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The Exponential and Logarithmic Functions on Commutative Banach Algebras

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Abstract

In this paper we define the two important functions exp and log on a commutative Banach algebra $\mathfrak{B}$ with unit e. Some important properties of these functions are established. The proofs given here have elementary character.

Keywords: Banach algebra, Banach space, series, differentiable function

1 Introduction

Recall that a Banach algebra is a set $\mathfrak{B}$ such that :

- $\mathfrak{B}$ is a Banach space, i.e. a complete normed vector space over the field $\mathbb{C}$ of complex numbers. This means that any Cauchy sequence converges on $\mathfrak{B}$.
- $\mathfrak{B}$ is a ring and $\| xy \|\leq\| x \|\| y \|$ for all $x, y \in \mathfrak{B}$.

If $\mathfrak{B}$ contains a unit element e satisfying $\| e \|= 1$, we shall say that $\mathfrak{B}$ is a Banach algebra with unit.
Many of the Banach spaces which occur in analysis are at the same time Banach algebras under a multiplication operation which is itself important for the analysis. A good example is the space of absolutely integrable functions on the infinite real line under convolution as multiplication. In spite of the fact that examples from analysis have always provided the main impetus to the study of Banach spaces, a comparable interest in Banach algebras was rather late in coming. Some of the reason lies no doubt in the absence of appropriate algebraic tools, since a large part of the earlier work in algebra was based on

finiteness conditions which rule out the most interesting examples.
It remained for the soviet mathematician Gelfand to lay the foundation for a general theory of Banach algebras in his now classical paper [3] on normed rings which was announced in 1939 [2] and appeared in 1941. Gelfand's innovation was a systematic use of elementary ideal theory coupled with the Mazur-Gelfand theorem which states that a normed division algebra (over the complex field) must be isomorphic with the complex field. His fundamental result was that a semi-simple commutative Banach algebra with an identity element is isomorphic to an algebra of continuous functions on a compact Hausdorff space. At the same time, Gelfand [4] used his theory to give an elegant proof of the well-known Wiener lemma that the reciprocal of a non-vanishing absolutely convergent Fourier series is also an absolutely convergent Fourier series. This proof attracted a great deal of attention to Banach algebras. Since the appearance of Gelfand's 1941 papers, there has been a rapid growth of interest in Banach algebras. The resulting development of the subject has brought the theory to a point where it is no longer just a promising tool in analysis but is an important field of research in its own right. Standing, as it were, between analysis and algebra (or perhaps more accurately, with feet in analysis and head in algebra), the theory of Banach algebras has developed roughly along two main lines representing respectively the analytic and algebraic influences. The analytic emphasis has been on the study of certain special Banach algebras, along with some generalizations of these algebras, and on extending certain portions of function theory and harmonic analysis to the more general situations offered by Banach algebras. On the other hand, the algebraic emphasis has naturally been on various aspects of structure theory. Of great importance here has been the growing interest of algebraists in algebras without finiteness restrictions. This development, which has been much stimulated by the study of Banach algebras, has supplied important new algebraic methods which are profitably applied to Banach algebras. It becomes increasingly evident that, in spite of the deep and continuing influence of analysis on the theory of Banach algebras, the essence of the subject as an independent discipline is to be found in its algebraic development.
In this paper $\mathfrak{B}$ will always denote a Banach algebra with unit e.
An element $x \in \mathfrak{B}$ is said to be invertible or regular, if there exists an element $y \in \mathfrak{B}$ such that $xy = e$. This element is unique and it is called the inverse of x and it is denoted by x^{-1}. If $x \in \mathfrak{B}$ and $\| e - x \| < 1$, then x is invertible [1].

2 Differentiable functions

Definition 2.1 *Let $a \in \mathfrak{B}$ and $f : B_\delta(a) \to \mathfrak{B}$ be a function defined on a neighborhood $B_\delta(a) = \{x \in \mathfrak{B} : \| x - a \| < \delta\}$ of a. We say that f is* *differentiable at* a *if there is a continuous linear map* $T_a : \mathfrak{B} \to \mathfrak{B}$ *such that*

$$f(a+h) - f(a) = T_a(h) + o(h), \text{ where } \lim_{h\to 0} \frac{\| o(h) \|}{\| h \|} = 0. \tag{2.1}$$

In this case, we write $f'(a) = T_a$ *or* $f'(a)(h) = T_a(h)$ *and we say that* $f'(a)$ *is the differential of* f *at* a.
We say that f *is differentiable on* $B_\delta(a)$ *if it is differentiable at each* $x \in B_\delta(a)$.

The proof of the following theorems may be found in [6].

Theorem 2.1 (Linearity) *If* $f : B_\delta(a) \to \mathfrak{B}$ *and* $g : B_\delta(a) \to \mathfrak{B}$ *are differentiable at* a*, then the function* $\varphi = f + g$ *defined by* $\varphi(x) = f(x) + g(x)$ *is differentiable at* a *and*

$$\varphi'(a) = f'(a) + g'(a).$$

Moreover, if $\lambda \in \mathbb{C}$ *and* $b \in \mathfrak{B}$*, then* $(\lambda f)'(a) = bf'(a)$ *and* $(bf)'(a) = bf'(a)$.

Theorem 2.2 (Product Rule) *If* $f : B_\delta(a) \to \mathfrak{B}$ *and* $g : B_\delta(a) \to \mathfrak{B}$ *are differentiable at* a*, then the function* $\varphi = fg$ *defined by* $\varphi(x) = f(x)g(x)$ *is differentiable at* a *and*

$$\varphi'(a) = g(a)f'(a) + f(a)g'(a).$$

Theorem 2.3 (Chain Rule) *If* $f : B_\delta(a) \to \mathfrak{B}$ *is differentiable at* a*,* $f(B_\delta(a)) \subseteq B\eta(b)$ *and* $g : B_\eta(b) \to \mathfrak{B}$ *is differentiable at* $b = g(a)$*, then the function* $\varphi = g \circ f : B_\delta(a) \to \mathfrak{B}$ *is differentiable at* a *and*

$$\varphi'(a) = (g \circ f)'(a) = g'(f(a)) \circ f'(a)(h).$$

Theorem 2.4 *Suppose that* $f'(x)(h) = 0$ *for each* x *in some convex open subset* E *of a Banach algebra* $\mathfrak{B}$*. Then* $f = \text{const}$ *on* E.

3 Exponential and logarithmic functions

We define two important functions $\exp(x)$ and $\log(x)$ from $\mathfrak{B}$ to $\mathfrak{B}$ as follows
:

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \quad \text{for any} \quad x \in \mathfrak{B} \tag{3.2}$$

and

$$\log(x) = -\sum_{n=1}^{\infty} \frac{(e-x)^n}{n} = -(e-x) - \frac{(e-x)^2}{2} - \frac{(e-x)^3}{3} - \cdots \quad \text{for} \quad \| e - x \| < 1. \tag{3.3}$$

Series (3.2) converges absolutely for all x, while series (3.3) converges absolutely for $\| e - x \| < 1$. This follows from the inequalities

$$\sum_{n=0}^{\infty} \| \frac{x^n}{n!} \| \leq \sum_{n=0}^{\infty} \frac{\| x \|^n}{n!} = \exp(\| x \|) < +\infty$$

and

$$\sum_{n=1}^{\infty} \| \frac{(e-x)^n}{n} \| \leq \sum_{n=1}^{\infty} \frac{\| e-x \|^n}{n} \leq \sum_{n=1}^{\infty} \| e-x \|^n = \frac{\| e-x \|}{1 - \| e-x \|} < +\infty.$$

In particular, $\exp(0) = e$ and $\log(e) = 0$. In the case when $\mathfrak{B}$ is the algebra of real numbers, $\exp(x)$ and $\log(x)$ are the usual exponential and logarithmic functions, respectively. We know that these functions are analytic as functions of real variable. We want to establish an analogous result for commutative Banach algebras.

Lemma 3.1 *Suppose that $b, y \in \mathfrak{B}$, $b \neq 0$, $\lambda > 0$ and $n \in \mathbb{N}$. If $\| y - b \| \leq \lambda \| b \|$, then we may find constants $C > 0$ and $\mu > 0$, which depend only on λ and b, such that*

$$\| y^n - b^n \| \leq C\mu^n \| y - b \| \tag{3.4}$$

If $\| b \| < 1$ and $0 < \lambda < \dfrac{1}{\| b \|} - 1$, then we may choose μ with $0 < \mu < 1$.

Proof. The inequality $\| y - b \| \leq \lambda \| b \|$ implies that $\| y \| \leq (1 + \lambda) \| b \|$. Define

$$\theta = 1 + \lambda \quad \text{and} \quad \mu = (1 + \lambda) \| b \| = \theta \| b \| .$$

It is clear that $\| y \| \leq \theta \| b \|$. On the other hand,

$$\begin{aligned}
\| y^n - b^n \| &\leq \| y - b \| \| y^{n-1} + y^{n-2}b + \cdots + y\, b^{n-2} + b^{n-1} \| \\
&\leq \| y - b \| \left(\| y \|^{n-1} + \| y \|^{n-2} \| b \| + \cdots + \| y \| \| b \|^{n-2} + \| b \|^{n-1} \right) \\
&\leq \| y - b \| \left((\theta \| b \|)^{n-1} + (\theta \| b \|)^{n-2} \| b \| + \cdots + \theta \| b \| \| b \|^{n-2} + \| b \|^{n-1} \right) \\
&= \| y - b \| \| b \|^{n-1} \left(\theta^{n-1} + \theta^{n-2} + \cdots + \theta + 1 \right) \\
&= \| y - b \| \| b \|^{n-1} \cdot \frac{\theta^n - 1}{\theta - 1} \leq \| y - b \| \| b \|^{n-1} \cdot \frac{\theta^n}{\theta - 1} \\
&= \frac{1}{\lambda \| b \|} (\theta \| b \|)^n \| y - b \| .
\end{aligned}$$

Thus,

$$\| y^n - b^n \| \leq C\mu^n \| y - b \|, \quad \text{where} \quad C = \frac{1}{\lambda \| b \|}.$$

This proves (3.4).

Finally, if $0 < \| b \| < 1$ and $0 < \lambda < \dfrac{1}{\| b \|} - 1$, then $0 < \mu = (\lambda + 1) \| b \| < 1$.

Lemma 3.2 *Let $h, u, t, x \in \mathfrak{B}$, $\lambda > 0$ and $n \in \mathbb{N}$.*
a) *If $\| t \| \leq \lambda \| u \|$ and $u \neq 0$, then*

$$\| (u+t)^n - u^n - ntu^{n-1} \| \leq C^2 \mu^n \| t \|^2 \quad \text{where } \mu = (1+\lambda) \| u \| \quad \text{and } C = (\lambda \| u \|)^{-1}. \tag{3.5}$$

b) *If $\| h \| \leq \| x \|$ and $x \neq 0$, then*

$$\| (x+h)^n - x^n - nhx^{n-1} \| \leq \mu^n \| x \|^{-2} \| h \|^2 \quad \text{where } \mu = 2 \| x \| . \tag{3.6}$$

In particular, the function $x \to x^n$ is differentiable at any $x \in \mathfrak{B}$ and its differential is $(x^n)'(h) = nx^{n-1}h$.
c) *If $\| e - x \| < 1$, $x \neq e$, $\| h \| \leq \lambda \| e - x \|$ and $0 < \lambda < \dfrac{1}{\| e - x \|} - 1$, then*

$$\| (e-x)^n - (e-x-h)^n - nh(e-x)^{n-1} \| \leq C^2 \mu^n \| h \|^2, \tag{3.7}$$

where $\mu = (1+\lambda) \| e - x \| < 1$ and $C = (\lambda \| e - x \|)^{-1}$.

Proof.
a) By Lemma 3.1, with $y = u+t$ and $b = u$, $\| (u+t)^{j-1} - u^{j-1} \| \leq C\mu^{j-1} \| t \|$ for $j = 1, 2, 3, \ldots$ so we have :

$$\| (u+t)^n - u^n - ntu^{n-1} \| = \| (u+t-u) \sum_{j=1}^{n} (u+t)^{j-1} u^{n-j} - ntu^{n-1} \| =$$

$$\| t \sum_{j=1}^{n} (u+t)^{j-1} u^{n-j} - t \sum_{j=1}^{n} u^{n-1} \| \leq \| t \| \sum_{j=1}^{n} \| (u+t)^{j-1} u^{n-j} - u^{n-1} \| \leq$$

$$\| t \| \sum_{j=1}^{n} \| u \|^{n-j} \| (u+t)^{j-1} - u^{j-1} \| \leq \| t \| \sum_{j=1}^{n} \| u \|^{n-j} C\mu^{j-1} \| t \| =$$

$$C \| t \|^2 \mu^{n-1} \sum_{j=1}^{n} \left(\frac{\| u \|}{\mu} \right)^{n-j} \leq C \| t \|^2 \mu^{n-1} \frac{1+\lambda}{\lambda} = C^2 \mu^n \| t \|^2 .$$

b) This follows from **a)** by letting $\lambda = 1$, $u = x$ and $t = h$.
c) This is a consequence of **a)** by taking $u = e - x$ and $t = -h$.

Theorem 3.5 *The function $\exp(x)$ defined by (3.2) is differentiable. More exactly,*

$$\| \exp(x+h) - \exp(x) - \exp(x)h \| \leq K \| h \|^2, \tag{3.8}$$

where $\| h \| \leq \| x \|$ and $K = \sum_{n=2}^{\infty} \dfrac{2^n \; \| x \|^{2n-2}}{n!}$. The derivative of $\exp(x)$ is given by

$$\exp'(x)(h) = \exp(x)h \text{ for any } x \in \mathfrak{B} \tag{3.9}$$

Proof. Let $x, h \in \mathfrak{B}$. The inequality (3.8) is evident if $x = 0$. In this case, $h = 0$. Suppose that $x \neq 0$ and $\| h \| \leq \| x \|$. We have

$$\exp(x+h) - \exp(x) - \exp(x)h = \sum_{n=0}^{\infty} \frac{(x+h)^n - x^n}{n!} - \sum_{n=0}^{\infty} \frac{hx^n}{n!} =$$

$$(x+h)^1 - x^1 - h + \sum_{n=2}^{\infty} \frac{(x+h)^n - x^n}{n!} - \sum_{n=1}^{\infty} \frac{hx^n}{n!} =$$

$$\sum_{n=2}^{\infty} \frac{(x+h)^n - x^n}{n!} - \sum_{n=2}^{\infty} \frac{hx^{n-1}}{(n-1)!} = \sum_{n=2}^{\infty} \frac{1}{n!} \left((x+h)^n - x^n - nhx^{n-1} \right),$$

so by Lemma 3.2, Part **b)**,

$$\| \exp(x+h) - \exp(x) - \exp(x)h \| \leq \sum_{n=2}^{\infty} \frac{1}{n!} \| (x+h)^n - x^n - nhx^{n-1} \| \leq$$

$$\sum_{n=2}^{\infty} \frac{1}{n!} \mu^n \| x \|^{n-2} \| h \|^2 = \frac{\| h \|^2}{\| x \|^2} \sum_{n=2}^{\infty} \frac{(\mu \| x \|)^n}{n!} = \sum_{n=2}^{\infty} \frac{2^n \| x \|^{2n-2}}{n!} \| h \|^2 = K \| h \|^2 .$$

Finally, equation (3.9) follows directly from (3.8), because if $o(h) = \exp(x + h) - \exp(x) - \exp(x)h$ and $h \neq 0$, then

$$\frac{\| o(h) \|}{\| h \|} \leq K \| h \| \to 0 \ (h \to 0).$$

Theorem 3.6 *The function* $\log(x)$ *defined by (3.3) is differentiable. More exactly,*

a) *If* $x \neq e$ *and* $\| e - x \| < 1$, *then*

$$\| \log(x+h) - \log(x) - x^{-1}h \| \leq L \| h \|^2, \tag{3.10}$$

where

$$\| h \| \leq \frac{1 - \| e - x \|}{2} \text{ and } L = \frac{4}{(1 - \| e - x \|)^2} \ln \left(\frac{2}{1 - \| e - x \|} \right).$$

b) *If* $x = e$ *and* $\| h \| \leq 1/2$ *,then*

$$\| \log(e+h) - \log(e) - e^{-1}h \| \leq 2 \| h \|^2 . \tag{3.11}$$

c) *The derivative of* $\log(x)$ *is given by*

$$\log'(x)(h) = x^{-1}h \text{ for any } x \in \mathfrak{B} \text{ such that } \| x - e \| < 1. \tag{3.12}$$

Proof.

a) Let $x \in A$ with $x \neq e$ and $\| e - x \| < 1$. Define $\lambda = \frac{1}{2}\left(\frac{1}{\| e - x \|} - 1\right)$ and consider $h \in \mathfrak{B}$ such that $\| h \| \leq \lambda \| e - x \|$. Observe that x^{-1} exists ,

$$x^{-1} = (e - (e - x))^{-1} = \sum_{n=1}^{\infty} (e - x)^{n-1}$$

and

$$\| e - (x + h) \| \leq \| e - x \| + \| h \| < 1,$$

so $\log(x + h)$ is defined. We have :

$$\begin{aligned} \log(x + h) - \log(x) - x^{-1}h &= -\sum_{n=1}^{\infty} \frac{(e - x - h)^n}{n} + \sum_{n=1}^{\infty} \frac{(e - x)^n}{n} - \sum_{n=1}^{\infty} h(e - x)^{n-1} \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \left((e - x)^n - (e - x - h)^n - nh(e - x)^{n-1}\right). \end{aligned}$$

By virtue of Lemma 3.2, Part **c)**,

$$\begin{aligned} \| \log(x + h) - \log(x) - x^{-1}h \| &\leq \sum_{n=1}^{\infty} \frac{1}{n} \| (e - x)^n - (e - x - h)^n - nh(e - x)^{n-1} \| \\ &= \sum_{n=1}^{\infty} \frac{C^2 \mu^n \| h \|^2}{n} = C^2 \| h \|^2 \sum_{n=1}^{\infty} \frac{\mu^n}{n} \\ &= -C^2 \| h \|^2 \ln(1 - \mu) = L \| h \|^2 . \end{aligned}$$

b) We have :

$$\begin{aligned} \| \log(e + h) - \log(e) - e^{-1}h \| &= \| \log(e + h) - h \| = \| -\sum_{n=1}^{\infty} \frac{(e - (e + h))^n}{n} - h \| \\ &= \| \sum_{n=2}^{\infty} \frac{(-1)^n h^n}{n} \| \leq \sum_{n=2}^{\infty} \| h \|^n = \| h \|^2 \sum_{n=2}^{\infty} \| h \|^{n-2} \\ &\leq \| h \|^2 \sum_{n=2}^{\infty} (1/2)^{n-2} = 2 \| h \|^2 . \end{aligned}$$

c) This is a direct consequence of **a)** and **b)**. In fact, from (3.10) and (3.11) we conclude that for any $x \in \mathfrak{B}$ such that $\| e - x \| < 1$ there exist a constant $D = D(x)$ and $\delta = \delta(x) > 0$ such that

$$\| \log(x + h) - \log(x) - x^{-1}h \| \leq D \| h \|^2 \quad \text{for } h \in \mathfrak{B} \text{ and } \| h \| \leq \delta. \quad (3.13)$$

From this estimation we get

$$\lim_{h\to 0}\frac{\| \log(x+h)-\log(x)-x^{-1}h \|}{\| h \|}=0.$$

From (3.8) and (3.13) we conclude that $\exp(x)$ and $\log(x)$ are continuous functions in their domain. The following theorem helps us to relate series and products.

Theorem 3.7 *Let $x, y \in \mathfrak{B}$. Then*

a) $\exp(x)\exp(y) = \exp(x+y)$. *In particular, the inverse of* $\exp(x)$ *is* $\exp(-x)$.

b)

$$\exp(\log(x)) = x \text{ if } \| e-x \| < 1. \tag{3.14}$$

c)

$$\log(\exp(x)) = x \text{ if } \| x \| < \ln 2. \tag{3.15}$$

d) *If* $\| e-x \| < 1$, $\| e-y \| < 1$ *and* $\| e-xy \| < 1$, *then* $\log(xy) = \log(x)+\log(y)$.
In general, if $a_1, a_2, \ldots, a_n \in \mathfrak{B}$, $\| e-a_i \| < 1$, $(i=1,2,\ldots,n)$ *and* $\| e-a_1a_2\cdots a_n \|$, *then*

$$\log(a_1a_2\cdots a_n) = \log(a_1)+\log(a_2)+\cdots+\log(a_n). \tag{3.16}$$

Proof.
a) Using the absolute convergence and the fact that in Banach spaces absolute convergence implies unconditional convergence, we may write

$$\begin{aligned}\exp(x+y) &= \sum_{n=0}^{\infty}\frac{(x+y)^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{n!}\binom{n}{k}x^{n-k}y^k \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{x^{n-k}}{(n-k)!}\frac{y^k}{k!} = \sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\frac{x^{n-k}}{(n-k)!}\frac{y^k}{k!} \\ &= \sum_{k=0}^{\infty}\frac{y^k}{k!}\sum_{n=k}^{\infty}\frac{x^{n-k}}{(n-k)!} = \exp(y)\exp(x) = \exp(x)\exp(y).\end{aligned}$$

b) Consider the function $f(x) = x\exp(-\log(x))$. By Lemma 3.2, Theorem 3.5 and 2.3 this function is differentiable for $\| e-x \| < 1$ and Theorem 2.2 allows us to write

$$f'(x) = \exp(-\log(x)) + x\exp(-\log(x))(-x^{-1}) = 0.$$

Consequently, by Theorem 2.4, $f = \text{const}$, say $f(x) = f(e) = e$, so $x\exp(-\log(x)) = e$ and by Part **a)**,

$$x = x\exp(-\log(x))\exp(\log(x)) = e\exp(\log(x)) = \exp(\log(x)).$$

c) Observe that

$$\| e - \exp(x) \| \leq \sum_{n=1}^{\infty} \frac{\| x \|^n}{n!} = \exp(\| x \|) - 1 < 1,$$

so the function $g(x) = \log(\exp(x)) - x$ is defined for $\| x \| < \ln 2$. We have :

$$g'(x)(h) = \exp(x)^{-1}\exp(x)(h) - h = h - h = 0.$$

This implies that $g(x) = \text{const}$, say $g(x) = g(0) = 0$ and then $\log(\exp(x)) = x$.
d) Observe that $\log(x)$, $\log(y)$ and $\log(xy)$ are defined. Fix y and consider the function $\varphi(x) = \log(xy) - \log(x) - \log(y)$. We have :

$$\varphi'(x) = y(xy)^{-1} - x^{-1} = yy^{-1}x^{-1} - x^{-1} = 0.$$

By Theorem 2.4, $\varphi(x) = \text{const}$, say $\varphi(x) = \varphi(e) = 0$ and then $\log(xy) = \log(x) + \log(y)$.

4 Concluding remarks

The definitions of exp and log we give here are the same as in Kulkarni [7]. Kulkarni proves the fundamental identities (3.14) and (3.15) for any real Banach algebra (not necessarily commutative) by using the spectral mapping theorem. We proved these fundamental identities by using facts from differential calculus. There are other approaches to the definition of these important functions. See, for example, [5] and [8].
The functions exp and log may be used to define other functions, for example $a^x = \exp(x\log(a))$ for a fixed a with $\| e - a \| < 1$ and $x \in \mathfrak{B}$. We also may use the properties of these functions to establish a relationship between series and infinite products on commutative Banach algebras.

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