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Generalised Growth Properties of Composite Entire and Meromorphic Functions

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Abstract

In this paper we study the generalised growth properties of composite entire and meromorphic functions using the generalised order and generalised lower order improving some earlier results .

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1 Introduction, Notations and Definitions.

Let f and g be two transcendental entire functions defined in the open complex plane $\mathbb{C}$.It is well known [2] that $\lim\limits_{r\to\infty}\frac{T(r,f\circ g)}{T(r,f)}=\infty$ and $\lim\limits_{r\to\infty}\frac{T(r,f\circ g)}{T(r,g)}=\infty$. Singh [9] proved some comparative growth properties of $\log T\left(r,f\circ g\right)$ and $T\left(r,f\right)$. But he [9] was unable to solve the growth properties of $\log T\left(r,f\circ g\right)$ and $T(r,g)$. However, some results on the comparative growth of $\log T\left(r,f\circ g\right)$ and $T\left(r,g\right)$ are proved in [5] . Further Datta [3] proved some results on the comparative growth of $\log T\left(r,f\circ g\right)$ with $T\left(r,f\right)\left\{\log T\left(r,f\right)\right\}^{k}$ and $T\left(r,g\right)\left\{\log T\left(r,g\right)\right\}^{k}$ respectively where f is taken to be meromorphic, g is entire and $k>0$. In this paper we generalize the results of Datta [3] under some different conditions. We also study the comparative growth of $\log^{[m]} T(r, f \circ g)$ with $T(r,f)\{\log T(r,f)\}^k$ and $T(r,g)\{\log T(r,g)\}^k$ respectively where f is taken to be meromorphic, g is entire $k > 0$ and m is a positive integer.

If f and g are of positive lower order then Song and Yang [11] proved that

$$\lim_{r\to\infty} \frac{\log^{[2]} M(r, f \circ g)}{\log^{[2]} M(r,f)} = \lim_{r\to\infty} \frac{\log^{[2]} M(r, f \circ g)}{\log^{[2]} M(r,g)} = \infty$$

$$\text{where } \log^{[k]} x = \log\left(\log^{[k-1]} x\right) \text{ for } k = 1, 2, 3, \ldots \text{ and } \log^{[0]} x = x.$$

Also in the sequel we use the following notation:

$$\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right) \text{ for } k = 1, 2, 3, \ldots \text{ and } \exp^{[0]} x = x.$$

Since $M(r,f)$ and $M(r,g)$ are increasing functions of r, Singh and Baloria [10] asked whether for sufficiently large $R = R(r)$

$$\lim_{r\to\infty} \frac{\log^{[2]} M(r, f \circ g)}{\log^{[2]} M(r,f)} < \infty \text{ and } \lim_{r\to\infty} \frac{\log^{[2]} M(r, f \circ g)}{\log^{[2]} M(r,g)} < \infty.$$

Singh and Baloria [10], Lahiri and Sharma[7] , Liao and Yang [8] worked on this question. In this paper we discuss on the comparative growth properties of $\log^{[m+1]} M(r, f \circ g)$ and $\log M(r,g)$ for any two entire functions f and g. We do not explain the standard notations and definitions in the theory of entire and meromorphic functions as those are available in [12] and[4].

Definition 1 *The generalised order $\rho_f^{(m)}$ and generalised lower order $\lambda_f^{(m)}$ of a meromorphic function f are defined as follows:*

$$\rho_f^{(m)} = \limsup_{r\to\infty} \frac{\log^{[m]} T(r,f)}{\log r}$$

and

$$\lambda_f^{(m)} = \liminf_{r\to\infty} \frac{\log^{[m]} T(r,f)}{\log r}.$$

If f is entire then

$$\rho_f^{(m)} = \limsup_{r \to \infty} \frac{\log^{[m+1]} M(r,f)}{\log r}$$

and

$$\lambda_f^{(m)} = \liminf_{r \to \infty} \frac{\log^{[m+1]} M(r,f)}{\log r}.$$

Definition 2 *The generalised type $\sigma_f^{(m)}$ of an entire function f is defined as*

$$\sigma_f^{(m)} = \limsup_{r \to \infty} \frac{\log^{[m]} M(r,f)}{r^{\rho_f^{(m)}}}, 0 < \rho_f^{(m)} < \infty.$$

2 Lemmas.

In this section we present some lemmas which will be needed in th sequel.

Lemma 1 [2] *If f and g are entire functions then for all sufficiently large values of r,*

$$M(r, f \circ g) \geq M\left(\frac{1}{8} M\left(\frac{r}{2}, g\right) - |g(0)|, f\right).$$

Lemma 2 [1] *If f is meromorphic and g is entire then for all sufficiently large values of r,*

$$T(r, f \circ g) \leq \{1 + o(1)\} \frac{T(r,g)}{\log M(r,g)} T(M(r,g), f).$$

Lemma 3 [6] *If f is a non constant entire function of finite order then*

$$\liminf_{r \to \infty} \frac{\log M(r,f)}{T(r,f)\{\log T(r,f)\}^k} = 0,$$

where $k > 0$.

Lemma 4 *Let f be an entire function such that $0<\rho_f^{(m)}<\infty$. If $\sigma_f^{(m)}$ and $\sigma_{f^{(k)}}^{(m)}$ be the respective generalised types of f and $f^{(k)}$ then*

$$\sigma_{f^{(k)}}^{(m)} \leq \left(2^k\right)^{\rho_f^{(m)}} \sigma_f^{(m)}$$

where $k=0,1,2,3,...$ and $m=0,1,2,3,...$
Proof. *It is known from Valiron $\{[12], p.35\}$ that*

$$\frac{1}{r}\left\{M(r,f)-|f(0)|\right\} \leq M(r,f) \leq \frac{1}{r}M(2r,f).$$

Noting that $\rho_{f^{(k)}}^{(m)}=\rho_f^{(m)}$ we get from the second part of the inequality for $k\geq 1$

$$M\left(r,f^{(k)}\right) \leq M\left(2^k r,f\right)$$

$$i.e., \ \frac{\log^{[m]} M\left(r,f^{(k)}\right)}{r^{\rho_{f^{(k)}}^{(m)}}} \leq \frac{\log^{[m]} M\left(2^k r,f\right)}{\left(2^k r\right)^{\rho_f^{(m)}}}.\left(2^k\right)^{\rho_f^{(m)}}$$

$$i.e., \ \limsup_{r\to\infty}\frac{\log^{[m]} M\left(r,f^{(k)}\right)}{r^{\rho_{f^{(k)}}^{(m)}}} \leq \left(2^k\right)^{\rho_f^{(m)}} \limsup_{r\to\infty}\frac{\log^{[m]} M\left(2^k r,f\right)}{\left(2^k r\right)^{\rho_f^{(m)}}}.$$

$$i.e., \ \sigma_{f^{(k)}}^{(m)} \leq \left(2^k\right)^{\rho_f^{(m)}} \sigma_f^{(m)}$$

which proves the lemma. ■

Lemma 5 *Let f be meromorphic and g be entire such that $\lambda_g^{(m)}<\infty$. If $\lambda_{f\circ g}^{(m)}=\infty$ then for every positive number A,*

$$\lim_{r\to\infty}\frac{\log^{[m]} T(r,f\circ g)}{\log^{[m+1]} M\left(r^A,g^{(k)}\right)}=\infty$$

where $k=0,1,2,3...$ and $m=1,2,3....$

Proof. Let us assume that the conclusion of the lemma do not hold . Then there exists a constant $B>0$ such that

$$\lim_{r\to\infty}\frac{\log^{[m]} T(r,f\circ g)}{\log^{[m+1]} M\left(r^B,g^{(k)}\right)}=\mu<\infty,$$

provided the limit exists. Then for all large r,

$$\log^{[m]} T(r, f\circ g) \leq (\mu+\epsilon)\log^{[m+1]} M\left(r^B, g^{(k)}\right).$$

Again for a sequence of values of r tending to infinity,

$$\log^{[m+1]} M\left(r^B, g^{(k)}\right) \leq \left(\lambda_{g^{(k)}}^{(m)}+\epsilon\right) B\log r.$$

Thus from above we get for a sequence of values of r tending to infinity ,

$$\log^{[m]} T(r, f\circ g) \leq (\mu+\epsilon)\left(\lambda_{g^{(k)}}^{(m)}+\epsilon\right) B\log r,$$

which implies that $\lambda_{f\circ g}^{(m)} < \infty$. This is a contradiction.Thus the lemma is proved. ■

3 Theorems.

In this section we present the main results of the paper.

Theorem 1 *Let f be meromorphic and g be non constant entire such that $\rho_f^{(m)}$ and $\rho_g^{(m)}$ are finite . Then*

$$\liminf_{r\to\infty}\frac{\log^{[m]} T(r, f\circ g)}{T(r,g)\{\log T(r,g)\}^k} = 0$$

where $k > 0$ and $m = 1, 2, 3, ...$

Proof. *By Lemma 2 and $T(r,g) \leq \log^+ M(r,g)$ we get for all sufficiently large values of r,*

$$\log^{[m]} T(r, f\circ g) \leq \left(\rho_f^{(m)}+\epsilon\right)\log M(r,g) + O(1)$$

$$i.e., \ \frac{\log^{[m]} T(r, f\circ g)}{T(r,g)\{\log T(r,g)\}^k} \leq \frac{\left(\rho_f^{(m)}+\epsilon\right)\log M(r,g)+O(1)}{T(r,g)\{\log T(r,g)\}^k}. \qquad (1)$$

Now by Lemma 3 it follows from (1) that

$$\liminf_{r\to\infty}\frac{\log^{[m]} T(r, f\circ g)}{T(r,g)\{\log T(r,g)\}^k} = 0.$$

This proves the theorem.

Remark 1 *Considering $f = g = \exp z$ one can easily verify that no term in the denominator of $\frac{\log^{[m]} T(r,f\circ g)}{T(r,g)\{\log T(r,g)\}^k}$ can be removed as we see in the following example.*

Example 1 *Let $f = g = \exp z$ and $m = k = 1$.*

Then $T(r,g) = \frac{r}{\pi}$ and $\log T(r,g) = \log r - \log \pi$.

Also

$$T(r, f\circ g) \sim \frac{\exp r}{(2\pi^3 r)^{\frac{1}{2}}}.$$

$$\begin{aligned} \textit{Therefore } \frac{\log^{[m]} T(r,f\circ g)}{T(r,g)\{\log T(r,g)\}^k} &= \frac{\log T(r,f\circ g)}{T(r,g)\log T(r,g)} \\ &= \frac{r - \frac{1}{2}\log r + O(1)}{\frac{r}{\pi}[\log r - \log \pi]} = \frac{\pi}{r}\left[\frac{r - \frac{1}{2}\log r + O(1)}{\log r - \log\pi}\right] \end{aligned}$$

$$\textit{i.e.,} \liminf_{r\to\infty} \frac{\log^{[m]} T(r,f\circ g)}{T(r,g)\{\log T(r,g)\}^k} = 0.$$

Remark 2 *The condition $\rho_f^{(m)} < \infty$ in Theorem 1 is necessary which is evident from the following example.*

Example 2 *Let $f = \exp^{[m+1]} z, g = z$, $m = 1$ and $k = 1$.*

Therefore

$$\begin{aligned} \rho_f^{(m)} &= \limsup_{r\to\infty} \frac{\log^{[m+1]} M(r,f)}{\log r} \\ &= \limsup_{r\to\infty} \frac{\log^{[m+1]} \exp^{[m+1]} r}{\log r} \\ &= \limsup_{r\to\infty} \frac{r}{\log r} = \infty \end{aligned}$$

and similarly $\rho_g^{(m)} = 0$. Since

$$T(r, f\circ g) \sim \frac{\exp r}{(2\pi^3 r)^{\frac{1}{2}}}$$

and $T(r,g) \le \log^+ M(r,g) = \log r$, it follows that

$$\frac{\log^{[m]} T(r,f\circ g)}{T(r,g)\{\log T(r,g)\}^k} = \frac{\log T(r,f\circ g)}{T(r,g)\log T(r,g)} \ge \frac{r - \frac{1}{2}\log r + O(1)}{\log r\left\{\log^{[2]} r\right\}}$$

which implies that $\lim_{r\to\infty} \frac{\log^{[m]} T(r,f\circ g)}{T(r,g)\{\log T(r,g)\}^k} = \infty$.

■

Theorem 2 *Let f and g be two entire functions such that $\rho_f^{(m)}$ and $\rho_g^{(m)}$ are finite. Also let $\lambda_f^{(m)} > \rho_g^{(m)}$. Then*

$$\liminf_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{T(r,f)\{\log T(r,f)\}^k} = 0$$

where $k > 0$ and $m = 1, 2, 3, ...$

Proof. *Since $\lambda_f^{(m)} > \rho_g^{(m)}$, we can choose $\epsilon\,(>0)$ in such a way that $\lambda_f^{(m)} - \epsilon > \rho_g^{(m)} + \epsilon$. By Lemma 2 and $T(r,g) \le \log^+ M(r,g)$ we obtain for all sufficiently large values of r,*

$$\log^{[m]} T(r, f\circ g) \le \left(\rho_f^{(m)} + \epsilon\right) \log M(r,g) + O(1)$$

$$\text{i.e., } \frac{\log^{[m]} T(r, f\circ g)}{T(r,f)\{\log T(r,f)\}^k} \le \frac{\left(\rho_f^{(m)} + \epsilon\right)\log M(r,f)}{T(r,f)\{\log T(r,f)\}^k} . \frac{\log M(r,g)}{\log M(r,f)} + \frac{O(1)}{T(r,f)\{\log T(r,f)\}^k}. \quad (2)$$

Again for all sufficiently large values of r,

$$\log M(r,g) \le \exp^{[m-1]}[r^{\rho_g^{(m)}+\epsilon}]$$

and

$$\log M(r,f) \ge \exp^{[m-1]}[r^{\lambda_f^{(m)}-\epsilon}].$$

Thus from (2) we obtain for all sufficiently large values of r,

$$\frac{\log^{[m]} T(r, f\circ g)}{T(r,f)\{\log T(r,f)\}^k} \le \frac{\left(\rho_f^{(m)} + \epsilon\right)\log M(r,f)}{T(r,f)\{\log T(r,f)\}^k} . \frac{\exp^{[m-1]}[r^{\rho_g^{(m)}+\epsilon}]}{\exp^{[m-1]}[r^{\lambda_f^{(m)}-\epsilon}]} + \frac{O(1)}{T(r,f)\{\log T(r,f)\}^k}.$$

$$\text{i.e., } \liminf_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{T(r,f)\{\log T(r,f)\}^k}$$

$$\le \left(\rho_f^{(m)} + \epsilon\right) \liminf_{r\to\infty} \frac{\log M(r,f)}{T(r,f)\{\log T(r,f)\}^k} . \liminf_{r\to\infty} \frac{\exp^{[m-1]}[r^{\rho_g^{(m)}+\epsilon}]}{\exp^{[m-1]}[r^{\lambda_f^{(m)}-\epsilon}]}. \quad (3)$$

Now in view of Lemma 3, the theorem follows from (3).

Remark 3 *Considering* $f = \exp z$, $g = z$ *and* $m = k = 1$ *and proceeding exactly as in Example 1 one can easily verify that no term in the denominator of* $\frac{\log^{[m]} T(r,f\circ g)}{T(r,f)\{\log T(r,f)\}^k}$ *can be removed.*

■

Theorem 3 *Let* f *and* g *be two entire functions such that* $0 < \lambda_f^{(m)} < \infty$ *and* $0 < \rho_g^{(m)} < \infty$ *Also let* $0 < \sigma_g^{(m)} < \infty$. *Then*

$$\limsup_{r\to\infty} \frac{\log^{[m+1]} M(r, f\circ g)}{\log M(r, g^{(k)})} \geq \frac{\lambda_f^{(m)}}{2^{(k+1)\rho_g^{(m)}}}$$

where $k = 0, 1, 2, 3, ...$*and* $m = 1, 2, 3, ...$

Proof. *Let* $0 < \epsilon < \min\left\{\lambda_f^{(m)}, \sigma_g^{(m)}\right\}$. *Then for a sequence of values of* r *tending to infinity we obtain that*

$$\log M\left(\frac{r}{2}, g\right) \geq \left(\sigma_g^{(m)} - \epsilon\right)\left(\frac{r}{2}\right)^{\rho_g^{(m)}}. \tag{4}$$

Again from Lemma 1 we get for all sufficiently large values of r,

$$\log^{[m+1]} M(r, f\circ g) \geq \left(\lambda_f^{(m)} - \epsilon\right)\log\frac{1}{8} + \left(\lambda_f^{(m)} - \epsilon\right)\log M\left(\frac{r}{2}, g\right). \tag{5}$$

Now for a sequence of values of r *tending to infinity it follows from* (4) *and* (5) *that*

$$\log^{[m+1]} M(r, f\circ g) \geq \left(\lambda_f^{(m)} - \epsilon\right)\log\frac{1}{8} + \left(\lambda_f^{(m)} - \epsilon\right)\left(\sigma_g^{(m)} - \epsilon\right)\left(\frac{r}{2}\right)^{\rho_g^{(m)}}. \tag{6}$$

Again by Lemma 4 we get for all sufficiently large values of r,

$$\log M\left(r, g^{(k)}\right) \leq \left(\sigma_{g^{(k)}}^{(m)} + \epsilon\right) r^{\rho_g^{(m)}} \leq \left(2^{k\rho_g^{(m)}}\sigma_g^{(m)} + \epsilon\right) r^{\rho_g^{(m)}}. \tag{7}$$

So from (6) *and* (7) *it follows that for a sequence of values of* r *tending to infinity,*

$$\frac{\log^{[m+1]} M(r, f\circ g)}{\log M(r, g^{(k)})} \geq \frac{\left(\lambda_f^{(m)} - \epsilon\right)\log\frac{1}{8} + \left(\lambda_f^{(m)} - \epsilon\right)\left(\sigma_g^{(m)} - \epsilon\right)\left(\frac{r}{2}\right)^{\rho_g^{(m)}}}{\left(2^{k\rho_g^{(m)}}\sigma_g^{(m)} + \epsilon\right) r^{\rho_g^{(m)}}}. \tag{8}$$

Since $\epsilon\,(> 0)$ *is arbitrary, we get from* (8) *that*

$$\limsup_{r\to\infty} \frac{\log^{[m+1]} M(r, f\circ g)}{\log M(r, g^{(k)})} \geq \frac{\lambda_f^{(m)}}{2^{(k+1)\rho_g^{(m)}}}.$$

Thus the theorem is established. ■

Theorem 4 *Let f be meromorphic and g be entire such that $0<\lambda_{f\circ g}^{(m)}\leq\rho_{f\circ g}^{(m)}<\infty$ and $0<\lambda_{g}^{(m)}\leq\rho_{g}^{(m)}<\infty$. Then for any positive number A,*

$$\frac{\lambda_{f\circ g}^{(m)}}{A\rho_{g}^{(m)}}\leq\liminf_{r\rightarrow\infty}\frac{\log^{[m]}T\left(r,f\circ g\right)}{\log^{[m]}T\left(r^{A},g^{(k)}\right)}\leq\frac{\lambda_{f\circ g}^{(m)}}{A\lambda_{g}^{(m)}}\leq\limsup_{r\rightarrow\infty}\frac{\log^{[m]}T\left(r,f\circ g\right)}{\log^{[m]}T\left(r^{A},g^{(k)}\right)}\leq\frac{\rho_{f\circ g}^{(m)}}{A\lambda_{g}^{(m)}},$$

where $k=0,1,2,\ldots$ and $m=1,2,3,\ldots$

Proof. For all large values of r,

$$\log^{[m]}T\left(r,f\circ g\right)\geq\left(\lambda_{f\circ g}^{(m)}-\epsilon\right)\log r \tag{9}$$

and

$$\log^{[m]}T\left(r^{A},g^{(k)}\right)\leq A\left(\rho_{g^{(k)}}^{(m)}+\epsilon\right)\log r=A\left(\rho_{g}^{(m)}+\epsilon\right)\log r. \tag{10}$$

Now from(9)and (10) it follows for all large values of r that

$$\frac{\log^{[m]}T\left(r,f\circ g\right)}{\log^{[m]}T\left(r^{A},g^{(k)}\right)}\geq\frac{\left(\lambda_{f\circ g}^{(m)}-\epsilon\right)}{A\left(\rho_{g}^{(m)}+\epsilon\right)}.$$

As $\epsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{r\rightarrow\infty}\frac{\log^{[m]}T\left(r,f\circ g\right)}{\log^{[m]}T\left(r^{A},g^{(k)}\right)}\geq\frac{\lambda_{f\circ g}^{(m)}}{A\rho_{g}^{(m)}}. \tag{11}$$

Again for a sequence of values of r tending to infinity,

$$\log^{[m]}T\left(r,f\circ g\right)\leq\left(\lambda_{f\circ g}^{(m)}+\epsilon\right)\log r \tag{12}$$

and for all large values of r,

$$\log^{[m]}T\left(r^{A},g^{(k)}\right)\geq A\left(\lambda_{g}^{(m)}-\epsilon\right)\log r. \tag{13}$$

So combining (12) and (13) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[m]}T\left(r,f\circ g\right)}{\log^{[m]}T\left(r^{A},g^{(k)}\right)}\leq\frac{\left(\lambda_{f\circ g}^{(m)}+\epsilon\right)}{A\left(\lambda_{g}^{(m)}-\epsilon\right)}.$$

Since $\epsilon\,(>0)$ is arbitrary,it follows that

$$\liminf_{r\to\infty}\frac{\log^{[m]}T\left(r,f\circ g\right)}{\log^{[m]}T\left(r^{A},g^{(k)}\right)}\leq\frac{\lambda_{f\circ g}^{(m)}}{A\lambda_{g}^{(m)}}. \tag{14}$$

Also for a sequence of values of r tending to infinity,

$$\log^{[m]}T\left(r^{A},g^{(k)}\right)\leq A\left(\lambda_{g}^{(m)}+\epsilon\right)\log r. \tag{15}$$

Now from (9) and (15) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log^{[m]}T\left(r,f\circ g\right)}{\log^{[m]}T\left(r^{A},g^{(k)}\right)}\geq\frac{\left(\lambda_{f\circ g}^{(m)}-\epsilon\right)}{A\left(\lambda_{g}^{(m)}+\epsilon\right)}.$$

Since $\epsilon\,(>0)$ is arbitrary,it follows that

$$\limsup_{r\to\infty}\frac{\log^{[m]}T\left(r,f\circ g\right)}{\log^{[m]}T\left(r^{A},g^{(k)}\right)}\geq\frac{\lambda_{f\circ g}^{(m)}}{A\lambda_{g}^{(m)}}. \tag{16}$$

Also for all large values of r,

$$\log^{[m]}T\left(r,f\circ g\right)\leq(\rho_{f\circ g}^{(m)}+\epsilon)\log r. \tag{17}$$

So from (13) and (17) it follows that for all large values of r,

$$\frac{\log^{[m]}T\left(r,f\circ g\right)}{\log^{[m]}T\left(r^{A},g^{(k)}\right)}\leq\frac{(\rho_{f\circ g}^{(m)}+\epsilon)}{A\left(\lambda_{g}^{(m)}-\epsilon\right)}.$$

As $\epsilon\,(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to\infty}\frac{\log^{[m]}T\left(r,f\circ g\right)}{\log^{[m]}T\left(r^{A},g^{(k)}\right)}\leq\frac{\rho_{f\circ g}^{(m)}}{A\lambda_{g}^{(m)}}. \tag{18}$$

Thus the theorem follows from(11), (14), (16) and(18).

Remark 4 *Considering* $f=z,\ g=\exp z,\ m=1$ *and* $A=1$ *one can easily verify that the sign* ‘ $\leq'$*cannot be replaced by* ‘ $<'$*only in Theorem* 4.

■

Theorem 5 *Let f be meromorphic and g be entire such that $0 < \lambda_{f\circ g}^{(m)} \leq \rho_{f\circ g}^{(m)} < \infty$ and $0 < \rho_g^{(m)} < \infty$. Then for any positive number A,*

$$\liminf_{r\to\infty}\frac{\log^{[m]} T\left(r, f\circ g\right)}{\log^{[m]} T\left(r^A, g^{(k)}\right)} \leq \frac{\rho_{f\circ g}^{(m)}}{A\rho_g^{(m)}} \leq \limsup_{r\to\infty}\frac{\log^{[m]} T\left(r, f\circ g\right)}{\log^{[m]} T\left(r^A, g^{(k)}\right)}$$

where $k = 0, 1, 2, 3...$ and $m = 1, 2, 3...$

Proof. For all large values of r,

$$\log^{[m]} T\left(r, f\circ g\right) \leq (\rho_{f\circ g}^{(m)} + \epsilon)\log r. \tag{19}$$

Also from the definition of generalised order we get for a sequence of values of r tending to infinity,

$$\log^{[m]} T\left(r^A, g^{(k)}\right) \geq A\left(\rho_g^{(m)} - \epsilon\right)\log r. \tag{20}$$

Now from (19) and (20) it follows that for a sequence of values of r tending to infinity,

$$\frac{\log^{[m]} T\left(r, f\circ g\right)}{\log^{[m]} T\left(r^A, g^{(k)}\right)} \leq \frac{(\rho_{f\circ g}^{(m)} + \epsilon)}{A\left(\rho_g^{(m)} - \epsilon\right)}.$$

As $\epsilon\,(> 0)$ is arbitrary, we obtain that

$$\liminf_{r\to\infty}\frac{\log^{[m]} T\left(r, f\circ g\right)}{\log^{[m]} T\left(r^A, g^{(k)}\right)} \leq \frac{\rho_{f\circ g}^{(m)}}{A\rho_g^{(m)}}. \tag{21}$$

Again for a sequence of values of r tending to infinity,

$$\log^{[m]} T\left(r, f\circ g\right) \geq (\rho_{f\circ g}^{(m)} - \epsilon)\log r \tag{22}$$

and from the definition of generalised order we get for all large values of r

$$\log^{[m]} T\left(r^A, g^{(k)}\right) \leq A\left(\rho_g^{(m)} + \epsilon\right)\log r. \tag{23}$$

So combining (22) and (23) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[m]} T\left(r, f\circ g\right)}{\log^{[m]} T\left(r^A, g^{(k)}\right)} \geq \frac{(\rho_{f\circ g}^{(m)} - \epsilon)}{A\left(\rho_g^{(m)} + \epsilon\right)}.$$

Since $\epsilon\,(> 0)$ is arbitrary,it follows that

$$\limsup_{r\to\infty}\frac{\log^{[m]} T\left(r, f\circ g\right)}{\log^{[m]} T\left(r^A, g^{(k)}\right)} \geq \frac{\rho_{f\circ g}^{(m)}}{A\rho_g^{(m)}}. \tag{24}$$

Thus the theorem follows from (21) and (24).

Remark 5 *Considering $f = z$, $g = \exp z$, $m = 1$ and $A = 1$ one can easily see that the sign '$\leq$' in Theorem 5 cannot be replaced by '$<$' only.*

Remark 6 *Combining Theorem 4 and Theorem 5 we may state the following theorem without proof.*

■

Theorem 6 *Let f be meromorphic and g be entire such that $0 < \lambda_{f\circ g}^{(m)} \leq \rho_{f\circ g}^{(m)} < \infty$ and $0 < \lambda_g^{(m)} \leq \rho_g^{(m)} < \infty$. Then for any positive number A,*

$$\begin{aligned}\liminf_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{\log^{[m]} T(r^A, g^{(k)})} &\leq \min\left\{\frac{\lambda_{f\circ g}^{(m)}}{A\lambda_g^{(m)}}, \frac{\rho_{f\circ g}^{(m)}}{A\rho_g^{(m)}}\right\} \\ &\leq \max\left\{\frac{\lambda_{f\circ g}^{(m)}}{A\lambda_g^{(m)}}, \frac{\rho_{f\circ g}^{(m)}}{A\rho_g^{(m)}}\right\} \\ &\leq \limsup_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{\log^{[m]} T(r^A, g^{(k)})}.\end{aligned}$$

Remark 7 *Considering $f = z$, $g = \exp z$, $m = 1$ and $A = 1$ one can easily verify that the sign '$\leq$' cannot be replaced by '$<$' only in Theorem 6.*

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