience, we put

$$\bar{A} \doteqdot \sum_{i\in H^*(\lambda_2)} \frac{r_{it}}{\underline{q}_i + \bar{q}_i} - \sum_{i\notin H^*(\lambda_2)} \frac{r_{it}}{\underline{q}_i + \bar{q}_i}$$

$$\bar{B} \doteqdot \sum_{i\in H^*(\lambda_2)} \frac{\omega_i \underline{q}_i}{\underline{q}_i + \bar{q}_i} + \sum_{i\notin H^*(\lambda_2)} \frac{\varphi_i \underline{q}_i}{\underline{q}_i + \bar{q}_i}$$

$$\bar{C} \doteqdot (\sum_{i\in H^*(\lambda_2)} \frac{1}{\underline{q}_i + \bar{q}_i} - \sum_{i\notin H^*(\lambda_2)} \frac{1}{\underline{q}_i + \bar{q}_i})^{-1}.$$

Thus, we know

$$\lambda_0 = [(1-\lambda_2)\bar{A} - \frac{\lambda_2}{2} - \bar{B}] \cdot \bar{C}$$

Then, one arrives at

$$\omega_i = \frac{(1-\lambda_2)[r_{it} - \bar{A}\cdot\bar{C}]}{\underline{q}_i} + \frac{\frac{\lambda_2}{2} + \bar{B}}{\underline{q}_i} \cdot \bar{C} - \frac{\gamma_i \cdot \frac{\underline{q}_i + \bar{q}_i}{2}}{\underline{q}_i}. \tag{3.2.52}$$

$$\varphi_i = \frac{-(1-\lambda_2)[r_{it} - \bar{A}\cdot\bar{C}]}{\underline{q}_i} - \frac{\frac{\lambda_2}{2} + \bar{B}}{\underline{q}_i} \cdot \bar{C} - \frac{\eta_i \cdot \frac{\underline{q}_i + \bar{q}_i}{2}}{\underline{q}_i}. \tag{3.2.53}$$

$$\gamma_i = \frac{2(1-\lambda_2)[r_{it} - \bar{A}\cdot\bar{C}]}{\underline{q}_i + \bar{q}_i} - \frac{2(\frac{\lambda_2}{2} + \bar{B})}{\underline{q}_i + \bar{q}_i} \cdot \bar{C} + \frac{2\omega_i \underline{q}_i}{\underline{q}_i + \bar{q}_i}. \tag{3.2.54}$$

$$\eta_i = \frac{-2(1-\lambda_2)[r_{it} - \bar{A}\cdot\bar{C}]}{\underline{q}_i + \bar{q}_i} - 2\frac{\frac{\lambda_2}{2} + \bar{B}}{\underline{q}_i + \bar{q}_i} \cdot \bar{C} - \frac{2\varphi_i \underline{q}_i}{\underline{q}_i + \bar{q}_i}. \tag{3.2.55}$$

We need to take the situation when the risk is always positive into account, that is $y > 0$. With (3.2.43) and (3.2.44), we make

$$\sum_{i\in H^*(\lambda_2)} \frac{1}{\underline{q}_i} - \sum_{i\notin H^*(\lambda_2)} \frac{1}{\underline{q}_i} > 0,$$

and

$$(\sum_{i\in H^*(\lambda_2)} \frac{1}{\underline{q}_i + \bar{q}_i} - \sum_{i\notin H^*(\lambda_2)} \frac{1}{\underline{q}_i + \bar{q}_i})^{-1} > (\sum_{i\in H^*(\lambda_2)} \frac{1}{\underline{q}_i} - \sum_{i\notin H^*(\lambda_2)} \frac{1}{\underline{q}_i})^{-1}.$$

Suppose $k_1 = \max\{l_2 : \sum\limits_{i=l_2+1}^{n} \frac{1}{\underline{q}_i} - \sum\limits_{i=l}^{l_2} \frac{1}{\underline{q}_i} > 0, (\sum\limits_{i=l_2+1}^{n} \frac{1}{\underline{q}_i+\bar{q}_i} - \sum\limits_{i=l}^{l_2} \frac{1}{\underline{q}_i+\bar{q}_i})^{-1} > (\sum\limits_{i=l_2+1}^{n} \frac{1}{\underline{q}_i} - \sum\limits_{i=l}^{l_2} \frac{1}{\underline{q}_i})^{-1}\}$. For $1 \leq l_2 \leq k_2+1$, we take $H^* = \{n, n-1, \cdots, l_2+1\}$ such that $\omega_i, \varphi_i, \gamma_i, \eta_i$ being non-negative, and then one can get:

(1) For $l_2 = 1$, then

$$\frac{(1-\lambda_2)(r_{it}-\bar{A})}{\underline{q}_i} + \frac{\frac{\lambda_2}{2} + \sum\limits_{i \in H^*(\lambda_2)} \frac{\underline{q}_i}{\underline{q}_i+\bar{q}_i}}{\underline{q}_i} - \frac{\underline{q}_i+\bar{q}_i}{2\underline{q}_i} \geq 0,$$

$$r_{it} - \sum_{i=1}^{n} \frac{r_{it}-r_1}{\underline{q}_i+\bar{q}_i} - \lambda_2 r_{it} - \lambda_2 \cdot \sum_{i=1}^{n} \frac{r_{it}-r_1}{\underline{q}_i+\bar{q}_i} + \frac{\lambda_2}{2} + \sum_{i=1}^{n} \frac{\underline{q}_i}{\underline{q}_i+\bar{q}_i} - \frac{\underline{q}_i+\bar{q}_i}{2} \geq 0,$$

$$\lambda_2 \leq \frac{r_{it} - \sum\limits_{i=1}^{n} \frac{r_{it}-r_1}{\underline{q}_i+\bar{q}_i} + \sum\limits_{i=1}^{n} \frac{\underline{q}_i}{\underline{q}_i+\bar{q}_i} - \frac{\underline{q}_i+\bar{q}_i}{2}}{r_{it} + \sum\limits_{i=1}^{n} \frac{r_{it}-r_1}{\underline{q}_i+\bar{q}_i} - \frac{1}{2}},$$

$$\frac{1}{\lambda_2-1} \geq \frac{r_{it} + \sum\limits_{i=1}^{n} \frac{r_{it}-r_1}{\underline{q}_i+\bar{q}_i} - \frac{1}{2}}{-2\sum\limits_{i=1}^{n} \frac{r_{it}-r_1}{\underline{q}_i+\bar{q}_i} + \sum\limits_{i=1}^{n} \frac{\underline{q}_i}{\underline{q}_i+\bar{q}_i} - \frac{\underline{q}_i+\bar{q}_i}{2} + \frac{1}{2}}.$$

(2) For $2 \leq l_2 \leq k_2+1$, then

$$\begin{aligned} &\frac{r_{it} + \sum\limits_{i=l_2+1}^{n} \frac{r_{it}-r_{l_2}}{\underline{q}_i+\bar{q}_i} + \sum\limits_{i=1}^{l_2} \frac{r_{l_2}-r_{it}}{\underline{q}_i+\bar{q}_i} - \frac{1}{2}}{-2(\sum\limits_{i=l_2+1}^{n} \frac{r_{it}-r_{l_2}}{\underline{q}_i+\bar{q}_i} + \sum\limits_{i=1}^{l_2} \frac{r_{l_2}-r_{it}}{\underline{q}_i+\bar{q}_i}) + \sum\limits_{i=1}^{n} \frac{\underline{q}_i}{\underline{q}_i+\bar{q}_i} - \frac{\underline{q}_i+\bar{q}_i}{2} + \frac{1}{2}} \\ &\leq \frac{1}{\lambda_2-1} \leq \frac{r_{it} + \sum\limits_{i=l_2}^{n} \frac{r_{it}-r_{l_2-1}}{\underline{q}_i+\bar{q}_i} + \sum\limits_{i=1}^{l_2-1} \frac{r_{l_2-1}-r_{it}}{\underline{q}_i+\bar{q}_i} - \frac{1}{2}}{-2(\sum\limits_{i=l_2}^{n} \frac{r_{it}-r_{l_2-1}}{\underline{q}_i+\bar{q}_i} + \sum\limits_{i=1}^{l_2-1} \frac{r_{l_2-1}-r_{it}}{\underline{q}_i+\bar{q}_i}) + \sum\limits_{i=1}^{n} \frac{\underline{q}_i}{\underline{q}_i+\bar{q}_i} - \frac{\underline{q}_i+\bar{q}_i}{2} + \frac{1}{2}} \end{aligned}$$

Thus, the solution to P_2 is as follows:

i) If there holds the following inequality

$$\frac{1}{\lambda_2-1} \geq \frac{r_{it} + \sum\limits_{i=1}^{n} \frac{r_{it}-r_1}{\underline{q}_i+\bar{q}_i} - \frac{1}{2}}{-2\sum\limits_{i=1}^{n} \frac{r_{it}-r_1}{\underline{q}_i+\bar{q}_i} + \sum\limits_{i=1}^{n} \frac{\underline{q}_i}{\underline{q}_i+\bar{q}_i} - \frac{\underline{q}_i+\bar{q}_i}{2} + \frac{1}{2}},$$

then we have

$$\underline{y}_2 = \frac{M}{\sum\limits_{i=1}^{n} \frac{1}{\underline{q}_i}}, \bar{y}_2 = \frac{M_0}{\sum\limits_{i=1}^{n} \frac{1}{\underline{q}_i+\bar{q}_i} - \sum\limits_{i=1}^{n} \frac{1}{\underline{q}_i}}, x_i = \frac{M_0}{\underline{q}_i \sum\limits_{i=1}^{n} \frac{1}{\underline{q}_i}}, 1 \le i \le n. \tag{3.2.56}$$

ii) When

$$\frac{r_{it} + \sum\limits_{i=l_2+1}^{n} \frac{r_{it}-r_{l_2}}{\underline{q}_i+\bar{q}_i} + \sum\limits_{i=1}^{l_2} \frac{r_{l_2}-r_{it}}{\underline{q}_i+\bar{q}_i} - \frac{1}{2}}{-2(\sum\limits_{i=l_2+1}^{n} \frac{r_{it}-r_{l_2}}{\underline{q}_i+\bar{q}_i} + \sum\limits_{i=1}^{l_2} \frac{r_{l_2}-r_{it}}{\underline{q}_i+\bar{q}_i}) + \sum\limits_{i=1}^{n} \frac{\underline{q}_i}{\underline{q}_i+\bar{q}_i} - \frac{\underline{q}_i+\bar{q}_i}{2} + \frac{1}{2}}$$

$$\le \frac{1}{\lambda_2 - 1} \le \frac{r_{it} + \sum\limits_{i=l_2}^{n} \frac{r_{it}-r_{l_2-1}}{\underline{q}_i+\bar{q}_i} + \sum\limits_{i=1}^{l_2-1} \frac{r_{l_2-1}-r_{it}}{\underline{q}_i+\bar{q}_i} - \frac{1}{2}}{-2(\sum\limits_{i=l_2}^{n} \frac{r_{it}-r_{l_2-1}}{\underline{q}_i+\bar{q}_i} + \sum\limits_{i=1}^{l_2-1} \frac{r_{l_2-1}-r_{it}}{\underline{q}_i+\bar{q}_i}) + \sum\limits_{i=1}^{n} \frac{\underline{q}_i}{\underline{q}_i+\bar{q}_i} - \frac{\underline{q}_i+\bar{q}_i}{2} + \frac{1}{2}},$$

then we get

$$\begin{aligned} &\underline{y}_2 = M_0(\sum_{l_2+1}^{n} \frac{1}{\underline{q}_i} - \sum_{i=1}^{l_2-1} \frac{1}{\underline{q}_i})^{-1} \\ &\bar{y}_2 = M_0[(\sum_{l_2+1}^{n} \frac{1}{\underline{q}_i + \bar{q}_i} - \sum_{i=1}^{l_2-1} \frac{1}{\underline{q}_i + \bar{q}_i})^{-1} - (\sum_{l_2+1}^{n} \frac{1}{\underline{q}_i} - \sum_{i=1}^{l_2-1} \frac{1}{\underline{q}_i})^{-1}] \\ &x_i = \begin{cases} \frac{\underline{y}+\bar{y}}{\underline{q}_i+\bar{q}_i}, 1 \le i \le l_2, \\ -\frac{\underline{y}+\bar{y}}{\underline{q}_i+\bar{q}_i}, l_2+1 \le i \le n. \end{cases} \end{aligned} \tag{3.2.57}$$

We will synthesize (3.2.32), (3.2.33), (3.2.56), (3.2.57), take the intersection of λ_1 and λ_2, and derive the interval solution of (3.2.5). Therefore, we get

Theorem 3.2.1 *i) If there holds*

$$\frac{1}{\lambda_1 - 1} \ge \frac{r_{it} - \sum\limits_{i=2}^{n} \frac{r_{it}-r_1}{\underline{q}_i} + 2\sum\limits_{i=2}^{n} \frac{r_{it}-r_1}{\underline{q}_i+\bar{q}_i}}{\frac{\underline{q}_1+\bar{q}_1}{2}}, \frac{1}{\lambda_2 - 1} \le \frac{r_{it} + \sum\limits_{i=2}^{n} \frac{r_{it}-r_1}{\underline{q}_i+\bar{q}_i} - \frac{1}{2}}{-2\sum\limits_{i=1}^{n} \frac{r_{it}-r_1}{\underline{q}_i+\bar{q}_i} + \sum\limits_{i=1}^{n} \frac{\underline{q}_i}{\underline{q}_i+\bar{q}_i} - \frac{\underline{q}_i+\bar{q}_i}{2} + \frac{1}{2}},$$

then the interval solution of (3.2.5):

$$\underline{y} = \frac{M_0}{\sum\limits_{i=1}^{n} \frac{1}{\underline{q}_i}}, \bar{y} = \frac{M_0}{\sum\limits_{i=1}^{n} \frac{1}{\underline{q}_i+\bar{q}_i} - \sum\limits_{i=1}^{n} \frac{1}{\underline{q}_i}}, x_i = \frac{\underline{y}}{\underline{q}_i}, 1 \le i \le n.$$

ii) If there holds

$$\frac{r_{it}-S_{1+1}+2T_{1+1}}{\frac{\underline{q}_i+\bar{q}_i}{2}} \leq \frac{1}{\lambda_1-1} \leq \frac{r_{it}-S_{1-1}+2T_{1-1}}{\frac{\underline{q}_i+\bar{q}_i}{2}},$$
$$\frac{r_{it}+T_{2+1}-\frac{1}{2}}{-2S_{2+1}+\sum_{i=1}^{n}\frac{\underline{q}_i}{\underline{q}_i+\bar{q}_i}-\frac{\underline{q}_i+\bar{q}_i}{2}+\frac{1}{2}} \leq \frac{r_{it}+T_{2-1}-\frac{1}{2}}{-2T_{2-1}+\sum_{i=1}^{n}\frac{\underline{q}_i}{\underline{q}_i+\bar{q}_i}-\frac{\underline{q}_i+\bar{q}_i}{2}+\frac{1}{2}},$$

then the interval solution of (3.2.5):

$$\begin{aligned}
&\underline{y} = M_0\cdot\max[(\sum_{l_1+1}^{n}\frac{1}{\underline{q}_i}-\sum_{i=1}^{l_1-1}\frac{1}{\underline{q}_i})^{-1},(\sum_{l_2+1}^{n}\frac{1}{\underline{q}_i}-\sum_{i=1}^{l_2-1}\frac{1}{\underline{q}_i})^{-1}];\\
&\bar{y} = M_0\cdot\min(Z_1,Z_2);\\
&x_i = \begin{cases} \frac{\underline{y}+\bar{y}}{\underline{q}_1+\bar{q}_i}, i<l_1<l_2,\\ -\frac{\underline{y}+\bar{y}}{\underline{q}_1+\bar{q}_i}, l_1<l_2<i\end{cases}
\end{aligned}$$

where $S_{k+1}=\sum_{i=l_k+1}^{n}\frac{r_{it}-r_{l_k}}{\underline{q}_i}+\sum_{i=1}^{l_k}\frac{r_{l_k}-r_{it}}{\underline{q}_i}$, $T_{k+1}=\sum_{i=l_k+1}^{n}\frac{r_{it}-r_{l_k}}{\underline{q}_i+\bar{q}_i}+\sum_{i=1}^{l_k}\frac{r_{l_k}-r_{it}}{\underline{q}_i+\bar{q}_i}$,

$S_{k-1}=\sum_{i=l_k}^{n}\frac{r_{it}-r_{l_k-1}}{\underline{q}_i}+\sum_{i=1}^{l_k-1}\frac{r_{l_k-1}-r_{it}}{\underline{q}_i}$, $T_{k-1}=\sum_{i=l_k}^{n}\frac{r_{it}-r_{l_k-1}}{\underline{q}_i+\bar{q}_i}+\sum_{i=1}^{l_k-1}\frac{r_{l_k-1}-r_{it}}{\underline{q}_i+\bar{q}_i}$, $k=1,2$.

$Z_1=[(\sum_{l_1}^{n}\frac{1}{\underline{q}_i+\bar{q}_i}-\sum_{i=1}^{l_1-1}\frac{1}{\underline{q}_i+\bar{q}_i})^{-1}-(\sum_{l_1}^{n}\frac{1}{\underline{q}_i}-\sum_{i=1}^{l_1-1}\frac{1}{\underline{q}_i})^{-1}]$,

$Z_2=[(\sum_{l_2}^{n}\frac{1}{\underline{q}_i+\bar{q}_i}-\sum_{i=1}^{l_2-1}\frac{1}{\underline{q}_i+\bar{q}_i})^{-1}-(\sum_{l_2}^{n}\frac{1}{\underline{q}_i}-\sum_{i=1}^{l_2-1}\frac{1}{\underline{q}_i})^{-1}]$.

Remark 3.2.1 *Notice that in order to solve the portfolio model based on Minimax rule with fuzzy interval, one can introduce order relation and turn (3.2.5) with fuzzy returns into P_1 and P_2.*

By comparison of the solutions of question one and two, we compute directly and get

$$\begin{aligned}
\underline{y}_2-\underline{y}_1 &= M_0(\sum_{l_2+1}^{n}\frac{1}{\underline{q}_i}-\sum_{i=1}^{l_2-1}\frac{1}{\underline{q}_i})^{-1}-M_0(\sum_{l_1+1}^{n}\frac{1}{\underline{q}_i}-\sum_{i=1}^{l_1-1}\frac{1}{\underline{q}_i})^{-1}\\
&= M_0\cdot\frac{(\sum_{l_1+1}^{n}\frac{1}{\underline{q}_i}-\sum_{i=1}^{l_1-1}\frac{1}{\underline{q}_i})-(\sum_{l_2+1}^{n}\frac{1}{\underline{q}_i}-\sum_{i=1}^{l_2-1}\frac{1}{\underline{q}_i})}{(\sum_{l_2+1}^{n}\frac{1}{\underline{q}_i}-\sum_{i=1}^{l_2-1}\frac{1}{\underline{q}_i})(\sum_{l_1+1}^{n}\frac{1}{\underline{q}_i}-\sum_{i=1}^{l_1-1}\frac{1}{\underline{q}_i})}\\
&= \frac{M_0}{\underline{q}_i}\cdot\frac{\sum_{l_1+1}^{l_2+1}\frac{1}{\underline{q}_i}+\sum_{l_1-1}^{l_2-1}\frac{1}{\underline{q}_i}}{(\sum_{i=1}^{n}\frac{1}{\underline{q}_i}-\sum_{i=1}^{l_2}\frac{1}{\underline{q}_i}-\sum_{i=1}^{l_2-1}\frac{1}{\underline{q}_i})(\sum_{i=1}^{n}\frac{1}{\underline{q}_i}-\sum_{i=1}^{l_1}\frac{1}{\underline{q}_i}-\sum_{i=1}^{l_1-1}\frac{1}{\underline{q}_i})}
\end{aligned}$$

$$\geq \frac{M_0}{\underline{q}_i} \cdot \frac{\sum\limits_{l_1+1}^{l_2+1} \frac{1}{\underline{q}_i} + \sum\limits_{l_1-1}^{l_2-1} \frac{1}{\underline{q}_i}}{(\sum\limits_{i=1}^{n} \frac{1}{\underline{q}_i} - \sum\limits_{i=1}^{l_2} \frac{1}{\underline{q}_i})(\sum\limits_{i=1}^{n} \frac{1}{\underline{q}_i} - \sum\limits_{i=1}^{l_1} \frac{1}{\underline{q}_i})} > 0.$$

$$\begin{aligned}
\bar{y}_2 - \bar{y}_1 &= M_0[(\sum_{l_2+1}^{n} \frac{1}{\underline{q}_i + \bar{q}_i} - \sum_{i=1}^{l_2-1} \frac{1}{\underline{q}_i + \bar{q}_i})^{-1} - (\sum_{l_2}^{n} \frac{1}{\underline{q}_i} - \sum_{i=1}^{l_2-1} \frac{1}{\underline{q}_i})^{-1}] \\
&- M_0[(\sum_{l_1+1}^{n} \frac{1}{\underline{q}_i + \bar{q}_i} - \sum_{i=1}^{l_1-1} \frac{1}{\underline{q}_i + \bar{q}_i})^{-1} - (\sum_{l_1}^{n} \frac{1}{\underline{q}_i} - \sum_{i=1}^{l_1-1} \frac{1}{\underline{q}_i})^{-1}] \\
&= \frac{M_0(\sum\limits_{l_1}^{l_2} \frac{1}{\underline{q}_i+\bar{q}_i} - \sum\limits_{l_1-1}^{l_2-1} \frac{1}{\underline{q}_i+\bar{q}_i})}{(\sum\limits_{i=1}^{n} \frac{1}{\underline{q}_i+\bar{q}_i} - 2\sum\limits_{i=1}^{l_2-1} \frac{1}{\underline{q}_i+\bar{q}_i})(\sum\limits_{i=1}^{n} \frac{1}{\underline{q}_i+\bar{q}_i} - 2\sum\limits_{i=1}^{l_2-1} \frac{1}{\underline{q}_i+\bar{q}_i})} \\
&+ \frac{M_0(\sum\limits_{l_1+1}^{l_2+1} \frac{1}{\underline{q}_i} + \sum\limits_{l_1-1}^{l_2-1} \frac{1}{\underline{q}_i})}{(\sum\limits_{i=1}^{n} \frac{1}{\underline{q}_i} - \sum\limits_{i=1}^{l_2} \frac{1}{\underline{q}_i} - \sum\limits_{i=1}^{l_2-1} \frac{1}{\underline{q}_i})(\sum\limits_{i=1}^{n} \frac{1}{\underline{q}_i} - \sum\limits_{i=1}^{l_1} \frac{1}{\underline{q}_i} - \sum\limits_{i=1}^{l_1-1} \frac{1}{\underline{q}_i})}
\end{aligned}$$

It is not hard to see that when $l_1 < l_2$, there holds $\underline{y}_2 > \underline{y}_1$, that is, the lower solution of P_2 is bigger than that of P_1. The upper solutions of P_1 and P_2 have relation with l_1, l_2, in fact, there is $l_1 < l_2$ such that $\bar{y}_2 < \bar{y}_1$. As a result, the interval of P_2 will be reduced to a smaller range, which tells us solving the problem as this is meaningful, of course, is also feasible.

Furthermore, the solution of P_2 will be substituted into (3.2.5), it holds

$$\underline{y} = \max[M_0(\sum_{l_1+1}^{n} \frac{1}{\underline{q}_i} - \sum_{i=1}^{l_1-1} \frac{1}{\underline{q}_i})^{-1}, M_0(\sum_{l_2+1}^{n} \frac{1}{\underline{q}_i} - \sum_{i=1}^{l_2-1} \frac{1}{\underline{q}_i})^{-1}] = \underline{y}_2.$$

It shows that the constraint of order relations change the lower bound of the solution into lower bound of P_2, narrow the scope of the solution and make it more near to the optimal solution. Thus, it is correct and reasonable to take the order relation into account.

3.2.4 Comparison of Solutions to Portfolio Models Without Transaction Costs

Theorem 3.2.2 *The optimal solution under the fuzzy interval is better than the optimal solution to the corresponding literature [39, 44].*

Proof. Denote by $y^0 = \lambda^0 \underline{y} + (1 - \lambda^0)\bar{y}$. Notice that $y = \frac{M_0}{\sum_{i=1}^{n} \frac{1}{q_i}}$ [39, 43] and by using

Theorem 3.2.1(i), we know that

$$\begin{aligned} y^0 - y &= M_0\Big(\frac{\lambda^0}{\sum\limits_{i=1}^n \frac{1}{\underline{q}_i}} - \frac{1-\lambda^0}{\sum\limits_{i=1}^n \frac{1}{\underline{q}_i+\bar{q}_i} - \sum\limits_{i=1}^n \frac{1}{\underline{q}_i}} - \frac{1}{\sum\limits_{i=1}^n \frac{1}{\underline{q}_i}}\Big) \\ &= M_0 \cdot \frac{\sum\limits_{i=1}^n \frac{1}{\underline{q}_i+\bar{q}_i}(\lambda^0 \sum\limits_{i=1}^n \frac{1}{q_i} - \sum\limits_{i=1}^n \frac{1}{\underline{q}_i}) + \sum\limits_{i=1}^n \frac{1}{\underline{q}_i}(\sum\limits_{i=1}^n \frac{1}{\underline{q}_i} - \sum\limits_{i=1}^n \frac{1}{q_i})}{\sum\limits_{i=1}^n \frac{1}{\underline{q}_i}(\sum\limits_{i=1}^n \frac{1}{\underline{q}_i+\bar{q}_i} - \sum\limits_{i=1}^n \frac{1}{\underline{q}_i}) \sum\limits_{i=1}^n \frac{1}{q_i}} \end{aligned}$$

By a direct computation and notice that $0 \le \lambda^0 \le 1$, we get

$$y^0 - y \le \frac{M_0(\sum\limits_{i=1}^n \frac{1}{\underline{q}_i} - \sum\limits_{i=1}^n \frac{1}{q_i})}{(\sum\limits_{i=1}^n \frac{1}{\underline{q}_i+\bar{q}_i} - \sum\limits_{i=1}^n \frac{1}{\underline{q}_i}) \sum\limits_{i=1}^n \frac{1}{q_i}} < 0.$$

Similarly, we can consider Theorem 3.2.1(ii) and compare the relation between y^0 and y. By a direct and complicated computation, we also get

$$y^0 - y = \frac{\lambda^0 M_0 - M_0}{\sum\limits_{i=1}^n \frac{1}{q_i}} \le 0.$$

This implies that regardless of any circumstances, as long as a linear combination of $\underline{y}, \bar{y}$, the combinative solution is superior to the original solution. In order to reduce the loss as much as possible, investment will be more inclined to models with interval solutions, as it allows investors to hedge against losses. □

Corollary 3.2.1 *The solution to (3.2.5) is stable, that is,* $|F_\lambda(x,y) - F_\lambda(x,y^0)| < \epsilon$ *when* $|q_i - [\lambda^0 \underline{q}_i + (1-\lambda^0)\bar{q}_i]| < \delta$ *for* $i = 1, 2, \cdots, n$.

In fact, when $|q_i - [\lambda^0 \underline{q}_i + (1-\lambda^0)\bar{q}_i]| < \delta$, we deduce $\frac{-\delta - q_i + \bar{q}_i}{\bar{q}_i - \underline{q}_i} < \lambda^0 < \frac{\delta - q_i + \bar{q}_i}{\bar{q}_i - \underline{q}_i}$. Substituting y^0 into the original objective function, we get

$$\begin{aligned} F_\lambda(x,y) &= \lambda y + (1-\lambda)(-\sum_{i=1}^n r_{it} x_i) = \lambda \frac{M_0}{\sum\limits_{i=1}^n \frac{1}{q_i}} + (1-\lambda)(-\sum_{i=1}^n r_{it} \frac{y}{q_i}), \\ F_\lambda(x,y^0) &= \lambda\Big[\lambda^0 \frac{M_0}{\sum\limits_{i=1}^n \frac{1}{\underline{q}_i}} + (1-\lambda^0)\frac{M_0}{\sum\limits_{i=1}^n \frac{1}{\underline{q}_i+\bar{q}_i} - \sum\limits_{i=1}^n \frac{1}{\underline{q}_i}}\Big] + (1-\lambda)(-\sum_{i=1}^n r_{it} \frac{\underline{y}}{\underline{q}_i}). \end{aligned}$$

Then, we, by a direct and not hard computation, get

$$\begin{aligned}|F_\lambda(x,y)-F_\lambda(x,y^0)| &= \Big|\frac{1}{1-\sum\limits_{i=1}^{n}\frac{1}{\underline{q}_i}\Big/\sum\limits_{i=1}^{n}\frac{1}{\sum\limits_{i=1}^{n}\underline{q}_i+\bar{q}_i}}\Big|\cdot\Big|\frac{1}{\sum\limits_{i=1}^{n}\frac{1}{q_i}}+\frac{\lambda^0}{\sum\limits_{i=1}^{n}\frac{1}{\underline{q}_i}}\Big| \\ &< \Big|\frac{1}{1-\sum\limits_{i=1}^{n}\frac{1}{\underline{q}_i}\Big/\sum\limits_{i=1}^{n}\frac{1}{\sum\limits_{i=1}^{n}\underline{q}_i+\bar{q}_i}}\Big|\cdot\Big[\Big|\frac{1}{\sum\limits_{i=1}^{n}\frac{1}{q_i}}\Big|+(1-\frac{\delta}{\bar{q}_i-q_i})\Big|\frac{1}{\sum\limits_{i=1}^{n}\frac{1}{\underline{q}_i}}\Big|\Big]\end{aligned}$$

By using the boundedness of $\Big|\sum\limits_{i=1}^{n}\frac{1}{\underline{q}_i}\Big/\sum\limits_{i=1}^{n}\frac{1}{\sum\limits_{i=1}^{n}\underline{q}_i+\bar{q}_i}\Big|$ and the hypotheses, it is not hard to see that Corollary 3.2.1 is tenable. □

Remark 3.2.2 *In this paper, the authors study only the existence and stability of fuzzy-interval-solutions to the portfolio model based on Minimax rule with fuzzy interval, but they do not pose the empirical results on this model. For the empirical analysis, they will study it in another paper (omitted here).*

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References

[1] R. Bellman and Zadeh L. A., "Decision making in a fuzzy environment," *Management Science*, 1970, **17**: 141-164

[2] X. Q. Cai, Teo K. L. and Yang X. Q., "Optimization under a minimax rule," *Management Science*, 2000, **46**: 957-972

[3] C. Carlsson, Fuller R. and Majlender P., "A possibilistic approach to selecting portfolio with highest utility score," *Fuzzy Sets and Systems,* 2002, **131**: 13-21

[4] C. D. Feinstein and M. N. Thapa, "A reformulation of a mean-absolute deviation portfolio optimization model," *Management Science,"* 1993, **39**: 1552-1553

[5] P. Chunhachinda, K. Dandapani, S. Hamid and A.J. Prakash, "Portfolio selection and skewness:evidence from international stock markets," *Journal of Banking and Finance,* 1997, **21**: 143-167

[6] T.M. Cover, "Universal portfolio," *Mathematical Finance,* 1991, **1**: 1-29

[7] X. L. Deng, Y. F. Zhang and P. B. Zhao, "Portfolio optomization based on spectral risk measures," *International J. of Mathematical Analysis,* 2009, **3**: 1657-1668

[8] X. T. Deng, Z. F. Li and S. Y. Wang, "A minimax portfolio selection strategy with equilibrium," *European Journal of Operational Research,* 2005, **166**: 278-292

[9] X. O. Guo, Y. L. Hu and P. B. Zhao, "Portfolio selection problems based on generalized CVaR," *Adv. Appl. Math. Sci.,* 2009, **1**: 323-334

[10] B. Hou, J. L. Liu and P. B. Zhao, "The evaluation of policy of portfolio problems," *Appl. Math. Sci.,* 2009, **3**: 2021-2034

[11] A. Kane, "Skewness preference and portfolio choice," *Journal of Financial and Quantitative Analysis,* 1982, **17**: 15-26

[12] K.K.Lai, S.Y.Wang, J.P.Xu, S.S.Zhu and Y. Fang, "A class of linear interval programming problems and its application to portfolio selection," *IEEE transactions on fuzzy systems,* 2002, **10**: 698-704

[13] H. Konno, " Piecewise linear risk function and portfolio optimization," *Journal of the Operations Research Society of Japan,* 1990, **33**: 139-156

[14] H. Konno and K. A. Suzuki, "Mean variance skewness optimization models," *Journal of the Operations Research Society of Japan,* 1995, **38**: 173-187

[15] H. Konno and Yamazaki, "Mean-absolute deviation portfolio optimization models and its applications to Tokoy stock market," *Management Science,* 1991, **37**: 519-531

[16] K. K. Lai, S. Y. Wang, J. H. Zeng and S. S. Zhu, "Portfolio selection models with transaction costs:crisp case and interval number case," in Li,D.(eds.). *Proceedings of the 5th International Conference On Optimization Techniques and Applications, Hong Kong,* 2001: 943-950,

[17] T. Leon, V. Liern and E. Vercher, "Fuzzy mathematical programming for portfolio management," *Financial Modelling,* 2000, **14**: 241-256

[18] T. Leon, V. Liern and E. Vercher, "Viability of infeasible portfolio selection problems:a fuzzy approach," *European Journal of Operational Research,* 2002, **139**: 178-189

[19] D. Li and W. K. Ng, "Optimal dynamic portfolio selection:multi-period mean-variance formulation," *Mathematical Finance,* 2000, **10**: 387-406

[20] J. C. T. Mao, "Models of capital budgeting,E-V vs E-S," *Journal of Financial and Quantitative Analysis,* 1970, **5**: 657-675

[21] H. M. Markowitz, " Portfolio selection," *Journal Finance,* 1952, **7**:77-91

[22] R. C. Merton, " An analytic derivation of the efficient portfolio frontier," *Financial Quant Analysts,* 1972, **7**: 1851-1872

[23] M. Boshernitzan, "A condition for minimal interval exchange maps to be uniquely ergodic," *Duke Mathematical Journal,* 1985, **52**: 723-752

[24] J. Mossin, "Optimal multi-period portfolio policies," *Journal of Business,* 1968, **41**: 215-229

[25] L. L. Pan and P. B. Zhao, "Dynamic portfolio choice problems with non-monotone utility functions, " *Appl. Math. Sci.,* 2008, **2**: 1761-1772

[26] M. A. Parra, A. B. Terol and M. V. R. Uria, "A fuzzy goal programming approach to portfolio seletion," *European Journal of Operational Research,* 2001, **133**: 187-297

[27] S. Ramaswamy, "Portfolio selection using fuzzy decision theory," *Working Paper of Bank for International Settlements,* 1998, **59**, http://www.bis.org/publ/work59.htm

[28] A. D. Roy, "Safety-first and the holding of assets," *Econometrics,* 1952, **20**: 431-449

[29] P. Samuelson, "The fundamental approximation theorem of portfolio analysis in terms of means variances and higher moments," *Review of Economic Studies,* 1958, **25**: 65-86

[30] W. Sharpe, "Capital asset prices:a theory of market equilibrium under conditions of risk," *The Journal of Finance,* 1964, **19**: 425-442

[31] M. G. Speranza, "Linear programming models for portfolio optimization," *Finance,* 1993, **14**: 107-123

[32] R. O. Swalm, "Utility theory-insights into risk taking," *Harvard Business Review,* 1966, **44**: 123-136

[33] H. Tanaka and P. Guo, "Portfolio selection based on upper and lower exponential possibility distributions," *European Journal of Operational Research,* 1999, **114**: 115-126

[34] H. Tanaka, P. Guo and I. B. Turksen, "Portfolio selection based on fuzzy possibilities and possibility distributions," *Fuzzy Sets and Systems,* 2000, **111**: 387-397

[35] K.-S. Tang, K.-F. Man, Z.-F. Liu and K. Sam, "Minimal fuzzy memberships and rules using hierarchical genetical gorithms," *Industrial Electronics,* 1998, **45**: 162-169

[36] W. G. Zhang, W. A. Liu and Y. L. Wang, "On admissible efficient portfolio selection:Models and algorithms," *Appl. Math. Comput.,* 2006, **176**: 208-218

[37] C. F. Yang and P. B. Zhao, "The portfolio problem under the variable rate of transaction costs," *Far East J. Appl. Math.,* 2008, **31**: 231-244

[38] A. Yoshimoto, "The mean-variance apporoach to portfolio optimization subject to transaction costs," *J. of the Operational Research Society of Japan,* 1996, **39**: 99-117

[39] A. Young, "Minimax Portfolio selection rule with linear programming solution," *Management science,* 1998, **44**: 673-683

[40] L. A. Zadeh, "Fuzzy sets," *Information and Contro,* 1965, **8**: 338-353

[41] L. A. Zadeh, "Fuzzy sets as a basis for a theory of possibility," *Fuzzy Sets and Systems,* 1978, **1**: 3-28

[42] X. Y. Zhou and D. Li, "Continuous-time mean-variance portfolio selection:a stochastic LQ framework," *Applied Mathematics and Optimization,"* 2000, **42**(1): 19-33

[43] Y. Fang and S. Y. Wang, "Fuzzy portfolio optimization: theory and methods," *Advanced Academic Publishers,* 2005

[44] J. F. Zhang, L. Zhao and W. Y. Chen, "The short selling permitted portfolio optimization under a minimax rule," *Systems Engineering-theory & Practice,* 2008, **28**(4): 12-18

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