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On a Result of Niino and Suita

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Abstract

In the paper we establish a consequence of a result of Niino and Suita[2] on the basis of weak type of entire functions.

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1 Introduction, Definitions and Notations.

Let f be an entire function defined in the open complex plane $\mathbb{C}$. The concept of weak type of entire and meromorphic functions was introduced by Datta and Jha [1]. In the paper we discuss about a consequence of a result of Niino and Suita[2] as an application of weak type of entire functions. We do not explain the standard notations and definitions in the theory of entire functions as those are available in [3] .

The following definitions are well known.

Definition 1 *The order ρ_f and lower order λ_f of an entire function f are* *defined as*

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r}$$

and

$$\lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r}$$

where $\log^{[k]} x = \log(\log^{[k-1]} x)$ *for* $k = 1, 2, 3, ...$ *and* $\log^{[0]} x = x$.

Definition 2 [1] *The weak type* τ_f *of an entire function* f *is defined by*

$$\tau_f = \liminf_{r\to\infty} \frac{\log M(r,f)}{r^{\lambda_f}}, \quad 0 < \lambda_f < \infty.$$

Also the quantity $\bar{\tau_f}$ *is defined as*

$$\bar{\tau_f} = \limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\lambda_f}}, \quad 0 < \lambda_f < \infty.$$

Niino and Suita[2] proved the following theorem:

Theorem A [2] Let f and g be two entire functions.If $M(r,g) > \frac{2+\varepsilon}{\varepsilon}|g(0)|$ for an $\varepsilon > 0$, then $T(r, f \circ g) < (1+\varepsilon)T(M(r,g), f)$. In particular if $g(0) = 0$ then $T(r, f \circ g) \le T(M(r,g), f)$ for all $r > 0$.

In the paper we prove a consequence of Theorem A as an application of Definition 2.

2 Theorem.

In this section we present the main result of the paper.

Theorem 1 *Let f and g be two entire functions of finite lower order such that* $0 < \lambda_g < \lambda_f$ *and* $M(r,g) > \frac{2+\varepsilon}{\varepsilon}|g(0)|$ *for an* $\varepsilon > 0$. *Then for all sufficiently large values of* r,

$$T(r, f \circ g) < T(M(r,f), g) \text{ for all } r > 0.$$

Proof. In view of Theorem A, we get for all sufficiently large values of r,

$$\begin{aligned} \log T(r, f \circ g) &\leq \log T(M(r,g), f) + O(1) \\ i.e., \quad \log T(r, f \circ g) &\leq (\rho_f + \varepsilon) \log M(r,g) + O(1) \\ i.e., \quad \log T(r, f \circ g) &\leq (\rho_f + \varepsilon)(\bar{\tau_g} + \varepsilon) r^{\lambda_g} + O(1) . \end{aligned} \tag{1}$$

Again from the definition of λ_g we obtain for all sufficiently large values of r,

$$\begin{aligned} \log T(M(r,f), g) &\geq (\lambda_g - \varepsilon) \log M(r,f) \\ i.e., \quad \log T(M(r,f), g) &\geq (\lambda_g - \varepsilon)(\tau_g - \varepsilon) r^{\lambda_f} . \end{aligned} \tag{2}$$

Now from (1) and (2) it follows for all sufficiently large values of r that

$$\frac{\log T(M(r,f), g)}{\log T(r, f \circ g)} \geq \frac{(\lambda_g - \varepsilon)(\tau_g - \varepsilon) r^{\lambda_f}}{(\rho_f + \varepsilon)(\bar{\tau_g} + \varepsilon) r^{\lambda_g} + O(1)}. \tag{3}$$

Since $\lambda_g < \lambda_f$, it follows from (3) that

$$\limsup_{r \to \infty} \frac{\log T(M(r,f), g)}{\log T(r, f \circ g)} = \infty. \tag{4}$$

From (4) we obtain for all sufficiently large values of r and $k > 1$,

$$\begin{aligned} \log T(M(r,f), g) &> k \log T(r, f \circ g) \\ i.e., \quad \log T(M(r,f), g) &> \log\{T(r, f \circ g)\}^k \\ i.e., \quad \log T(M(r,f), g) &> \log T(r, f \circ g) \\ i.e., \quad T(M(r,f), g) &> T(r, f \circ g). \end{aligned}$$

This proves the theorem. ■

Remark 1 *If in particular $g(0) = 0$ and the other conditions remain the same then also Theorem 1 holds.*

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