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Approximation Methods for Solving Equilibrium Problems and Variational Inequality Problems of an Infinite Family of Nonexpansive Mappings[1]

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Abstract

In this paper, we introduce an iterative scheme by a modified hybrid extragradient method for finding a common element of the set of fixed points of a countable family of nonexpansive mappings, the set of solutions of an equilibrium problem and the set of solutions of the variational inequality for α-inverse-strongly monotone mappings in a real Hilbert space. We show that the iterative sequence converges strongly to a common element of the above three sets, which solves some fixed

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point problems, variational inequality problems and equilibrium problems by using the hybrid method in mathematical programming which are connected with optimization problems. The results extend and improve the recent result of Kumam [11], Shinzato and Takahashi [21], Tada and Takahashi [24] and Takahashi et. al. [27].

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1 Introduction

Throughout this paper, we always assume that H is a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. Let C be a closed convex subset of H and let P_C be the metric projection of H onto C. Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. Let $D : C \to H$ be a nonlinear mapping. Then, we consider the following equilibrium problem: Find $z \in C$ such that

$$F(z, y) + \langle Dz, y - z \rangle \geq 0, \forall y \in C \tag{1.1}$$

The set of such $z \in C$ is denoted by EP, i.e.,

$$EP = \{z \in C : F(z, y) + \langle Dz, y - z \rangle \geq 0, \forall y \in C\}. \tag{1.2}$$

In the case of $D \equiv 0$, EP is denoted by $EP(F)$, that is, to find an element $z \in C$ such that $F(z, y) \geq 0, \ \forall y \in C$. This problem contains fixed point problems, includes as special cases numerous problems in physics, optimization, and economics. Some methods have been proposed to solve the equilibrium problem, please consult [4, 5, 22]. Let A of C in to H be a nonlinear mapping. The classical *variational inequality problem* is to find $u \in C$ such that $\langle v - u, Au \rangle \geq 0$ for all $v \in C$. We denoted by $VI(C, A)$ the set of solutions of this variational inequality problem. The variational inequality has been extensively studied in the literature; see [31, 33] and the references therein.

The above formulation (1.1) was shown in [3] to cover monotone inclusion problems, saddle point problems, variational inequality problems, minimization problems, optimization problems, variational inequality problems, vector equilibrium problems, Nash equilibria in noncooperative games. In addition, there are several other problems, for example, the complementarity problem, fixed point problem and optimization problem, which can also be written in the form of an EP(F). In other words, the EP(F) is an unifying model for several problems arising in physics, engineering, science, optimization, economics, etc. In the last two decades, many papers have appeared in the literature on the existence of solutions of EP(F); see, for example [3, 5, 14, 22] and references therein. Some solution methods have been proposed to solve the EP(F); see, for example, [4, 5, 22, 24, 25] and references therein. In 2005, Combettes and Hirstoaga [4] introduced an iterative scheme of finding the best approximation to the initial data when EP(F) is nonempty and they also proved a strong convergence theorem. They also studied the strong convergence of the sequences generated by their algorithm to a solution of $EP(F)$ which is also a fixed point of a nonexpansive mapping on a closed convex subset of a Hilbert space.

Recall, a mapping $S : C \to C$ is said to be *nonexpansive* if

$$\|Sx - Sy\| \leq \|x - y\|,$$

for all $x, y \in C$. We denote by F(S) the set of fixed points of S. If C is bounded closed convex and S is a nonexpansive mapping of C into itself, then F(S) is nonempty (see [9]). We write $x_n \to x$ ($x_n \rightharpoonup x$, resp.) if $\{x_n\}$ converges (weakly, resp.) to x. A mapping A of C into H is called *monotone* if

$$\langle Au - Av, u - v\rangle \geq 0,$$

A mapping A of C into H is called *α-inverse-strongly-monotone* if there exists a positive real number α such that

$$\langle Au - Av, u - v\rangle \geq \alpha\|Au - Av\|^2, \tag{1.3}$$

for all $u, v \in C$. It is obvious that any α-inverse-strongly-monotone mapping A is monotone and Lipschitz continuous.

In 1953, Mann [13] introduced the iteration as follows: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Sx_n \tag{1.4}$$

where the initial guess element $x_0 \in C$ is arbitrary and $\{\alpha_n\}$ is a real sequence in $[0, 1]$. The Mann iteration has been extensively investigated for nonexpansive mappings. One of the fundamental convergence results is proved by Reich [18]. In an infinite-dimensional Hilbert space, the Mann iteration can conclude only weak convergence [6]. Attempts to modify the Mann iteration method (1.4) so that strong convergence is guaranteed have recently been made. Generally speaking, the algorithm suggested by Takahashi and Toyoda [26] is based on two well-known types of methods, namely, on the projection-type methods for solving variational inequality problems and so-called hybrid or outer-approximation methods for solving fixed point problems. The idea of "hybrid" or "outer-approximation" types of methods was originally introduced by Haugazeau in 1968; see [2] for more details.

Recently, for finding an element of $EP(F) \cap F(S)$, Tada and Takahashi [24] introduced the following iterative scheme by the hybrid method in a Hilbert space: $x_0 = x \in H$ and let

$$\begin{cases} u_n \in C \text{ such that } \quad F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ w_n = (1-\alpha_n)x_n + \alpha_n S u_n, \\ C_n = \{z \in H : \|w_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{1.5}$$

for every $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$ and $\{r_n\} \subset (0, \infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$. Further, they proved $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F(S) \cap EP(F)$, where $z = P_{F(S)\cap EP(F)}x_0$.

On the other hand, for finding an element of $F(S) \cap VI(C, A)$ under the assumption that a set $C \subset H$ is closed and convex, a mapping S of C into itself is nonexpansive and a mapping A of C into H is α -inverse-strongly-monotone, Takahashi and Toyoda [26] introduced the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) S P_C(x_n - \lambda_n A x_n) \tag{1.6}$$

for every $n = 0, 1, 2, ...$, where $x_0 = x \in C, \{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. They shown that, if $F(S) \cap VI(C, A) \neq \emptyset$, then such a sequence $\{x_n\}$ converges weakly to some $z \in P_{F(S)\cap VI(C,A)}x$.

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see [28, 29, 30] and the references therein. Convex minimization problems have a great impact and influence in the development of almost all branches of pure and applied sciences.

Recently, motivated by the idea of Korpelevich's extragradient method [10], Nadezhkina and Takahashi [15] introduced an iterative scheme for finding an element of $F(S) \cap VI(C, A)$ and the weak convergence theorem is presented. Very recently, Yao, Liou and Yao [32] introduced the following iterative scheme for finding an element of $F(S) \cap VI(C, A)$ under some mild conditions. Let C be a closed convex subset of a real Hilbert space H, $A : C \to H$ a monotone, L-Lipschitz continuous mapping and S a nonexpansive mapping of C into itself such that $F(S) \cap VI(C, A) \neq \emptyset$. Suppose $x_1 = u \in C$ and $\{x_n\}, \{y_n\}$ are given by

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(x_n - \lambda_n A y_n), \forall n \in \mathbb{N}, \end{cases} \tag{1.7}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq [0, 1]$ and $\{\lambda_n\} \subseteq (0, 1)$ satisfy some parameters controlling conditions. They proved that the sequence $\{x_n\}$ defined by (1.7) converges strongly to a common element of $F(S) \cap VI(C, A)$.

In 2007, Takahashi, Takeuchi and Kubota [27] proved the following strong convergence theorem by using the hybrid method in mathematical programming. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, define a sequence as follows:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)S_n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases} \tag{1.8}$$

where $0 \leq \alpha_n < \alpha < 1$ for all $n \in \mathbb{N}$. They proved a strongly convergence theorem in a Hilbert space under $\{S_n\}$ satisfy the following conditions: Suppose that $\sum_n^\infty \sup\{\|S_{n+1}z - S_n z\| : z \in B\} < \infty$ for any bounded subset B of C. Let S be a mapping of C into itself defined by $Sy = \lim_{n\to\infty} S_n y$ for all $y \in C$ and suppose that $F(S) = \cap_{n=1}^\infty F(S_n)$.

In this paper, motivated and inspired by the above results, we introduce a new following iterative scheme (3.1) below for finding a common element of the set of fixed points of a countable family of nonexpansive mappings, the set of solutions of an equilibrium problem and the set of solutions of the variational inequality for α-inverse-strongly monotone mappings in a Hilbert space. Consequently, we prove a strong convergence theorem by the new hybrid iterative algorithm method in the mathematical programming which solves some fixed point problems, equilibrium problems and variational inequality problems under some parameters controlling conditions. Our results are extend and improve the recent result of Kumam[11, 12], Shinzato and Takahashi [21], Tada and Takahashi[24] and Takahashi et. al. [27].

2 Preliminaries

Let H be a real Hilbert space. Then

$$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle \tag{2.1}$$

and

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2 \tag{2.2}$$

for all $x, y \in H$ and $\lambda \in [0, 1]$. It is also known that H satisfies the *Opial's condition* [16], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$. Hilbert space H satisfies the *Kadec-Klee property* [7, 23], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ together imply $\|x_n - x\| \to 0$.

Let C be a closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \quad \text{for all } y \in C.$$

P_C is called the *metric projection* of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2 \tag{2.3}$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \tag{2.4}$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2 \tag{2.5}$$

for all $x \in H, y \in C$.

In the context of the variational inequality problem, this implies that

$$u \in VI(C, A) \Leftrightarrow u = P_C(u - \lambda A u), \text{ for all } \lambda > 0. \tag{2.6}$$

We also have that, for all $u, v \in C$ and $\lambda > 0$,

$$\begin{aligned} \|(I - \lambda A)u - (I - \lambda A)v\|^2 &= \|(u - v) - \lambda(Au - Av)\|^2 \\ &= \|u - v\|^2 - 2\lambda\langle u - v, Au - Av\rangle \\ &\quad + \lambda^2\|Au - Av\|^2 \\ &\leq \|u - v\|^2 + \lambda(\lambda - 2\alpha)\|Au - Av\|^2. \end{aligned} \tag{2.7}$$

So, if $\lambda \leq 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping from C to H.

A set valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H, \langle x - y, f - h\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let A be an inverse-strongly monotone mapping of C into H and let $N_C v$ be the normal cone to C at $v \in C$, i.e.,

$$N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \forall u \in C\}$$

and define

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$; see [19, 20].

The following lemmas will be useful for proving the convergence result of this paper.

Lemma 2.1. [17, Lemma 3.1] *Let C be a nonempty bounded closed convex subset of Hilbert space H and $\{T_n\}$ a sequence of mappings of C into itself. Suppose that*

$$\lim_{k,l\to\infty} \rho_l^k = 0$$

where $\rho_l^k = \sup\{\|T_k z - T_l z\| : z \in C\} < \infty$, for all $k, l \in \mathbb{N}$. Then for each $x \in C$, $\{T_n x\}$ converges strongly to some point of C. Moreover, let T be a mapping from C in to itself defined by

$$Tx = \lim_{n\to\infty} T_n x, \textit{for all } x \in C.$$

Then $\limsup_{n\to\infty}\{\|Tz - T_n z\| : z \in C\} = 0$.

For solving the equilibrium problem, let us assume that the bifunction F satisfies the following conditions (see [3]):

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \le 0$ for any $x, y \in C$;

(A3) F is upper-hemicontinuous, i.e., for each $x, y, z \in C$,

$$\limsup_{t\to 0^+} F(tz + (1-t)x, y) \le F(x, y);$$

(A4) $F(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$.

The following lemma appears implicitly in [3]

Lemma 2.2. [3] *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbf{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0 \textit{ for all } y \in C.$$

The following lemma was also given in [4].

Lemma 2.3. [4] *Assume that $F : C \times C \to \mathbf{R}$ satisfies (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \forall y \in C\}$$

for all $z \in H$. Then, the following hold:

(i) *T_r is single-valued;*

(ii) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$, $\|T_r x - T_r y\|^2 \le \langle T_r x - T_r y, x - y\rangle$;*

(iii) $F(T_r) = EP(F)$;

(iv) *$EP(F)$ is closed and convex.*

3 Strong convergence theorems

In this section, we show a strong convergence theorem which solves the problem of finding a common element of the set of fixed points of a countable family of nonexpansive mappings, the set of solutions of an equilibrium problem and the set of solutions of the variational inequality of an α-inverse-strongly monotone mapping in a Hilbert space by the new hybrid method in the mathematical programming.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying* (A1)-(A4). *Let A be an α-inverse-strongly monotone mapping of C into H and let D be an β-inverse-strongly monotone mapping of C into H, respectively. Let $\{S_n\}$ be a sequence of nonexpansive mapping from C into H such that $\Omega := \cap_{n=1}^{\infty} F(S_n) \cap VI(C, A) \cap EP \neq \emptyset$. For $x_0 \in H$, $C_1 = C$, $x_1 = P_{C_1}x_0$, define sequences $\{x_n\}$ and $\{u_n\}$ of C as follows:*

$$\begin{cases} u_n \in C \text{ such that} \\ F(u_n, y) + \langle Dx_n, y - u_n\rangle + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ z_n = P_C(u_n - \lambda_n Au_n), \\ y_n = \alpha_n x_n + (1 - \alpha_n)S_n P_C(z_n - \lambda_n Az_n), \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases} \tag{3.1}$$

for every $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0, 1)$, $\{\lambda_n\} \subset [a, b] \subset (0, 2\alpha)$ and $\{r_n\} \subset [c, d] \subset (0, 2\beta)$ satisfies $\liminf_{n\to\infty} r_n > 0$. Suppose that $\sum_{k,l}^{\infty} \sup\{\|S_k z - S_l z\| : z \in B\} < \infty$ for any bounded subset B of C. Let S be a mapping of C into itself defined by $Sy = \lim_{n\to\infty} S_n y$ for all $y \in C$ and suppose that $F(S) = \cap_{n=1}^{\infty} F(S_n)$. Then $\{x_n\}$ converges strongly to $P_\Omega x_0$.

Proof. We show first that the sequence $\{x_n\}$ is well defined. From the definition of C_n, it is obvious that C_n is closed for all $n \geq 0$. Since $C_n = \{z \in C : \|y_n - x_n\|^2 + 2\langle y_n - x_n, x_n - z\rangle \leq 0\}$, we deduce that C_n is convex for all $n \geq 0$. Next we show by mathematical induction that $\Omega \subset C_n$ for all $n \in \mathbb{N}$. It is obvious that $\Omega \subset C = C_1$. Suppose that $\Omega \subset C_k$ for each $k \in \mathbb{N}$. Hence, for $v \in \Omega \subset C_k$ and let $\{T_{r_n}\}$ be a sequence of mappings defined as in Lemma 2.3. Then $v = P_C(v - \lambda_n Av) = T_{r_n}(I - r_n D)v$. Put $v_n = P_C(z_n - \lambda_n Az_n)$ and from $u_n = T_{r_n}(x_n - r_n Dx_n)$, we have

$$\begin{aligned} \|v_n - v\| &= \|P_C(z_n - \lambda_n Az_n) - P_C(v - \lambda_n Av)\| \\ &\leq \|(z_n - \lambda_n Az_n) - (v - \lambda_n Av)\| \\ &= \|(I - \lambda_n A)z_n - (I - \lambda_n A)v\| \end{aligned}$$

$$\leq \|z_n - v\| \tag{3.2}$$

since $r_n \leq 2\beta$ by (2.7), then $I - r_n D$ is nonexpansive, for all $n \in \mathbb{N}$ and from T_{r_n} is firmly nonexpansive, we get

$$\begin{aligned}
\|z_n - v\| &= \|P_C(u_n - \lambda_n Au_n) - P_C(v - \lambda_n Av)\| \\
&\leq \|(u_n - \lambda_n Au_n) - (v - \lambda_n Av)\| \\
&\leq \|u_n - v\| \qquad (3.3)\\
&= \|T_{r_n}(x_n - r_n Dx_n) - T_{r_n}(v - r_n Dv)\| \\
&\leq \|(x_n - r_n Dx_n) - (v - r_n Dv)\| \\
&\leq \|x_n - v\| \qquad (3.4)
\end{aligned}$$

which implies that

$$\|v_n - v\| \leq \|x_n - v\|$$

for every $n \in \mathbb{N}$. Thus, we obtain

$$\begin{aligned}
\|y_n - v\| &= \|\alpha_n x_n + (1 - \alpha_n) S_n P_C(z_n - \lambda_n Az_n) - v\| \\
&\leq \alpha_n \|x_n - v\| + (1 - \alpha_n)\|S_n v_n - v\| \\
&\leq \alpha_n \|x_n - v\| + (1 - \alpha_n)\|v_n - v\| \\
&\leq \alpha_n \|x_n - v\| + (1 - \alpha_n)\|x_n - v\| \\
&= \|x_n - v\|. \qquad (3.5)
\end{aligned}$$

So, we have $v \in C_n$. This implies that

$$\Omega \subset C_n \quad \text{for all } n \in \mathbb{N}. \tag{3.6}$$

This implies that $\{x_n\}$ is well-defined. From Lemma 2.2, the sequence $\{u_n\}$ is also well defined. From $x_n = P_{C_n} x_0$, we have

$$\langle x_0 - x_n, x_n - y\rangle \geq 0$$

for each $y \in C_n$. Using $\Omega \subset C_n$, we also have

$$\langle x_0 - x_n, x_n - u\rangle \geq 0 \text{ for each } u \in \Omega \text{ and } n \in \mathbb{N}.$$

Then, for $u \in \Omega$, we obtain

$$\begin{aligned}
0 &\leq \langle x_0 - x_n, x_n - u\rangle \\
&= \langle x_0 - x_n, x_n - x_0 + x_0 - u\rangle \\
&= -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\|x_0 - u\|.
\end{aligned}$$

This implies that

$$\|x_0 - x_n\| \leq \|x_0 - u\| \text{ for all } \ u \in \Omega \ \text{ and } \ n \in \mathbb{N}.$$

Since $x_n = P_{C_n}x_0$ and $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, we get

$$\langle x_0 - x_n, x_n - x_{n+1}\rangle \geq 0. \tag{3.7}$$

It follow that, for $n \in \mathbb{N}$,

$$\begin{aligned} 0 &\leq \langle x_0 - x_n, x_n - x_{n+1}\rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1}\rangle \\ &= -\|x_0 - x_n\|^2 + \langle x_0 - x_n, x_0 - x_{n+1}\rangle \\ &\leq -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\|x_0 - x_{n+1}\| \end{aligned}$$

and hence

$$\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\|.$$

Since $\{\|x_n - x_0\|\}$ is bounded, then $\lim_{n\to\infty}\|x_n - x_0\|$ exists.
Next, we can show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. In deed, from (3.7) we have

$$\begin{aligned} \|x_n - x_{n+1}\|^2 &= \|x_n - x_0 + x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &= -\|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_n + x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &\leq -\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2. \end{aligned}$$

Since $\lim_{n\to\infty}\|x_0 - x_n\|$ exists, this implies that

$$\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0. \tag{3.8}$$

Since $x_{n+1} \in C_n$, we have

$$\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \leq 2\|x_n - x_{n+1}\|.$$

By (3.8), we have

$$\lim_{n\to\infty}\|x_n - y_n\| = 0, \tag{3.9}$$

and we also have

$$\|y_n - x_n\| = (1 - \alpha_n)\|S_n v_n - x_n\| \to 0 \text{ as } n \to \infty. \tag{3.10}$$

Next, we show that $\|x_n - u_n\| \to 0$. For $v \in \Omega$, from nonexpansive of P_C and $I - \lambda_n A$, we get

$$\|v_n - v\|^2 = \|P_C(z_n - \lambda_n A z_n) - P_C(v - \lambda_n A v)\|^2$$

$$\begin{aligned} &\le \|z_n - v\|^2 \\ &\le \|u_n - v\|^2. \end{aligned} \tag{3.11}$$

By Lemma 2.3 we apply (2.7) and definition of $\{u_n\}$ that

$$\begin{aligned} \|y_n - v\|^2 &= \alpha_n\|x_n - v\|^2 + (1-\alpha_n)\|S_n v_n - v\|^2 - (1-\alpha_n)\alpha_n\|x_n - S_n v_n\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|v_n - v\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|u_n - v\|^2 \\ &= \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|T_{r_n}(x_n - r_n Dx_n) - T_{r_n}(v - r_n Dv)\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|(I - r_n D)x_n - (I - r_n D)v\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)[\|x_n - v\|^2 + r_n(r_n - 2\beta)\|Dx_n - Dv\|^2] \\ &= \|x_n - v\|^2 + c(d - 2\beta)\|Dx_n - Dv\|^2, \end{aligned}$$

and hence

$$\begin{aligned} c(2\beta - d)\|Dx_n - Dv\|^2 &\le \|x_n - v\|^2 - \|y_n - v\|^2 \\ &= \|x_n - y_n\|(\|x_n - v\| + \|y_n - v\|). \end{aligned}$$

From (3.9), we have $\|Dx_n - Dv\| \to 0$, as $n \to \infty$.

Since $\lambda_n \le 2\beta$ then $I - r_n D$ is nonexpansive, for all $n \in \mathbb{N}$ and T_{r_n} is firmly nonexpansive we have

$$\begin{aligned} \|u_n - v\|^2 &= \|T_{r_n}(x_n - r_n Dx_n) - T_{r_n}(v - r_n Dv)\|^2 \\ &\le \langle (x_n - r_n Dx_n) - (p - r_n Dv), u_n - v\rangle \\ &= \frac{1}{2}(\|(x_n - r_n Dx_n) - (v - r_n Dv)\|^2 \\ &\quad + \|u_n - v\|^2 - \|(x_n - r_n Dx_n) - (p - r_n Dv) - (u_n - v)\|^2) \\ &\le \frac{1}{2}(\|x_n - v\|^2 \\ &\quad + \|u_n - v\|^2 - \|(x_n - u_n) - r_n(Dx_n - Dv)\|^2) \\ &= \frac{1}{2}(\|x_n - v\|^2 + \|u_n - v\|^2 - \|x_n - u_n\|^2 \\ &\quad + 2r_n\langle x_n - u_n, Dx_n - Dv\rangle - r_n^2\|Dx_n - Dv\|^2). \end{aligned}$$

Thus, we obtain

$$\|u_n - v\|^2 \le \|x_n - v\|^2 - \|x_n - u_n\|^2 + 2r_n\langle x_n - u_n, Dx_n - Dv\rangle - r_n^2\|Dx_n - Dv\|^2, \tag{3.12}$$

and hence

$$\|u_n - v\|^2 \le \|x_n - v\|^2 - \|x_n - u_n\|^2 + 2r_n\|x_n - u_n\|\|Dx_n - Dv\|. \tag{3.13}$$

By (3.11) and (3.13), it follows that

$$\begin{aligned}\|y_n - v\|^2 &= \alpha_n\|x_n - v\|^2 + (1-\alpha_n)\|Sv_n - v\|^2 - \alpha_n(1-\alpha_n)\|x_n - Sv_n\|^2 \\ &\leq \alpha_n\|x_n - v\|^2 + (1-\alpha_n)\|v_n - v\|^2 \\ &\leq \alpha_n\|x_n - v\|^2 + (1-\alpha_n)\|u_n - v\|^2 \\ &\leq \alpha_n\|x_n - v\|^2 + (1-\alpha_n)\{\|x_n - v\|^2 \\ &\quad -\|x_n - u_n\|^2 + 2r_n\|x_n - u_n\|\|Bx_n - Bv\|\} \qquad (3.14)\end{aligned}$$

Since $\{\alpha_n\} \subset [0,1)$, we get

$$\begin{aligned}\|x_n - u_n\|^2 &\leq \|x_n - v\|^2 - \|y_n - v\|^2 + 2r_n(1-\alpha_n)\|x_n - u_n\|\|Dx_n - Dv\| \\ &\leq \|x_n - y_n\|\{\|x_n - v\| + \|y_n - v\|\} + 2r_n\|x_n - u_n\|\|Dx_n - Dv\|.\end{aligned}$$

From this (3.9) and $\|Dx_n - Dv\| \to 0$, imply

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \qquad (3.15)$$

Since $\liminf_{n\to\infty} r_n > 0$, we also have

$$\lim_{n\to\infty}\|\frac{x_n - u_n}{r_n}\| = \lim_{n\to\infty}\frac{1}{r_n}\|x_n - u_n\| = 0. \qquad (3.16)$$

From (3.9) and (3.15), we get

$$\|u_n - y_n\| \leq \|u_n - x_n\| + \|x_n - y_n\| \to 0.$$

Next, we show that $\|u_n - z_n\| \to 0$. For $v \in \Omega$, from (2.7) and (3.2), we compute that

$$\begin{aligned}\|y_n - v\|^2 &\leq \alpha_n\|x_n - v\|^2 + (1-\alpha_n)\|S_nv_n - v\|^2 \\ &\leq \alpha_n\|x_n - v\|^2 + (1-\alpha_n)\|v_n - v\|^2 \\ &\leq \|x_n - v\|^2 + (1-\alpha_n)\|z_n - v\|^2 \\ &= \|x_n - v\|^2 + (1-\alpha_n)\|P_C(u_n - \lambda_nAu_n) - P_C(v - \lambda_nAv)\|^2 \\ &\leq \|x_n - v\|^2 + (1-\alpha_n)\|(I-\lambda_nA)u_n - (I-\lambda_nA)v\|^2 \\ &\leq \|x_n - v\|^2 + (1-\alpha_n)(\|u_n - v\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Au_n - Av\|^2) \\ &\leq \|x_n - v\|^2 + \|u_n - v\|^2 + (1-\alpha_n)a(b-2\alpha)\|Au_n - Av\|^2.\end{aligned}$$

Therefore, we have

$$\begin{aligned}&-(1-\alpha_n)a(b-2\alpha)\|Au_n - Av\|^2 \\ &\leq \|x_n - v\|^2 + \|u_n - v\|^2 - \|y_n - v\|^2 \\ &\leq \|x_n - v\|^2 + (\|u_n - v\| + \|y_n - v\|)\|u_n - y_n\|. \qquad (3.17)\end{aligned}$$

Since $\alpha_n \in [0,1), a, b \in (0, 2\alpha)$ and $\|u_n - y_n\| \to 0$, we obtain $\|Au_n - Av\| \to 0$. On the other hand, by (2.3) and (3.2), we have

$$\begin{aligned}
\|z_n - v\|^2 &= \|P_C(u_n - \lambda_n Au_n) - P_C(v - \lambda_n Av)\|^2 \\
&\le \langle (u_n - \lambda_n Au_n) - (v - \lambda_n Av), z_n - v\rangle \\
&= (1/2)\{\|(u_n - \lambda_n Au_n) - (v - \lambda_n Av)\|^2 + \|z_n - v\|^2 \\
&\quad - \|[(u_n - \lambda_n Au_n) - (v - \lambda_n Av)] - (z_n - v)\|^2\} \\
&\le (1/2)\{\|u_n - v\|^2 + \|z_n - v\|^2 - \|(u_n - z_n) - \lambda_n(Au_n - Av)\|^2\} \\
&= (1/2)\{\|u_n - v\|^2 + \|z_n - v\|^2 - \|(u_n - z_n)\|^2 \\
&\quad + 2\lambda_n \langle u_n - z_n, Au_n - Av\rangle - \lambda_n^2 \|Au_n - Av\|^2\}
\end{aligned}$$

So, we obtain

$$\begin{aligned}
\|z_n - v\|^2 &\le \|u_n - v\|^2 - \|u_n - z_n\|^2 \\
&\quad + 2\lambda_n \langle u_n - v_n, Au_n - Av\rangle - \lambda_n^2 \|Au_n - Av\|^2
\end{aligned}$$

and hence

$$\begin{aligned}
\|y_n - v\|^2 &= \|\alpha_n x_n + (1-\alpha_n) S_n v_n - v\|^2 \\
&\le \alpha_n \|x_n - v\|^2 + (1-\alpha_n)\|v_n - v\|^2 \\
&\le \alpha_n \|x_n - v\|^2 + (1-\alpha_n)\|z_n - v\|^2 \\
&\le \|x_n - v\|^2 + \|u_n - v\|^2 - \|u_n - z_n\|^2 \\
&\quad + 2\lambda_n \langle u_n - v_n, Au_n - Av\rangle - \lambda_n^2 \|Au_n - Av\|^2.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
\|u_n - z_n\|^2 &\le \|x_n - v\|^2 + \|u_n - v\|^2 - \|y_n - v\|^2 \\
&\quad + 2\lambda_n \langle u_n - z_n, Au_n - Av\rangle - \lambda_n^2 \|Au_n - Av\|^2 \\
&\le \|x_n - v\|^2 + (\|u_n - v\|^2 - \|y_n - v\|^2) \\
&\quad + 2\lambda_n \langle u_n - z_n, Au_n - Av\rangle - \lambda_n^2 \|Au_n - Av\|^2 \\
&\le \|x_n - v\|^2 + (\|u_n - v\| - \|y_n - v\|) \times \|u_n - y_n\| \\
&\quad + 2\lambda_n \langle u_n - z_n, Au_n - Av\rangle - \lambda_n^2 \|Au_n - Av\|^2.
\end{aligned}$$

Since $\|u_n - y_n\| \to 0$ and $\|Au_n - Av\| \to 0$, we obtain

$$\lim_{n\to\infty} \|u_n - z_n\| = 0. \tag{3.18}$$

From (3.15) and (3.18), we also have

$$\begin{aligned}
\|x_n - v_n\| &\le \|x_n - u_n\| + \|u_n - z_n\| + \|z_n - v_n\| \\
&\le \|x_n - u_n\| + \|u_n - z_n\| + \|P_C(u_n - \lambda_n Au_n) - P_C(z_n - \lambda_n Az_n)\|
\end{aligned}$$

$$\begin{aligned}
&\leq \|x_n - u_n\| + \|u_n - z_n\| + \|(u_n - \lambda_n A u_n) - (z_n - \lambda_n A z_n)\| \\
&\leq \|x_n - u_n\| + \|u_n - z_n\| + \|u_n - z_n\| \\
&\leq \|x_n - u_n\| + 2\|u_n - z_n\| \to 0 \text{ as } n \to \infty. \qquad (3.19)
\end{aligned}$$

On the other hand, we note that

$$\begin{aligned}
\|S_n x_n - x_n\| &\leq \|S_n x_n - S_n v_n\| + \|S_n v_n - x_n\| \\
&\leq \|x_n - v_n\| + \|S_n v_n - x_n\|.
\end{aligned}$$

By (3.10) and (3.19), this imply that

$$\lim_{n\to\infty} \|S_n x_n - x_n\| = 0. \qquad (3.20)$$

It now follows from(3.19) and (3.20) that

$$\begin{aligned}
\|S_n v_n - v_n\| &\leq \|S_n v_n - S_n x_n\| + \|S_n x_n - x_n\| + \|x_n - v_n\| \\
&\leq 2\|v_n - x_n\| + \|S_n x_n - x_n\| \to 0 \text{ as } n \to \infty. \qquad (3.21)
\end{aligned}$$

Applying Lemma 2.1 and (3.21), we have

$$\begin{aligned}
\|S v_n - v_n\| &\leq \|S v_n - S_n v_n\| + \|S_n v_n - v_n\| \\
&\leq \sup\{\|Sv - S_n v\| : v \in \{v_n\}\} + \|S_n v_n - v_n\| \to 0 \text{ as } n \to \infty.
\end{aligned}$$

It follows from the last inequality, (3.15) and (3.18), that

$$\|x_n - S v_n\| \leq \|x_n - u_n\| + \|u_n - v_n\| + \|v_n - S v_n\| \to 0 \text{ as } n \to \infty.$$

Since $\{v_n\}$ is bounded, there exists a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ which converges weakly to z. From (3.19), we obtain also that $v_{n_i} \rightharpoonup z$. Since $v_{n_i} \subset C$ and C is closed and convex, we obtain $z \in C$. From $\|Sv_n - v_n\| \to 0$, we obtain $S v_{n_i} \rightharpoonup z$. Let us show that $z \in EP$. Since $u_n = T_{r_n}(x_n - r_n D x_n)$ and

$$F(u_n, y) + \langle D x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C.$$

It follows by (A2) that

$$\langle D x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n)$$

and hence

$$\langle D x_{n_i}, y - u_{n_i}\rangle + \langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \geq F(y, u_{n_i}) \qquad (3.22)$$

Put $y_t = ty + (1-t)z$ for all $t \in (0, 1]$ and $y \in C$. Since $y \in C$ and $z \in C$, we have $y_t \in C$. So, from (3.15), we have

$\langle y_t - u_{n_i}, Dy_t\rangle - \langle y_t - u_{n_i}, By_t\rangle = 0 \geq -\langle y_t - u_{n_i}, Dx_{n_i}\rangle - \langle y - u_{n_i}, \frac{u_{n_i}-x_{n_i}}{r_{n_i}}\rangle + F(y_t, u_{n_i})$
and hence
$\langle y_t - u_{n_i}, Dy_t\rangle \geq \langle y_t - u_{n_i}, Dy_t\rangle - \langle y_t - u_{n_i}, Dx_{n_i}\rangle - \langle y - u_{n_i}, \frac{u_{n_i}-x_{n_i}}{r_{n_i}}\rangle + F(y_t, u_{n_i})$
$= \langle y_t - u_{n_i}, Dy_t - Du_{n_i}\rangle + \langle y_t - u_{n_i}, Du_{n_i} - Dx_{n_i}\rangle - \langle y - u_{n_i}, \frac{u_{n_i}-x_{n_i}}{r_{n_i}}\rangle + F(y_t, u_{n_i})$.
Since $\|u_{n_i} - x_{n_i}\| \to 0$, it follows that $\|Du_{n_i} - Dx_{n_i}\| \to 0$. Further, from monotonicity of D, we get

$$\langle y_t - u_{n_i}, Dy_t - Du_{n_i}\rangle \geq 0.$$

So, from (A4), we have

$$\langle y_t - z, Dy_t\rangle \geq F(y_t, z), \tag{3.23}$$

as $i \to \infty$. From (A1), (A4) and (3.23), we have

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, z) \leq tF(y_t, y) + (1-t)\langle y_t - z, Dy_t\rangle$$
$$\leq tF(y_t, y) + (1-t)t\langle y - z, Dy_t\rangle$$

and hence

$$0 \leq F(y_t, y) + (1-t)\langle y - z, Dy_t\rangle.$$

Letting $t \to 0$, we have for each $y \in C$, $0 \leq F(z, y) + \langle y - z, Dz\rangle$. This implies that $z \in EP$. Let us show that $z \in VI(C, A)$. Define

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone. Let $(u, v) \in G(T)$. Since $u - Av \in N_C v$ and $z_n \in C$, we have $\langle v - z_n, u - Av\rangle \geq 0$. On the other hand, from $z_n = P_C(u_n - \lambda_n Au_n)$, we have that

$$\langle v - z_n, z_n - (u_n - \lambda_n Au_n)\rangle \geq 0,$$

and hence,

$$\langle v - z_n, \frac{(z_n - u_n)}{\lambda_n} + Au_n\rangle \geq 0$$

Therefore, we have

$$\begin{aligned}
\langle v - v_{n_i}, u)\rangle &\geq \langle v - v_{n_i}, Av\rangle \\
&\geq \langle v - v_{n_i}, Av\rangle - \langle v - v_{n_i}, \frac{(v_{n_i} - u_{n_i})}{\lambda_{n_i}} + Au_{n_i}\rangle \\
&= \langle v - v_{n_i}, Av - Au_{n_i} - \frac{(v_{n_i} - u_{n_i})}{\lambda_{n_i}}\rangle \\
&= \langle v - v_{n_i}, Av - Av_{n_i}\rangle + \langle v - v_{n_i}, Av_{n_i} - Au_{n_i}\rangle
\end{aligned}$$

$$-\langle v - v_{n_i}, \frac{(v_{n_i} - u_{n_i})}{\lambda_{n_i}} \rangle$$
$$\geq \langle v - v_{n_i}, Av_{n_i} - Au_{n_i} \rangle - \langle v - v_{n_i}, \frac{(v_{n_i} - u_{n_i})}{\lambda_{n_i}} \rangle,$$

which together with $\|z_n - u_n\| \to 0$ and A is Lipschitz continuous implies that

$$\langle v - z, u \rangle \geq 0.$$

Since T is maximal monotone, we have $z \in T^{-1}0$ and hence $z \in VI(C, A)$. Finally, we can show that $z \in F(S)$. Assume $z \notin F(S)$. From Opial's condition, we have

$$\begin{aligned} \liminf_{i\to\infty} \|v_{n_i} - z\| &< \liminf_{i\to\infty} \|v_{n_i} - Sz\| \\ &= \liminf_{i\to\infty} \|v_{n_i} - Sv_{n_i} + Sv_{n_i} - Sz\| \\ &= \liminf_{i\to\infty} \|Sv_{n_i} - Sz\| \\ &\leq \liminf_{i\to\infty} \|v_{n_i} - z\| \end{aligned}$$

This is a contradiction. Thus, we obtain $z \in F(S)$. Hence $z \in \Omega$.

Finally, we show that $x_n \to z$, where $z = P_\Omega x_0$. Since $x_n = P_{C_n} x_0$ and $z \in \Omega \subset C_n$, we have

$$\|x_n - x_0\| \leq \|z - x_0\|.$$

It follows from $z' = P_\Omega x_0$ and the lower semicontinuity of the norm that

$$\|z' - x_0\| \leq \|z - x_0\| \leq \liminf_{i\to\infty} \|x_{n_i} - x_0\| \leq \limsup_{i\to\infty} \|x_{n_i} - x_0\| \leq \|z' - x_0\|.$$

Thus, we obtain that $\lim_{k\to\infty} \|x_{n_i} - x_0\| = \|z - x_0\| = \|z' - x_0\|$. Using the Kadec-Klee property of H, we obtain that

$$\lim_{k\to\infty} x_{n_i} = z = z'.$$

Since $\{x_{n_i}\}$ is an arbitrary weakly convergent subsequence of $\{x_n\}$, we can conclude that $\{x_n\}$ converges strongly to z, where $z = P_\Omega x_0$. □

As direct consequences of Theorem 3.1, we can obtain the following results.

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying* (A1)-(A4) *and let A be an α-inverse-strongly monotone mapping of C into H. Let S be a nonexpansive mappings from C into H such that $\Omega := F(S) \cap VI(C, A) \cap EP(F) \neq \emptyset$. For*

$C_1 = C$, $x_1 = P_{C_1}x_0$, *define sequences* $\{x_n\}$ *and* $\{u_n\}$ *of* C *as follows:*

$$\begin{cases} u_n \in C \text{ such that } \quad F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ z_n = P_C(u_n - \lambda_n A u_n), \\ y_n = \alpha_n x_n + (1-\alpha_n) S P_C(z_n - \lambda_n A z_n), \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases} \tag{3.24}$$

for every $n \in \mathbb{N}$, *where* $\{\alpha_n\} \subset (0,1)$ *and* $\{\lambda_n\} \subset [a,b] \subset (0, 2\alpha)$ *and* $\{r_n\} \subset (0,\infty)$ *satisfies* $\liminf_{n\to\infty} r_n > 0$. *Then* $\{x_n\}$ *converges strongly to* $P_\Omega x_0$.

Proof. Setting $D \equiv 0$ and $S_n = S$, for all $n \in \mathbb{N}$ in Theorem 3.1, we have the desired result easily. □

Theorem 3.3. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* F *be a bifunction from* $C \times C$ *into* $\mathbb{R}$ *satisfying* (A1)-(A4) *and let* D *be an* β*-inverse-strongly monotone mapping of* C *into* H. *Let* S *be a nonexpansive mappings from* C *into* H *such that* $\mathrm{F}(S) \cap EP \neq \emptyset$. *For* $C_1 = C$, $x_1 = P_{C_1}x_0$, *define sequences* $\{x_n\}$ *and* $\{u_n\}$ *of* C *as follows:*

$$\begin{cases} u_n \in C \text{ such that } \quad F(u_n, y) + \langle Dx_n, y - u_n\rangle + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n) S u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases}$$

for every $n \in \mathbb{N}$, *where* $\{\alpha_n\} \subset (0,1)$, $\{y_n\} \subset [c,d] \subset (0, 2\beta)$ *and* $\{r_n\} \subset (0,\infty)$ *satisfies* $\liminf_{n\to\infty} r_n > 0$. *Then* $\{x_n\}$ *converges strongly to* $P_{\mathrm{F}(S)\cap EP}x_0$.

Proof. Putting $A \equiv 0$ and $S_n = S$, for all $n \in \mathbb{N}$ in Theorem 3.1, we obtain the desired result easily. □

Theorem 3.4. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* F *be a bifunction from* $C \times C$ *into* $\mathbb{R}$ *satisfying* (A1)-(A4) *and let* D *be an* β*-inverse-strongly monotone mapping of* C *into* H. *Let* S *be a nonexpansive mappings from* C *into* H *such that* $\mathrm{F}(S) \cap VI(C,D) \neq \emptyset$. *For* $C_1 = C$, $x_1 = P_{C_1}x_0$, *define sequences* $\{x_n\}$ *and* $\{u_n\}$ *of* C *as follows:*

$$\begin{cases} \langle Dx_n, y - u_n\rangle + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n) S u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases}$$

for every $n \in \mathbb{N}$*, where* $\{\alpha_n\} \subset (0,1)$*,* $\{r_n\} \subset [c,d] \subset (0,2\beta)$ *and* $\{r_n\} \subset (0,\infty)$ *satisfies* $\liminf_{n\to\infty} r_n > 0$*. Then* $\{x_n\}$ *converges strongly to* $P_{\mathrm{F}(S)\cap VI(C,D)}x_0$*.*

Proof. Putting $F(x,y) = 0, \forall x,y \in C$ in Theorem 3.3. The conclusion of Theorem 3.4 can be obtained from Theorem 3.3 immediately. □

Theorem 3.5. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H*. Let* F *be a bifunction from* $C \times C$ *into* $\mathbb{R}$ *satisfying* (A1)-(A4) *such that* $EP(F) \neq \emptyset$*. For* $C_1 = C$*,* $x_1 = P_{C_1}x_0$*, define sequences* $\{x_n\}$ *and* $\{u_n\}$ *of* C *as follows:*

$$\begin{cases} u_n \in C \text{ such that } \ F(u_n,y) + \frac{1}{r_n}\langle y-u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases}$$

for every $n \in \mathbb{N}$*, where* $\{\alpha_n\} \subset (0,1)$ *and* $\{r_n\} \subset (0,\infty)$ *satisfies* $\liminf_{n\to\infty} r_n > 0$*. Then* $\{x_n\}$ *converges strongly to* $P_{EP(F)}x_0$*.*

Proof. Putting $S \equiv I$ (the identity), $D \equiv 0$ (the zero operator) and $\alpha_n = 0$ in Theorem 3.3, we obtain the Theorem 3.5. □

Theorem 3.6. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H*. Let* F *be a bifunction from* $C \times C$ *into* $\mathbb{R}$ *satisfying* (A1)-(A4) *and let* A *be an* α*-inverse-strongly monotone mapping of* C *into* H*. Let* S *be a nonexpansive mappings from* C *into* H *such that* $\bigcap_{n=1}^{\infty} \mathrm{F}(S) \cap VI(C,A) \neq \emptyset$*. For* $C_1 = C$*,* $x_1 = P_{C_1}x_0$*, define sequences* $\{x_n\}$ *and* $\{u_n\}$ *of* C *as follows:*

$$\begin{cases} z_n = P_C(u_n - \lambda_n A u_n), \\ y_n = \alpha_n x_n + (1-\alpha_n)S_n P_C(z_n - \lambda_n A z_n), \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases} \tag{3.25}$$

for every $n \in \mathbb{N}$*, where* $\{\alpha_n\} \subset (0,1)$*,* $\{\lambda_n\} \subset [a,b] \subset (0,2\alpha)$ *and* $\{r_n\} \subset (0,\infty)$ *satisfies* $\liminf_{n\to\infty} r_n > 0$*. Suppose that* $\sum_{k,l}^{\infty} \sup\{\|S_k z - S_l z\| : z \in B\} < \infty$ *for any bounded subset* B *of* C*. Let* S *be a mapping of* C *into itself defined by* $Sy = \lim_{n\to\infty} S_n y$ *for all* $y \in C$ *and suppose that* $F(S) = \cap_{n=1}^{\infty} F(S_n)$*. Then* $\{x_n\}$ *converges strongly to* $P_{\bigcap_{n=1}^{\infty}\mathrm{F}(S)\cap VI(C,A)}x_0$*.*

Proof. Putting $D \equiv 0, F(x,y) = 0, \forall x,y \in C$ and $r_n = 1, \forall n \geq 1$ in Theorem 3.1, we have $u_n = P_C x_n$ in Theorem 3.1. The conclusion of Theorem 3.4 can be obtained from Theorem 3.1 immediately. □

Remark 3.7. Theorem 3.1 generalize, improve and extend the results of Kumam [11, Theorem 3.1] and Shinzato and Takahashi [21, Theorem 3.1] to a countable family of nonexpansive mappings.

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