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Row (Column) Bounded Operators on Operator Spaces

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Abstract. We have explored some properties of row (column) bounded operators and their relations to completely bounded operators and bounded operators on operator spaces and C*-algebras. Among other results, we obtained an equivalence of a row bounded operator with a bounded bilinear operator through the Haagerup tensor product of C*-algebras. Also, we have computed norms for some row bounded operators on matrix algebras.

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1. INTRODUCTION

The theory of completely bounded operators in operator algebras and operator spaces which has its base in Stinespring's [22] pioneering work and Arveson's [2] fundamental work on completely positive maps became important in the early 80's through the independent work of Wittstock ([24], [25]), Haagerup [11] and Paulsen [14]. An extensive study of these operators has been made by many authors and a detailed account can be found in Effros [6], Christensen and Sinclair [5] and Paulsen [15], This study has added a new dimension to most of well-known results of Functional Analysis. A very important class of operators known as row (column) bounded operators lies in between the set of completely bounded operators and the set of bounded operators; and a very less information is known about them. These operators are appeared in a cohomology problem for von Neumann algebras [21] and in

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a recent work of Arveson's [1] in C*-algebras and multivariable operator theory. In this paper our efforts are to explore various properties of row (column) bounded operators and their relations to the theory of completely bounded operators and bounded operators. Section 2 contains definitions and notations required for proving the results whereas in the third section, in addition to computing norms of some row bounded operators on matrix algebras, we have obtained an equivalence of a row bounded operator on a C*-algebra with the corresponding bounded bilinear operator on the Haagerup tensor product of the given C*-algebra.

2. PRELIMINARIES

Let $M_n(X)$ be the space of $n \times n$ - matrices with entries from a complex vector space X. In case $X = \mathbb{C}$, we will write simply M_n. We can identify M_n with $B(\mathbb{C}^n, \mathbb{C}^n)$, the set of all bounded linear operators from an n-dimensional Hilbert space $\mathbb{C}^n$ to $\mathbb{C}^n$. A matrically normed space is a vector space X together with the sequence of norms $\|.\|_n$ on $M_n(X)$ having the following properties:

$$\|x \oplus o\|_{n+m} = \|x\|_n \tag{I}$$

$$\|\alpha x\|_n \leq \|\alpha\| \, \|x\|_n \text{ and } \|x\alpha\|_n \leq \|x\|_n \|\alpha\|, \tag{II}$$

for all $x \in M_n(X), \alpha \in M_n$ and the zero element $o \in M_m(X)$. A matricially normed space is L^∞ if it satisfies the following condition:

(III)

$$\text{if } x \in M_n(X) \text{ and } y \in M_m(X), \text{then } \|x \oplus y\|_{n+m} = \max\{\|x\|_n, \|y\|_m\},$$

where $x \oplus y = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \in M_{n+m}(X)$.Let X and Y be matricially normed spaces and let $\phi : X \to Y$ be a linear map. Then for each $n \in \mathbb{N}$, we define $\phi_n : M_n(X) \to M_n(Y)$ by $\phi_n([x_{ij}]) = [\phi(x_{ij})]$, for every $[x_{ij}] \in M_n(X)$. If $\sup\{\|\phi_n\| : n \in \mathbb{N}\}$ is finite, then ϕ is known as a completely bounded operator and the completely bounded norm of ϕ is given by $\|\phi\|_{cb} = \sup\{\|\phi_n\| : n \in \mathbb{N}\}$.If $\|\phi\|_{cb} \leq 1$, then we call ϕ as a complete contraction and if each $\phi_n : M_n(X) \to M_n(Y)$ is an isometry, we call ϕ as a complete isometry.

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H with a natural operator norm. Then by a concrete operator space we mean a linear subspace of $B(H)$ with the sequence of operator norms over $B(H)$. A matricially normed space X is called an operator space if X is completely isometric to a concrete operator space. Thus every subspace of a C^*-algebra is an operator space because any faithful *-representation from a C^*-algebra to $B(H)$ is a complete isometry. In [20], Ruan has shown that a matricially normed space X is an operator space if and only if X is $L^\infty-$ matricially normed space. For more details on operator spaces we refer to ([3], [4], [8], [9], [10], [15], [16], [17], [18], [19]).

Let X and Y be C^*-algebras and let $X \otimes Y$ denote the algebraic tensor product of X and Y. Then the norm $\|.\|_h$ of an element $z \in X \otimes Y$ is defined as

$$\|z\|_h = \inf\left\{\left\|\sum_{i=1}^{n} x_i x_i^*\right\|^{\frac{1}{2}} \left\|\sum_{i=1}^{n} y_i^* y_i\right\|^{\frac{1}{2}} : z = \sum_{i=1}^{n} x_i \otimes y_i\right\}.$$

This norm is called the Haagerup norm and it was introduced by Effros and Kishimoto [7]. The Haagerup tensor product $X \otimes_h Y$ is the completion of $X \otimes Y$ in this norm. Let $R_n(X)$ be the space of all rows of length n with entries from X and $C_n(X)$ be the space of all columns of length n with entries from X. For each $n \in \mathbb{N}$, we define a row norm $\|\cdot\|_n$ on $R_n(X)$ as

$$\|[x_{1i}]\|_n = \left\|\sum_{i=1}^{n} x_{1i} x_{1i}^*\right\|^{\frac{1}{2}}, \forall\, [x_{1i}] \in R_n\,(X)\,.$$

The space X together with a family of row norms $\{\|\cdot\|_n : n \in \mathbb{N}\}$ is called a row space. In duality, we can define a column norm $\|\cdot\|_n$ on $C_n(X)$ as

$$\|[x_{i1}]\|_n = \left\|\sum_{i=1}^{n} x_{i1}^* x_{i1}\right\|^{\frac{1}{2}}, \forall\, [x_{i1}] \in C_n\,(X)\,.$$

The space X together with a family of column norms $\{\|\cdot\|_n : n \in \mathbb{N}\}$ is called a column space. It can be seen that both row and column norms are related with the Haagerup norm in the following way:

$$\|[x_{1i}]\|_n = \left\|\sum_{i=1}^{n} x_{1i} \otimes x_{1i}^*\right\|_h^{\frac{1}{2}} \text{ and } \|[x_{i1}]\|_n = \left\|\sum_{i=1}^{n} x_{i1}^* \otimes x_{i1}\right\|_h^{\frac{1}{2}}.$$

Let $\phi : X \to Y$ be a linear operator. Then for every $n \in \mathbb{N}$, we define $\phi_n : R_n(X) \to R_n(Y)$ as

$\phi_n([x_{1i}]) = [\phi(x_{1i})], \forall [x_{1i}] \in R_n\,(X)$ .Let $\|\phi_n\|_r = \sup\{\|\phi_n\,([x_{1i}])\| : [x_{1i}] \in R_n\,(X)$ and $\|[x_{1i}]\|_n \leq 1\}$. Now, if $\sup\{\|\phi_n\|_r : n \in \mathbb{N}\}$ is finite, then we call ϕ as a row bounded operator and the row bounded norm of ϕ is given by $\|\phi\|_{rb} = \sup\{\|\phi_n\|_r : n \in \mathbb{N}\}$ .If $\|\phi\|_{rb} \leq 1$, then we say that ϕ is a row contraction. Also, if each $\phi_n : R_n\,(X) \to R_n\,(Y)$ is an isometry then ϕ is called a row isometry. By duality, we can have similar definitions for column bounded operators. From these definitions, it is clear that every completely bounded operator is row (column) bounded and every row (column) bounded operator is bounded. In fact, we have the following inequalities:

$$\|\phi\| \leq \|\phi\|_{rb} \leq \|\phi\|_{cb}\,. \tag{2.1}$$

But in general the inequalities in (2.1) cannot be reversed. For example, let $B(H)$ be the C^*-algebra of all bounded linear operators on an infinite dimensional separable Hilbert space H with a countable orthonormal basis $\{\xi_i\}_{i=1}^{\infty}$. Then we can express every operator $T \in B(H)$ as an infinite matrix whose

(i,j)-th entry is the inner product $< T\xi_j, \xi_i >$. Let $\phi : B(H) \to B(H)$ be the transpose map given as

$$\phi\left([x_{ij}]_{i,j=1}^{\infty}\right) = [x_{ji}]_{j,i=1}^{\infty}, \forall\, [x_{ij}]_{i,j=1}^{\infty} \in B(H).$$

Then clearly ϕ is an isometry and moreover it is a well known example of non-completely bounded operator. But it can be easily seen that ϕ is not even row bounded operator. Further, if we take
$S = \left\{[x_{ij}]_{i,j=1}^{\infty} \in B(H) : x_{ij} = 0, \text{ for all } i > 1 \text{ and } j = 1, 2, \ldots\right\}$ as a subspace of $B(H)$ and if $\phi : S \to B(H)$ is the transpose map, then ϕ becomes row contraction which is not completely bounded. Also, the inverse open mapping theorem is not true for row bounded operators. For example, let

$$X = \left\{[\alpha_{1i}]_{i=1}^{\infty} : \sum_{i=1}^{\infty} |\alpha_{1i}|^2 < \infty\right\} \text{ and } Y = \left\{[\beta_{i1}]_{i=1}^{\infty} : \sum_{i=1}^{\infty} |\beta_{i1}|^2 < \infty\right\}.$$

Then clearly X and Y are closed subspaces of $B(H)$. Let $\phi : X \to Y$ be the transpose map. Then clearly ϕ is bijective and an isometry. We have already seen that ϕ is a row bounded operator but not completely bounded. In fact, it readily follows that ϕ is not even column bounded operator. But it is straightforward to see that the inverse of ϕ is not row bounded.

Also, in general we observe that row bounded operator and column bounded operator are adjoint to each other. That is , if $\phi : X \to B(H)$ is a linear map and $\phi^* : X \to B(H)$ is defined as $\phi^*(x) = \phi(x^*)^*$ every $x \in X$, then clearly ϕ is row bounded if and only if ϕ^* is column bounded. In particular, if the operator ϕ is self-adjoint or symmetric (that is $\phi(x^*) = \phi(x)^*$, for every $x \in X$),then ϕ is row bounded if and only if ϕ is column bounded.

3. ROW (COLUMN) BOUNDED OPERATORS

Let $X \subseteq B(H)$ be an operator space. We define a row matricially structure on $M_n(X)$ for all n as: For $[x_{ij}] \in M_n(X)$, define $\|[x_{ij}]\|_{row} = \sup\{\|[\phi(x_{ij})]\| : \phi : X \to B(K), \|\phi\|_{rb} \leq 1\}$.By using Ruan's theorem [20], it can be easily seen that X together with this new row matricial structure$\|\cdot\|_{row}$ is again an operator space.Now the following theorem is straightforward.

Theorem 3.1. *Let $X \subseteq B(H)$ and let $X_{row} = (X, \|\cdot\|_{row})$ be an operator space with $\|\cdot\|_{row}$ as a row matricial norm structure on $M_n(X)$. Then $\phi : X \to B(K)$ is row bounded if and only if $\phi : X_{row} \to B(K)$ is completely bounded. Moreover, $\|\phi\|_{rb} = 1$ if and only if $\|\phi\|_{cb} = 1$.*

In the next theorem we shall show that a row bounded operator is lifted to a bounded bilinear operator through the Haagerup tensor product of C^*-algebras.

Theorem 3.2. *Let X be a C^*-algebra and let $\phi : X \to B(H)$ be a linear map. Then ϕ is row bounded if and only if the map $\psi : X \otimes_h X \to B(H)$,* *defined by* $\psi(x,y) = \phi(x)\phi(y^*)^*$, *for every* $x, y \in X$, *is continuous. Moreover,* $\|\psi\| = \|\phi\|_{rb}^2$.

Proof. Suppose that $\phi : X \to B(H)$ is row bounded. Let $u \in X \otimes_h X$ be such that $u = \sum_{i=1}^{n} x_i \otimes y_i$. Then we have

$$\begin{aligned}
\|\psi(u)\| &= \left\|\psi\left(\sum_{i=1}^{n} x_i \otimes y_i\right)\right\| = \left\|\sum_{i=1}^{n} \psi(x_i \otimes y_i)\right\| = \left\|\sum_{i=1}^{n} \phi(x_i)\,\phi(y_i^*)^*\right\| \\
&= \left\|[\phi(x_1), \ldots, \phi(x_n)] \begin{bmatrix} \phi(y_1^*)^* \\ \phi(y_2^*)^* \\ \vdots \\ \phi(y_n^*)^* \end{bmatrix}\right\| \\
&\leq \|[\phi(x_1), \ldots, \phi(x_n)]\| \left\|\begin{bmatrix} \phi(y_1^*)^* \\ \vdots \\ \vdots \\ \phi(y_n^*)^* \end{bmatrix}\right\| \\
&= \|[\phi(x_1), \ldots, \phi(x_n)]\| \, \|[\phi(y_1^*), \ldots, \phi(y_n^*)]^*\| \\
&= \|[\phi(x_1), \ldots, \phi(x_n)]\| \, \|[\phi(y_1^*), \ldots, \phi(y_n^*)]\| \\
&\leq \|\phi\|_{rb}^2 \, \|[x_1, \ldots, x_n]\| \, \|[y_1^*, \ldots, y_n^*]\| \\
&= \|\phi\|_{rb}^2 \left\|\sum_{i=1}^{n} x_i x_i^*\right\|^{\frac{1}{2}} \left\|\sum_{i=1}^{n} y_i^* y_i\right\|^{\frac{1}{2}}.
\end{aligned}$$

Now the infimum of the right hand side is equal to $\|\phi\|_{rb}^2 \|u\|_h$ by definition of $\|\cdot\|_h$. Thus we have $\|\psi\| \leq \|\phi\|_{rb}^2$. This proves that ψ is bounded. Conversely, suppose that ψ is bounded. To show that $\phi : X \to B(H)$ is row bounded, let $[x_1, \ldots, x_n] \in R_n(X)$ be such that $\|[x_1, \ldots, x_n]\| \leq 1$. Then $\|[x_1, \ldots, x_n]\|^2 \leq 1$ implies that $\left\|\sum_{i=1}^{n} x_i x_i^*\right\| \leq 1$. Now

$$\begin{aligned}
\|\phi_n([x_1, \ldots, x_n])\|^2 = \|[\phi(x_1), \ldots, \phi(x_n)]\|^2 &= \left\|\sum_{i=1}^{n} \phi(x_i)\,\phi(x_i)^*\right\| \\
&= \left\|\sum_{i=1}^{n} \psi(x_i, x_i^*)\right\| \\
&\leq \|\psi\| \left\|\sum_{i=1}^{n} (x_i \otimes x_i^*)\right\|_h = \|\psi\| \left\|\sum_{i=1}^{n} x_i x_i^*\right\| \leq \|\psi\|.
\end{aligned}$$

Thus it follows that $\|\phi\|_{rb}^2 \leq \|\psi\|$. This proves that ϕ is row bounded and $\|\phi\|_{rb}^2 = \|\psi\|$. With this the proof of the theorem is complete. □

Now, in the following results our efforts are to compute norms of some row bounded operators in matrix algebras. We begin with the following theorem.

Theorem 3.3. *Let X and Y be operator spaces and let $\phi : X \to Y$ be a row bounded operator. Then $\phi_n : M_n(X) \to M_n(Y)$ has*

$$\|\phi_n\|_{rb} \leq \sqrt{n}\, \|\phi\|_{rb} \ .$$

Proof. For each $n \in \mathbb{N}, \phi_n : M_n(X) \to M_n(Y)$ is given by $\phi_n([x_{ij}]) = [\phi(x_{ij})]$, for every $[x_{ij}] \in M_n(X)$. Fix $k \in \mathbb{N}$. Then $(\phi_n)_k : R_k\left(M_n\left(X\right)\right)$ $\to R_k\left(M_n\left(Y\right)\right)$ is given by $(\phi_n)_k\left([x^1, \ldots\ldots, x^k]\right) = [\phi_n(x^1), \ldots, \phi_n(x^k)]$, for every $[x^1, \ldots, x^k] \in R_k\left(M_n\left(X\right)\right)$. Let $\left\|\left[x^1, \ldots, x^k\right]\right\| \leq 1$. Then we have $\left\|\sum_{l=1}^{k} \left[x_{ij}^l\right] \left[x_{ij}^l\right]^*\right\| \leq 1$. Further, it implies that $\left\|\left[x_{ij}^l\right]\right\| \leq 1$, for every $l = 1, \ldots k$ and in turn we get $\left\|\left[x_{i1}^l, \ldots, x_{in}^l\right]\right\| \leq 1$, for every $i = 1, \ldots, n$ and $l = 1, ..., k$. Now, we have

$$\begin{aligned}
\left\|(\phi_n)_k\left(\left[x^1, \ldots, x^k\right]\right)\right\|^2 &= \left\|\left[\phi_n\left(x^1\right), \ldots, \phi_n\left(x^k\right)\right]\right\|^2 \\
&= \left\|\left[\phi\left(x_{ij}^1\right)\right], \ldots, \left[\phi\left(x_{ij}^k\right)\right]\right\|^2 \\
&= \left\|\sum_{l=1}^{k} \left[\phi\left(x_{ij}^l\right)\right] \left[\phi\left(x_{ij}^l\right)\right]^*\right\| \\
&= \left\|\sum_{l=1}^{k} \left(\sum_{s=1}^{n} \phi\left(x_{is}^l\right) \phi\left(x_{js}^l\right)^*\right)\right\| \\
&\leq \left\|\left(\sum_{l=1}^{k}\sum_{s=1}^{n} \phi\left(x_{is}^l\right) \phi\left(x_{js}^l\right)^*\right)\right\| \\
&\leq n \max_{i,,j} \left\|\sum_{l=1}^{k}\sum_{s=1}^{n} \phi\left(x_{is}^l\right) \phi\left(x_{js}^l\right)^*\right\| \\
&\leq n \max_{i} \left\|\sum_{l=1}^{k}\sum_{s=1}^{n} \phi\left(x_{is}^l\right) \phi\left(x_{is}^l\right)^*\right\| \\
&\leq n \max_{i} \|\phi\|_{rb}^2 \left\|\sum_{l=1}^{k}\sum_{s=1}^{n} x_{is}^l x_{is}^{l*}\right\| \\
&\leq n \max_{i} \|\phi\|_{rb}^2 .
\end{aligned}$$

Since $\left\|\sum_{l=1}^{k}\sum_{s=1}^{n} x_{is}^l x_{is}^{l*}\right\| \leq \left\|\sum_{l=1}^{k} \left[x_{ij}^l\right] \left[x_{ij}^l\right]^*\right\| \leq 1$, therefore we have $\|(\phi_n)_k\|_r$

$\leq \sqrt{n}\,\|\phi\|_{rb}$, for every $k \in \mathbb{N}$. Thus $\|\phi_n\|_{rb} \leq \sqrt{n}\,\|\phi\|_{rb}$. This completes the proof of the theorem. □

In the above theorem if we replace the spaces X and Y by the $n \times n$-matrix algebra M_n, and take the transpose map ϕ, then we can compute the best possible constants for the row bounded norm of ϕ which is proved in the following corollary.

Corollary 3.1. *Let $\phi : M_n \to M_n$ be the transpose map. Then the norm of the row map ϕ_k is given by*

$$\|\phi_k\|_r = \begin{cases} \sqrt{k}, 1 \leq k \leq n \\ \sqrt{n}, k > n \end{cases}.$$

Proof. The map $\phi_k : R_k(M_n) \to R_k(M_n)$ is given by $\phi_k([x_{1i}]) = [\phi(x_{1_i})], \forall [x_{1i}] \in R_k(M_n)$. Let $\|[x_{1i}]\| \leq 1$. Then $\|[x_{1i}]\|^2 = \left\|\sum_{i=1}^{k} x_{1i}x_{1i}^*\right\|$ implies that $\|x_{1i}\| \leq 1$, for every $i = 1, 2, \ldots, k$. Since ϕ is an isometry, we have $\|\phi(x_{1i})\| \leq 1, \forall i = 1, \ldots, k$. Now

$$\begin{aligned} \|\phi_k([x_{1i}])\| &= \|[\phi(x_{1i})]\| \\ &= \left\|\sum_{i=1}^{k} \phi(x_{1i})\,\phi(x_{1i})^*\right\|^{\frac{1}{2}} \leq \left(\sum_{i=1}^{k} \|\phi(x_{1i})\|^2\right)^{\frac{1}{2}} \leq \sqrt{k}. \end{aligned}$$

This implies that $\|\phi_k\|_r \leq \sqrt{k}$. Let $[e_{ij}]$ be the standard matrix units in M_n and let $k \leq n$. Let $[e_{i1}]$ be a row of units in $R_k(M_n)$. Then it can be seen that

$$\|[e_{i1}]\| = \left\|\sum_{i=1}^{k} e_{i1}e_{i1}^*\right\|^{\frac{1}{2}} = 1$$

and$\|\phi_k([e_{i1}])\| = \|[e_{1i}]\| = \left\|\sum_{i=1}^{k} e_{1i}e_{1i}^*\right\|^{\frac{1}{2}} = \sqrt{k}$.This implies that $\|\phi_k\|_r \geq \sqrt{k}$ and hence $\|\phi_k\|_r = \sqrt{k}$.On the other hand, for any $k \in \mathbb{N}$, let $[x_{1i}]$ be a row in $R_k(M_n) \subseteq M_k(M_n)$ such that $\|[x_{1i}]\| \leq 1$. Now, by a well-known "Canonical shuffle" method [15] which is just a permutation, we can pass from the matrix C^*-algebra $M_k(M_n)$ to the matrix C^*-algebra $M_n(M_k)$ with norm preserving because the canonical shuffle is a *-isomorphism. So, the row $[x_{1i}]$ has the same norm when it is considered in $M_n(M_k)$ after permutation. Therefore, by considering the row $\phi_k([x_{1i}])$ as an element of $M_n(M_k)$ and using isometry of ϕ, we can have

$$\|\phi_k([x_{1i}])\| = \|[\phi(x_{1i})]\| = \left\|\sum_{i=1}^{n}\phi(x_{1i})\,\phi(x_{1i})^*\right\|^{\frac{1}{2}} \leq \left(\sum_{i=1}^{n}\|\phi(x_{1i})\|^2\right)^{\frac{1}{2}}$$
$$\leq \sqrt{n}.$$

Thus $\|\phi_k\|_r \leq \sqrt{n}$, for every $k \in \mathbb{N}$. Hence it is immediate that for $k > n, \|\phi_k\|_r = \sqrt{n}$. This completes the proof of the corollary. □

Remark 3.1. *In [23], Tamiyama has computed the completely bounded norm of the transpose map $\phi : M_n \to M_n$ as*

$$\|\phi_k\| = \begin{cases} k \;, & \text{if } 1 \leq k \leq n, \\ n \;, & \text{if } k > n. \end{cases}$$

Thus from the above corollary, we have observed that the constants for the row bounded norm of the transpose map $\phi : M_n \to M_n$ are sharper than the constants for the completely bounded norm of ϕ.

In the next theorem we shall extract information of the completely bounded norm of a given row contraction map between C^*-algebras.

Theorem 3.4. *Let X and Y be C^*-algebras and let $\phi : X \to Y$ be a row contraction. Then $\|\phi_n\| \leq \sqrt{n}, \forall n \in \mathbb{N}$.*

Proof. For each $n \in \mathbb{N}, \phi_n : M_n(X) \to M_n(Y)$ is given by

$$\phi_n([x_{ij}]) = [\phi(x_{ij})], \forall [x_{ij}] \in M_n(X).$$

Let $[x_{ij}]$ be any matrix in $M_n(X)$. Then by considering $M_n(X)$ as a C^*-subalgebra of $M_n(B(H)) = B(H^n) = B(H \oplus \ldots \oplus H)$,we can regard the matrix $[x_{ij}]$ as an operator from n-fold direct sum $H \oplus \ldots \oplus H$ to itself. Let $(e_{11}, e_{21}, \ldots, e_{n1})$ be a vector in $H \oplus \ldots \oplus H$ such that $\|(e_{11}, e_{21}, \ldots, e_{n1})\| \leq 1$. Then

$$\|[x_{ij}]\,[e_{i1}]\|^2 = \|[x_{1j}]\,[e_{i1}]\|^2 + \|[x_{2j}]\,[e_{i1}]\|^2 + \cdots\cdots + \|[x_{nj}]\,[e_{i1}]\|^2$$
$$\leq \|[x_{1j}]\|^2\,\|[e_{i1}]\|^2 + \|[x_{2j}]\|^2\,\|[e_{i1}]\|^2 + \cdots + \|[x_{nj}]\|^2\,\|[e_{i1}]\|^2$$
$$= \left[\sum_{i=1}^{n}\|[x_{ij}]\|^2\right]\|[e_{i1}]\|^2.$$

Further, it implies that $\|[x_{ij}]\|^2 \leq \sum_{i=1}^{n}\|[x_{ij}]\|^2$ .That is, $\|[x_{ij}]\| \leq \left[\sum_{i=1}^{n}\|[x_{ij}]\|^2\right]^{1/2}$.

Now, we are given that $\|\phi\|_{rb} \leq 1$. This implies that $\|\phi_n\|_r \leq 1, \forall n \in \mathbb{N}$. Further, it follows that $\|[\phi(x_{1j})]\| \leq 1$, for every $[x_{1j}] \in R_n(X)$ such that

$\|[x_{1j}]\| \leq 1$. Now to show that $\|\phi_n\| \leq \sqrt{n}$, for every $n \in \mathbb{N}$,let $[x_{ij}] \in M_n(X)$ be such that $\|[x_{ij}]\| \leq 1$. From this, it is immediate that $\|[x_{ij}]\| \leq 1$, for every $i = 1, \ldots, n$. Further, it yields that $\|[\phi(x_{ij})]\| \leq 1$, for every $i = 1, \ldots, n$. Thus it follows that $\|\phi_n([x_{ij}])\| = \|[\phi(x_{ij})]\| \leq \left[\sum_{i=1}^{n} \|[\phi(x_{ij})]\|^2\right]^{\frac{1}{2}} \leq \sqrt{n}$.That is, $\|\phi_n\| \leq \sqrt{n}, \forall n \in \mathbb{N}$. This completes the proof of the theorem. □

If the C^*-algebras X and Y of the above theorem are replaced by the matrix algebra M_n, then we can compute the best possible constants for the related norm and which we shall prove in the following corollary.

Corollary 3.2. *Let $\phi : M_n \to M_n$ be a row contraction map. Then*

$$\|\phi_k\| \leq \begin{cases} \sqrt{k} & , \quad \text{if } 1 \leq k \leq n, \\ \sqrt{n} & , \quad \text{if } k > n. \end{cases}$$

Proof. If $k \leq n$, then it is immediate from the above theorem that $\|\phi_k\| \leq \sqrt{k}$. Now, suppose that $k > n$. Let $\phi_k : M_k(M_n) \to M_k(M_n)$ be given by

$$\phi_k([x_{ij}]) = [\phi(x_{ij})], \forall [x_{ij}] \in M_k(M_n).$$

Let $[x_{ij}] \in M_k(M_n)$ be such that $\|[x_{ij}]\| \leq 1$. Now, since $M_k(M_n)$ and $M_n(M_k)$ are *-isomorphic to each other by canonical shuffle method [15], therefore for $[x_{ij}]$ in $M_n(M_k)$, we have $\|[x_{ij}]\| \leq 1$. From this, it readily follows that $\|[x_{i1}, x_{i2} \ldots x_{in}]\| \leq 1$, for every $i = 1, \ldots, n$. By the given hypothesis, we have $\|[\phi(x_{i1}), \phi(x_{i2}), \ldots, \phi(x_{in})]\| \leq 1$, for every $i = 1, \ldots, n$. Now by considering $\phi_k([x_{ij}])$ as the element of $M_n(M_k)$, it follows that

$$\|\phi_k([x_{ij}])\| = \|[\phi(x_{ij})]\| \leq \left[\sum_{i=1}^{n} \|[\phi(x_{i1}), \phi(x_{i2}), \ldots, \phi(x_{in})]\|^2\right]^{\frac{1}{2}} \leq \sqrt{n}.$$

Thus $\|\phi_k\| \leq \sqrt{n}$, for $k > n$. This completes the proof of the corollary. □

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