Generalised Growth Properties of Composite Entire and Meromorphic Functions

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Abstract

In this paper we study the generalised growth properties of composite entire and meromorphic functions using the generalised order and generalised lower order improving some earlier results.

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1 Introduction, Notations and Definitions.

Let $f$ and $g$ be two transcendental entire functions defined in the open complex plane $\mathbb{C}$. It is well known [2] that $\lim_{r \to \infty} \frac{T(r, f \circ g)}{T(r, f)} = \infty$ and $\lim_{r \to \infty} \frac{T(r, f \circ g)}{T(r, g)} = \infty$. Singh [9] proved some comparative growth properties of $\log T(r, f \circ g)$ and $T(r, f)$. But he [9] was unable to solve the growth properties of $\log T(r, f \circ g)$ and $T(r, g)$. However, some results on the comparative growth of $\log T(r, f \circ g)$ and $T(r, g)$ are proved in [5]. Further Datta [3] proved some results on the comparative growth of $\log T(r, f \circ g)$ with $T(r, f) \{\log T(r, f)\}^k$ and $T(r, g) \{\log T(r, g)\}^k$ respectively where $f$ is taken to be meromorphic, $g$ is entire and $k > 0$. In this paper we generalize the results of Datta [3] under some different
conditions. We also study the comparative growth of \( \log^{[m]} T (r, f \circ g) \) with \( T (r, f) \{ \log T (r, f) \}^k \) and \( T (r, g) \{ \log T (r, g) \}^k \) respectively where \( f \) is taken to be meromorphic, \( g \) is entire \( k > 0 \) and \( m \) is a positive integer.

If \( f \) and \( g \) are of positive lower order then Song and Yang [11] proved that

\[
\lim_{r \to \infty} \frac{\log^{[2]} M (r, f \circ g)}{\log^{[2]} M (r, f)} = \lim_{r \to \infty} \frac{\log^{[2]} M (r, f \circ g)}{\log^{[2]} M (r, g)} = \infty
\]

where \( \log^{[k]} x = \log \left( \log^{[k-1]} x \right) \) for \( k = 1, 2, 3, \ldots \) and \( \log^{[0]} x = x \).

Also in the sequel we use the following notation:

\[
\exp^{[k]} x = \exp \left( \exp^{[k-1]} x \right) \text{ for } k = 1, 2, 3, \ldots \text{ and } \exp^{[0]} x = x.
\]

Since \( M (r, f) \) and \( M (r, g) \) are increasing functions of \( r \), Singh and Baloria [10] asked whether for sufficiently large \( R = R (r) \)

\[
\lim_{r \to \infty} \frac{\log^{[2]} M (r, f \circ g)}{\log^{[2]} M (r, f)} < \infty \text{ and } \lim_{r \to \infty} \frac{\log^{[2]} M (r, f \circ g)}{\log^{[2]} M (r, g)} < \infty.
\]

Singh and Baloria [10], Lahiri and Sharma[7], Liao and Yang [8] worked on this question. In this paper we discuss on the comparative growth properties of \( \log^{[m+1]} M (r, f \circ g) \) and \( \log M (r, g) \) for any two entire functions \( f \) and \( g \). We do not explain the standard notations and definitions in the theory of entire and meromorphic functions as those are available in [12] and[4].

**Definition 1** The generalised order \( \rho_f^{(m)} \) and generalised lower order \( \lambda_f^{(m)} \) of a meromorphic function \( f \) are defined as follows:

\[
\rho_f^{(m)} = \limsup_{r \to \infty} \frac{\log^{[m]} T (r, f)}{\log r}
\]

and

\[
\lambda_f^{(m)} = \liminf_{r \to \infty} \frac{\log^{[m]} T (r, f)}{\log r}.
\]

If \( f \) is entire then
\[ \rho_f^{(m)} = \limsup_{r \to \infty} \frac{\log^{[m+1]} M(r, f)}{\log r} \]

and

\[ \lambda_f^{(m)} = \liminf_{r \to \infty} \frac{\log^{[m+1]} M(r, f)}{\log r} . \]

**Definition 2** The generalised type \( \sigma_f^{(m)} \) of an entire function \( f \) is defined as

\[ \sigma_f^{(m)} = \limsup_{r \to \infty} \frac{\log^{[m]} M(r, f)}{r^{\rho_f^{(m)}}}, 0 < \rho_f^{(m)} < \infty. \]

## 2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

**Lemma 1** [2] If \( f \) and \( g \) are entire functions then for all sufficiently large values of \( r \),

\[ M(r, f \circ g) \geq M\left(\frac{1}{8} M \left(\frac{T}{2}, g\right) - |g(0)|, f\right). \]

**Lemma 2** [1] If \( f \) is meromorphic and \( g \) is entire then for all sufficiently large values of \( r \),

\[ T(r, f \circ g) \leq \{1 + o(1)\} \frac{T(r, g)}{\log M(r, g)} T(M(r, g), f). \]

**Lemma 3** [6] If \( f \) is a non constant entire function of finite order then

\[ \liminf_{r \to \infty} \frac{\log M(r, f)}{T(r, f) \{\log T(r, f)\}^k} = 0, \]

where \( k > 0 \).
Lemma 4. Let \( f \) be an entire function such that \( 0 < \rho_f^{(m)} < \infty \). If \( \sigma_f^{(m)} \) and \( \sigma_{f(k)}^{(m)} \) be the respective generalised types of \( f \) and \( f^{(k)} \) then

\[
\sigma_{f(k)}^{(m)} \leq (2^k) \rho_f^{(m)} \sigma_f^{(m)}
\]

where \( k = 0, 1, 2, 3, \ldots \) and \( m = 0, 1, 2, 3, \ldots \)

**Proof.** It is known from Valiron \([12], p. 35\) that

\[
\frac{1}{r} \{ M(r, f) - |f(0)| \} \leq M(r, f) \leq \frac{1}{r} M(2r, f).
\]

Noting that \( \rho_{f(k)}^{(m)} = \rho_f^{(m)} \) we get from the second part of the inequality for \( k \geq 1 \)

\[
M(r, f^{(k)}) \leq M(2^k r, f)
\]

i.e.,

\[
\frac{\log[r] M(r, f^{(k)})}{r^{\rho_{f(k)}^{(m)}}} \leq \frac{\log[r] M(2^k r, f)}{(2^k)^{\rho_f^{(m)}}} \left( \frac{1}{r} \right)^{\rho_f^{(m)}}
\]

i.e.,

\[
\limsup_{r \to \infty} \frac{\log[r] M(r, f^{(k)})}{r^{\rho_{f(k)}^{(m)}}} \leq (2^k)^{\rho_f^{(m)}} \limsup_{r \to \infty} \frac{\log[r] M(2^k r, f)}{(2^k)^{\rho_f^{(m)}}}.
\]

i.e.,

\[
\sigma_{f(k)}^{(m)} \leq (2^k)^{\rho_f^{(m)}} \sigma_f^{(m)}
\]

which proves the lemma. \( \blacksquare \)

Lemma 5. Let \( f \) be meromorphic and \( g \) be entire such that \( \lambda_g^{(m)} < \infty \). If \( \lambda_{fog}^{(m)} = \infty \) then for every positive number \( A \),

\[
\lim_{r \to \infty} \frac{\log[r] T(r, f \circ g)}{\log[r] M(r^A, g^{(k)})} = \infty
\]

where \( k = 0, 1, 2, 3, \ldots \) and \( m = 1, 2, 3, \ldots \)

**Proof.** Let us assume that the conclusion of the lemma do not hold. Then there exists a constant \( B > 0 \) such that

\[
\lim_{r \to \infty} \frac{\log[r] T(r, f \circ g)}{\log[r] M(r^B, g^{(k)})} = \mu < \infty,
\]
provided the limit exists. Then for all large \( r \),
\[
\log^{[m]} T(r, f \circ g) \leq (\mu + \epsilon) \log^{[m+1]} M(r^B, g^{(k)}) .
\]
Again for a sequence of values of \( r \) tending to infinity,
\[
\log^{[m+1]} M(r^B, g^{(k)}) \leq \left( \lambda_{g^{(k)}}^{(m)} + \epsilon \right) B \log r.
\]
Thus from above we get for a sequence of values of \( r \) tending to infinity ,
\[
\log^{[m]} T(r, f \circ g) \leq (\mu + \epsilon) \left( \lambda_{g^{(k)}}^{(m)} + \epsilon \right) B \log r,
\]
which implies that \( \lambda_{f \circ g}^{(m)} < \infty \). This is a contradiction. Thus the lemma is proved.

\[
\begin{align*}
\text{3 Theorems.} \\
\text{In this section we present the main results of the paper.}
\end{align*}
\]

**Theorem 1** Let \( f \) be meromorphic and \( g \) be non constant entire such that \( \rho_f^{(m)} \) and \( \rho_g^{(m)} \) are finite. Then
\[
\liminf_{r \to \infty} \frac{\log^{[m]} T(r, f \circ g)}{T(r, g) \{ \log T(r, g) \}^k} = 0
\]
where \( k > 0 \) and \( m = 1, 2, 3, ... \)

**Proof.** By Lemma 2 and \( T(r, g) \leq \log^+ M(r, g) \) we get for all sufficiently large values of \( r \),
\[
\log^{[m]} T(r, f \circ g) \leq \left( \rho_f^{(m)} + \epsilon \right) \log M(r, g) + O(1)
\]
i.e.,
\[
\frac{\log^{[m]} T(r, f \circ g)}{T(r, g) \{ \log T(r, g) \}^k} \leq \frac{\left( \rho_f^{(m)} + \epsilon \right) \log M(r, g) + O(1)}{T(r, g) \{ \log T(r, g) \}^k}. \tag{1}
\]
Now by Lemma 3 it follows from (1) that
\[
\liminf_{r \to \infty} \frac{\log^{[m]} T(r, f \circ g)}{T(r, g) \{ \log T(r, g) \}^k} = 0.
\]
This proves the theorem.
**Remark 1** Considering \( f = g = \exp z \) one can easily verify that no term in the denominator of \( \frac{\log^{[m]} T(r, f \circ g)}{T(r, g) \{ \log T(r, g) \}^k} \) can be removed as we see in the following example.

**Example 1** Let \( f = g = \exp z \) and \( m = k = 1 \).

Then \( T(r, g) = \frac{r}{\pi} \) and \( \log T(r, g) = \log r - \log \pi \).

Also \( T(r, f \circ g) \sim \exp r \frac{r}{(2\pi^3 r)^{\frac{1}{2}}} \).

Therefore \( \frac{\log^{[m]} T(r, f \circ g)}{T(r, g) \{ \log T(r, g) \}^k} = \frac{\log T(r, f \circ g)}{T(r, g) \log T(r, g)} \)

\[ = \frac{r - \frac{1}{2} \log r + O(1)}{\frac{r}{\pi} \{ \log r - \log \pi \}} = \frac{\pi}{r} \left[ \frac{r - \frac{1}{2} \log r + O(1)}{\log r - \log \pi} \right] \]

i.e., \( \liminf_{r \to \infty} \frac{\log^{[m]} T(r, f \circ g)}{T(r, g) \{ \log T(r, g) \}^k} = 0 \).

**Remark 2** The condition \( \rho_f^{(m)} < \infty \) in Theorem 1 is necessary which is evident from the following example.

**Example 2** Let \( f = \exp^{[m+1]} z, g = z, m = 1 \) and \( k = 1 \).

Therefore

\[ \rho_f^{(m)} = \limsup_{r \to \infty} \frac{\log^{[m+1]} M(r, f)}{\log r} \]

\[ = \limsup_{r \to \infty} \frac{\log^{[m+1]} \exp^{[m+1]} r}{\log r} \]

\[ = \limsup_{r \to \infty} \frac{r}{\log r} = \infty \]

and similarly \( \rho_g^{(m)} = 0 \). Since \( T(r, f \circ g) \sim \exp r \frac{r}{(2\pi^3 r)^{\frac{1}{2}}} \) and \( T(r, g) \leq \log^+ M(r, g) = \log r \), it follows that

\[ \frac{\log^{[m]} T(r, f \circ g)}{T(r, g) \{ \log T(r, g) \}^k} = \frac{\log T(r, f \circ g)}{T(r, g) \log T(r, g)} \geq \frac{r - \frac{1}{2} \log r + O(1)}{\log r \{ \log^2 r \}} \]

which implies that \( \lim_{r \to \infty} \frac{\log^{[m]} T(r, f \circ g)}{T(r, g) \{ \log T(r, g) \}^k} = \infty \).
Theorem 2 Let \( f \) and \( g \) be two entire functions such that \( \rho_f^{(m)} \) and \( \rho_g^{(m)} \) are finite. Also let \( \lambda_f^{(m)} > \rho_g^{(m)} \). Then

\[
\liminf_{r \to \infty} \frac{\log^{[m]} T(r, f \circ g)}{T(r, f) \{\log T(r, f)\}^k} = 0
\]

where \( k > 0 \) and \( m = 1, 2, 3, \ldots \).

Proof. Since \( \lambda_f^{(m)} > \rho_g^{(m)} \), we can choose \( \epsilon > 0 \) in such a way that \( \lambda_f^{(m)} - \epsilon > \rho_g^{(m)} + \epsilon \). By Lemma 2 and \( T(r, g) \leq \log^+ M(r, g) \) we obtain for all sufficiently large values of \( r \),

\[
\log^{[m]} T(r, f \circ g) \leq \left( \rho_f^{(m)} + \epsilon \right) \log M(r, g) + O(1)
\]

i.e.,

\[
\frac{\log^{[m]} T(r, f \circ g)}{T(r, f) \{\log T(r, f)\}^k} \leq \frac{\left( \rho_f^{(m)} + \epsilon \right) \log M(r, f)}{T(r, f) \{\log T(r, f)\}^k} \frac{\log M(r, g)}{\log M(r, f)} + \frac{O(1)}{T(r, f) \{\log T(r, f)\}^k}.
\]

(2)

Again for all sufficiently large values of \( r \),

\[
\log M(r, g) \leq \exp^{[m-1]} r^{[\rho_g^{(m)} + \epsilon]}
\]

and

\[
\log M(r, f) \geq \exp^{[m-1]} r^{[\lambda_f^{(m)} - \epsilon]}
\]

Thus from (2) we obtain for all sufficiently large values of \( r \),

\[
\frac{\log^{[m]} T(r, f \circ g)}{T(r, f) \{\log T(r, f)\}^k} \leq \frac{\left( \rho_f^{(m)} + \epsilon \right) \log M(r, f)}{T(r, f) \{\log T(r, f)\}^k} \frac{\exp^{[m-1]} r^{[\rho_g^{(m)} + \epsilon]}}{\exp^{[m-1]} r^{[\lambda_f^{(m)} - \epsilon]}} + \frac{O(1)}{T(r, f) \{\log T(r, f)\}^k}.
\]

i.e.,

\[
\liminf_{r \to \infty} \frac{\log^{[m]} T(r, f \circ g)}{T(r, f) \{\log T(r, f)\}^k} \leq \left( \rho_f^{(m)} + \epsilon \right) \liminf_{r \to \infty} \frac{\log M(r, f)}{T(r, f) \{\log T(r, f)\}^k} \liminf_{r \to \infty} \frac{\exp^{[m-1]} r^{[\rho_g^{(m)} + \epsilon]}}{\exp^{[m-1]} r^{[\lambda_f^{(m)} - \epsilon]}}. \tag{3}
\]

Now in view of Lemma 3, the theorem follows from (3).
Remark 3 Considering \( f = \exp z, \ g = z \) and \( m = k = 1 \) and proceeding exactly as in Example 1 one can easily verify that no term in the denominator of \( \frac{\log^{[m]} T(r, f \circ g)}{T(r, f) \{ \log T(r, f) \}^k} \) can be removed.

\[ \Box \]

Theorem 3 Let \( f \) and \( g \) be two entire functions such that \( 0 < \lambda_f^{(m)} < \infty \) and \( 0 < \rho_g^{(m)} < \infty \). Also let \( 0 < \sigma_g^{(m)} < \infty \). Then

\[
\limsup_{r \to \infty} \frac{\log^{[m+1]} M(r, f \circ g)}{\log M(r, g^{(k)})} \geq \frac{\lambda_f^{(m)}}{2(2k+1)\rho_g^{(m)}}.
\]

where \( k = 0, 1, 2, 3, ... \) and \( m = 1, 2, 3, ... \)

Proof. Let \( 0 < \epsilon < \min \{ \lambda_f^{(m)}, \sigma_g^{(m)} \} \). Then for a sequence of values of \( r \) tending to infinity we obtain that

\[
\log M\left(\frac{r}{2}, g\right) \geq (\sigma_g^{(m)} - \epsilon) \left( \frac{r}{2} \right)^{\rho_g^{(m)}}.
\]  

(4)

Again from Lemma 1 we get for all sufficiently large values of \( r \),

\[
\log^{[m+1]} M(r, f \circ g) \geq (\lambda_f^{(m)} - \epsilon) \log \frac{1}{8} + (\lambda_f^{(m)} - \epsilon) \log M\left(\frac{r}{2}, g\right).
\]  

(5)

Now for a sequence of values of \( r \) tending to infinity it follows from (4) and (5) that

\[
\log^{[m+1]} M(r, f \circ g) \geq (\lambda_f^{(m)} - \epsilon) \log \frac{1}{8} + (\lambda_f^{(m)} - \epsilon) (\sigma_g^{(m)} - \epsilon) \left( \frac{r}{2} \right)^{\rho_g^{(m)}}.
\]  

(6)

Again by Lemma 4 we get for all sufficiently large values of \( r \),

\[
\log M(r, g^{(k)}) \leq (\sigma_g^{(m)} + \epsilon) r^{\rho_g^{(m)}} \leq \left( \frac{2^{k+1}}{\sigma_g^{(m)}} \right) (\sigma_g^{(m)} + \epsilon) r^{\rho_g^{(m)}}.
\]  

(7)

So from (6) and (7) it follows that for a sequence of values of \( r \) tending to infinity,

\[
\log^{[m+1]} M(r, f \circ g) \geq \frac{(\lambda_f^{(m)} - \epsilon) \log \frac{1}{8} + (\lambda_f^{(m)} - \epsilon) (\sigma_g^{(m)} - \epsilon) \left( \frac{r}{2} \right)^{\rho_g^{(m)}}}{\left( \frac{2^{k+1}}{\sigma_g^{(m)}} \right) (\sigma_g^{(m)} + \epsilon) r^{\rho_g^{(m)}}}.
\]  

(8)

Since \( \epsilon (> 0) \) is arbitrary, we get from (8) that

\[
\limsup_{r \to \infty} \frac{\log^{[m+1]} M(r, f \circ g)}{\log M(r, g^{(k)})} \geq \frac{\lambda_f^{(m)}}{2(2k+1)\rho_g^{(m)}}.
\]

Thus the theorem is established. \( \Box \)
Theorem 4 Let $f$ be meromorphic and $g$ be entire such that $0 < \lambda_{fog}^{(m)} \leq \rho_{fog}^{(m)} < \infty$ and $0 < \lambda_{g}^{(m)} \leq \rho_{g}^{(m)} < \infty$. Then for any positive number $A$,

$$\frac{\lambda_{fog}^{(m)}}{A\rho_{g}^{(m)}} \leq \liminf_{r \to \infty} \frac{\log^{[m]} T (r, f \circ g)}{\log^{[m]} T (r^A, g^{(k)})} \leq \frac{\lambda_{fog}^{(m)}}{A\lambda_{g}^{(m)}} \leq \limsup_{r \to \infty} \frac{\log^{[m]} T (r, f \circ g)}{\log^{[m]} T (r^A, g^{(k)})} \leq \frac{\rho_{fog}^{(m)}}{A\lambda_{g}^{(m)}},$$

where $k = 0, 1, 2, ...$ and $m = 1, 2, 3, ...$

Proof. For all large values of $r$,

$$\log^{[m]} T (r, f \circ g) \geq \left( \lambda_{fog}^{(m)} - \epsilon \right) \log r \tag{9}$$

and

$$\log^{[m]} T (r^A, g^{(k)}) \leq A \left( \rho_{g}^{(m)} + \epsilon \right) \log r = A \left( \rho_{g}^{(m)} + \epsilon \right) \log r. \tag{10}$$

Now from (9) and (10) it follows for all large values of $r$ that

$$\frac{\log^{[m]} T (r, f \circ g)}{\log^{[m]} T (r^A, g^{(k)})} \geq \frac{\lambda_{fog}^{(m)} - \epsilon}{A \left( \rho_{g}^{(m)} + \epsilon \right)}.$$ 

As $\epsilon (> 0)$ is arbitrary, we obtain that

$$\liminf_{r \to \infty} \frac{\log^{[m]} T (r, f \circ g)}{\log^{[m]} T (r^A, g^{(k)})} \geq \frac{\lambda_{fog}^{(m)}}{A\rho_{g}^{(m)}}. \tag{11}$$

Again for a sequence of values of $r$ tending to infinity,

$$\log^{[m]} T (r, f \circ g) \leq \left( \lambda_{fog}^{(m)} + \epsilon \right) \log r \tag{12}$$

and for all large values of $r$,

$$\log^{[m]} T (r^A, g^{(k)}) \geq A \left( \lambda_{g}^{(m)} - \epsilon \right) \log r. \tag{13}$$

So combining (12) and (13) we get for a sequence of values of $r$ tending to infinity,

$$\frac{\log^{[m]} T (r, f \circ g)}{\log^{[m]} T (r^A, g^{(k)})} \leq \frac{\lambda_{fog}^{(m)} + \epsilon}{A \left( \lambda_{g}^{(m)} - \epsilon \right)}.$$ 


Since $\epsilon (> 0)$ is arbitrary, it follows that
\[
\liminf_{r \to \infty} \frac{\log^{|m|} T(r, f \circ g)}{\log^{|m|} T(r^A, g^{(k)})} \leq \frac{\lambda_{fog}^{(m)}}{\lambda_g^{(m)} A}.
\] (14)

Also for a sequence of values of $r$ tending to infinity,
\[
\log^{|m|} T(r^A, g^{(k)}) \leq A \left( \lambda_g^{(m)} + \epsilon \right) \log r.
\] (15)

Now from (9) and (15) we obtain for a sequence of values of $r$ tending to infinity,
\[
\frac{\log^{|m|} T(r, f \circ g)}{\log^{|m|} T(r^A, g^{(k)})} \geq \frac{\lambda_{fog}^{(m)} - \epsilon}{A \left( \lambda_g^{(m)} + \epsilon \right)}.
\]

Since $\epsilon (> 0)$ is arbitrary, it follows that
\[
\limsup_{r \to \infty} \frac{\log^{|m|} T(r, f \circ g)}{\log^{|m|} T(r^A, g^{(k)})} \geq \frac{\lambda_{fog}^{(m)}}{\lambda_g^{(m)} A}.
\] (16)

Also for all large values of $r$,
\[
\log^{|m|} T(r, f \circ g) \leq \left( \rho_{fog}^{(m)} + \epsilon \right) \log r.
\] (17)

So from (13) and (17) it follows that for all large values of $r$,
\[
\frac{\log^{|m|} T(r, f \circ g)}{\log^{|m|} T(r^A, g^{(k)})} \leq \frac{\rho_{fog}^{(m)} + \epsilon}{A \left( \lambda_g^{(m)} - \epsilon \right)}.
\]

As $\epsilon (> 0)$ is arbitrary, we obtain that
\[
\limsup_{r \to \infty} \frac{\log^{|m|} T(r, f \circ g)}{\log^{|m|} T(r^A, g^{(k)})} \leq \frac{\rho_{fog}^{(m)}}{\lambda_g^{(m)} A}.
\] (18)

Thus the theorem follows from (11), (14), (16) and (18).

**Remark 4** Considering $f = z$, $g = \exp z$, $m = 1$ and $A = 1$ one can easily verify that the sign $' \leq ' cannot be replaced by $' < ' only in Theorem 4.
Theorem 5 Let $f$ be meromorphic and $g$ be entire such that $0 < \lambda_{fog}^{(m)} \leq \rho_{fog}^{(m)} < \infty$ and $0 < \rho_g^{(m)} < \infty$. Then for any positive number $A$,
\[
\liminf_{r \to \infty} \frac{\log[m] T(r, f \circ g)}{\log[m] T(r^A, g^{(k)})} \leq \frac{\rho_{fog}^{(m)}}{A \rho_g^{(m)}} \leq \limsup_{r \to \infty} \frac{\log[m] T(r, f \circ g)}{\log[m] T(r^A, g^{(k)})}
\]
where $k = 0, 1, 2, 3...$ and $m = 1, 2, 3...$

Proof. For all large values of $r$,
\[
\log[m] T(r, f \circ g) \leq (\rho_{fog}^{(m)} + \epsilon) \log r.
\]
Also from the definition of generalised order we get for a sequence of values of $r$ tending to infinity,
\[
\log[m] T(r^A, g^{(k)}) \geq A (\rho_g^{(m)} - \epsilon) \log r.
\]
Now from (19) and (20) it follows that for a sequence of values of $r$ tending to infinity,
\[
\frac{\log[m] T(r, f \circ g)}{\log[m] T(r^A, g^{(k)})} \leq \frac{(\rho_{fog}^{(m)} + \epsilon)}{A (\rho_g^{(m)} - \epsilon)}.
\]
As $\epsilon (> 0)$ is arbitrary, we obtain that
\[
\liminf_{r \to \infty} \frac{\log[m] T(r, f \circ g)}{\log[m] T(r^A, g^{(k)})} \leq \frac{\rho_{fog}^{(m)}}{A \rho_g^{(m)}}.
\]
Again for a sequence of values of $r$ tending to infinity,
\[
\log[m] T(r, f \circ g) \geq (\rho_{fog}^{(m)} - \epsilon) \log r
\]
and from the definition of generalised order we get for all large values of $r$
\[
\log[m] T(r^A, g^{(k)}) \leq A (\rho_g^{(m)} + \epsilon) \log r.
\]
So combining (22) and (23) we get for a sequence of values of $r$ tending to infinity,
\[
\frac{\log[m] T(r, f \circ g)}{\log[m] T(r^A, g^{(k)})} \geq \frac{(\rho_{fog}^{(m)} - \epsilon)}{A (\rho_g^{(m)} + \epsilon)}.
\]
Since $\epsilon (> 0)$ is arbitrary, it follows that
\[
\limsup_{r \to \infty} \frac{\log[m] T(r, f \circ g)}{\log[m] T(r^A, g^{(k)})} \geq \frac{\rho_{fog}^{(m)}}{A \rho_g^{(m)}}.
\]
Thus the theorem follows from (21) and (24).
Remark 5 Considering \( f = z, g = \exp z, m = 1 \) and \( A = 1 \) one can easily see that the sign \( \leq \) in Theorem 5 cannot be replaced by \( < \) only.

Remark 6 Combining Theorem 4 and Theorem 5 we may state the following theorem without proof.

**Theorem 6** Let \( f \) be meromorphic and \( g \) be entire such that \( 0 < \lambda_f^{(m)} \leq \rho_f^{(m)} < \infty \) and \( 0 < \lambda_g^{(m)} \leq \rho_g^{(m)} < \infty \). Then for any positive number \( A \),

\[
\liminf_{r \to \infty} \frac{\log^{[m]} T(r, f \circ g)}{\log^{[m]} T(r, A, g(k))} \leq \min \left\{ \frac{\lambda_f^{(m)}}{A\lambda_g^{(m)}}, \frac{\rho_f^{(m)}}{A\rho_g^{(m)}} \right\}
\]

\[
\leq \max \left\{ \frac{\lambda_f^{(m)}}{A\lambda_g^{(m)}}, \frac{\rho_f^{(m)}}{A\rho_g^{(m)}} \right\}
\]

\[
\leq \limsup_{r \to \infty} \frac{\log^{[m]} T(r, f \circ g)}{\log^{[m]} T(r, A, g(k))}.
\]

Remark 7 Considering \( f = z, g = \exp z, m = 1 \) and \( A = 1 \) one can easily verify that the sign \( \leq \) cannot be replaced by \( < \) only in Theorem 6.

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**References**


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