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New Properties for Certain Integral Operators

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Abstract

The purpose of the present paper is to use the so-called pre-Schwarzian derivatives to obtain some properties of certain integral operator. We first establish the relationships between the two integral operators F_n *and* $F_{\gamma_1,\ldots,\gamma_n}$, which were given by Breaz and Breaz [1], and Breaz et.al [2] respectively, under the familiar classes of starlike of order α, $S^*(\alpha)$ and convex functions of order α, $K(\alpha)$. Furthermore, some other properties of the integral operator $F_{\gamma_1,\ldots,\gamma_n}$ by using the concept of the norm and pre-Schwarzian derivatives are obtained.

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1 Introduction

Let $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane C and let H denotes the space of all holomorphic functions on U. Here we think of H as a topological vector space endowed with the topology of uniform convergence over compact subsets of U. For example, a sequence $\{f_i\}$ of holomorphic functions that converges uniformly on compact sets has a holomorphic functions as its limit. Further, let $\mathcal{A}$ denotes the class of functions normalized by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U}$ and satisfy the condition $f(0) = f'(0) - 1 = 0$. We also denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions which are also univalent in $\mathcal{U}$.

A function $f \in \mathcal{A}$ is the convex function of order $\alpha, 0 \leq \alpha < 1$ if f satisfies the following inequality

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > \alpha, \quad z \in \mathcal{U}$$

and we denote this class by $\mathcal{K}(\alpha)$.

Similarly, if $f \in \mathcal{A}$ satisfies the following inequality:

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in \mathcal{U}$$

for some α, $0 \leq \alpha < 1$, then f is said to be starlike of order α and we denote this class by $\mathcal{S}^*(\alpha)$. We note that $f \in \mathcal{K} \Leftrightarrow zf'(z) \in \mathcal{S}^*$, $z \in \mathcal{U}$. In particular case, the classes $\mathcal{K}(0) = \mathcal{K}$ and $\mathcal{S}^*(0) = \mathcal{S}^*$ are familiar classes of starlike and convex functions in $\mathcal{U}$.

A holomorphic function f on the unit disk U is said to be uniformly locally univalent if it is univalent on each hyperbolic disk $D(a, \rho) = \{z \in U : |(z-a)/(z-\overline{a})| < \tanh \rho\}$, with radius ρ and center $a \in U$ for a positive constant ρ.

For a locally univalent holomorphic function f, we define

$$T_f = \frac{f''}{f'},$$

which is said to be pre-Schwarzian derivative (or nonlinearity). For a locally univalent function f in U, we define the norm of T_f by

$$\|T_f\| = \sup_{|z| \in U} (1 - |z|^2) |T_f|$$

It is well-known [8] that a holomorphic function f on the unit disk is uniformly locally univalent if and only if the pre-Schwarzian derivative $T_f = \frac{f''}{f'}$ is hyperbolic bounded, i.e., the norm

$$\|T_f\| = \sup_{|z| \in U} (1 - |z|^2) |T_f|,$$

is finite. It is well-known that from Becker's univalence criterion [5]: *every analytic function f in U with $\|T_f\| \leq 1$ is in fact univalent in U.* Conversely, $\|T_f\| \leq 6$ holds if f univalent.

Lemma 1.1. [**3**]. Let $\alpha \in [0, 1[$. Then $f \in S^*(\alpha)$ if and only if $g \in S^*$, where $g(z) = z\left[\frac{f(z)}{z}\right]^{\frac{1}{1-\alpha}}$, $z \in U$.

The branch of the power function is chosen such that $\left.\left[\frac{f(z)}{z}\right]^{\frac{1}{1-\alpha}}\right|_{z=0} = 1$.

Theorem 1.1. [**5, 6, 10**]. Let f be analytic and locally univalent in U. Then

i) If $\| T_f \| \leq 1$, then f is univalent, and ii) If $\| T_f \| \leq 2$, then f is bounded.

Theorem 1.2. [**7**]. Let $0 \leq \alpha < 1$ *and* $f \in S$.

1) If f is starlike of order α, *i.e*, $\Re\, zf'(z)/f(z) > \alpha$, then $\| T_f \| \leq 6 - 4\alpha$.

2) If f is convex of order α, *i.e*, $\Re\, zf(z)/f'(z) + 1 > \alpha$, then $\| T_f \| \leq 4(1 - \alpha)$.

The constants are sharp.

The study of the integral operators has been rapidly investigated by many authors in the field of univalent functions. The integral operator

$$\Upsilon[f](z) = \int_0^z \frac{f(\zeta)}{\zeta} d\zeta,$$

was introduced by Alexander [4]. Note that $f \in S^*(\alpha) \Leftrightarrow \Upsilon[f] \in K(\alpha)$.

For the complex number γ, Kim and Merkes [9] considered the nonlinear integral transform $\Upsilon_\gamma[f](z)$, defined by

$$\Upsilon_\gamma[f](z) = \int_0^z \left(\frac{f(\zeta)}{\zeta}\right)^\gamma d\zeta.$$

For $f_i(z) \in A$ and $\gamma_i > 0$, for all $i \in \{1, 2, 3, ..., n\}$, Breaz and Breaz [1], introduced the following integral operator

$$F_n[f](z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\gamma_1} ... \left(\frac{f_n(t)}{t}\right)^{\gamma_n} dt. \tag{1.2}$$

In [11] Kim, Ponnusamy and Sugawa defined the following integral operator

$$I_\gamma[f](z) = \int_0^z [f'(t)]^\gamma dt,$$

for $\gamma \in \mathcal{C},\ f \in A$.

Recently Breaz et.al [2] introduced the following integral operator

$$F_{\gamma_1,\dots,\gamma_n}[f](z) = \int_0^z [f_1'(t)]^{\gamma_1} \dots [f_n'(t)]^{\gamma_n} dt. \tag{1.3}$$

In this paper, we first establish the relationships between the two integral operators F_n and $F_{\gamma_1,\dots,\gamma_n}$ which defined as in (1.2) and (1.3) respectively, under the familiar classes of starlike of order α, $S^*(\alpha)$ and convex functions of order α, $K(\alpha)$.

Furthermore, some other properties of the integral operator $F_{\gamma_1,\dots,\gamma_n}$ are obtained by using the concept of the norm and pre-Schwarzian derivatives .

2 Main results

Theorem 2.1. For $\gamma_i \in \mathcal{R},\ \ \gamma_i > 0,\ \ 0 \le \alpha_i < 1,\ i \in \{1, 2, 3, ..., n\}$, we have

$$F_n(S^*) = F_{(1-\alpha_1)\gamma_1,\dots,(1-\alpha_n)\gamma_n}(K),$$

where, F_n and $F_{\gamma_1,\dots,\gamma_n}$ are the integral operators defined as in (1.2) and (1.3) respectively, and S^* and K are the classes of starlike and convex functions respectively.

Proof. Let $f \in F_n\ (S^*)$, then there exist $g_i(z) \in S^*(\alpha_i)$, for $i = \{1, 2, 3, ..., n\}$ such that

$$f(z) = \int_0^z \left(\frac{g_1(t)}{t}\right)^{\gamma_1} \dots \left(\frac{g_n(t)}{t}\right)^{\gamma_n} dt.$$

Since $g_i(z) \in S^*(\alpha_i)$, for $i = \{1, 2, 3, ..., n\}$, then by apply Lemma 1.1, there exist $s_i(z) \in S^*$, for $i = \{1, 2, 3, ..., n\}$, such that

$$\frac{s_i(z)}{z} = \left(\frac{g_i(z)}{z}\right)^{\frac{1}{1-\alpha_i}}, \text{ for } i = \{1, 2, 3, ..., n\ \}.$$

Therefore

$$f(z) = \int_0^z \left(\frac{s_1(t)}{t}\right)^{(1-\alpha_1)\gamma_1} \dots \left(\frac{s_n(t)}{t}\right)^{(1-\alpha_n)\gamma_n} dt.$$

By using the Alexander relation between the classes S^* *and* K, there exist $u(z) \in K$ such that $s(z) = zu'(z)$, then

$$f(z) = \int_0^z \left(u_1'(z)\right)^{(1-\alpha_1)\gamma_1} ... \left(u_n'(z)\right)^{(1-\alpha_n)\gamma_n} dt.$$

Then $f(z) \in F_{(1-\alpha_1)\gamma_1,...,(1-\alpha_n)\gamma_n}(K)$. As a result $F_n\ (S^*) \subset F_{(1-\alpha_1)\gamma_1,...,(1-\alpha_n)\gamma_n}(K)$.

Conversely, let $f(z) \in F_{(1-\alpha_1)\gamma_1,...,(1-\alpha_n)\gamma_n}(K)$. Then there exist $u_i(z) \in K$, for $i = \{1, 2, 3, ..., n\}$ such that

$$f(z) = \int_0^z \left(u_1'(z)\right)^{(1-\alpha_1)\gamma_1} ... \left(u_n'(z)\right)^{(1-\alpha_n)\gamma_n} dt.$$

Since $u_i(z) \in K$, for $i = \{1, 2, 3, ..., n\}$,then $s_i(z) = zu_i'(z) \in S^*$.

$$f(z) = \int_0^z \left(\frac{s_1(t)}{t}\right)^{(1-\alpha_1)\gamma_1} ... \left(\frac{s_n(t)}{t}\right)^{(1-\alpha_n)\gamma_n} dt.$$

Since $s_i(z) \in S^*$, for $i = \{1, 2, 3, ..., n\}$, then by apply Lemma 1.1, there exist $g_i(z) \in S^*(\alpha_i)$, for $i = \{1, 2, 3, ..., n\}$, such that

$$\frac{s_i(z)}{z} = \left(\frac{g_i(z)}{z}\right)^{\frac{1}{1-\alpha_i}}, \text{ for } i = \{1, 2, 3, ..., n\ .\}$$

Then

$$f(z) = \int_0^z \left(\frac{g_1(t)}{t}\right)^{\gamma_1} ... \left(\frac{g_n(t)}{t}\right)^{\gamma_n} dt.$$

Thus $f \in F_n\ (S^*)$, and therefore

$$F_{(1-\alpha_1)\gamma_1,...,(1-\alpha_n)\gamma_n}(K) \subset F_n\ (S^*).$$

From the above proof, we obtain that $F_n\ (S^*) = F_{(1-\alpha_1)\gamma_1,...,(1-\alpha_n)\gamma_n}(K)$.

Letting $n = 1$ in Theorems 2.1, we have

Corollary 2.1 For $\gamma_1 \in \mathcal{R}$, $\gamma_1 > 0$, $0 \le \alpha_1 < 1$, we have

$$F_1(S^*) = F_{(1-\alpha_1)\gamma_1}(K).$$

Now by using the concept of norm and the so-called pre-Schwarzian derivative and applying the theorems 1.1 and 1.2, we introduce some properties for the integral operator $F_{\gamma_1,\ldots,\gamma_n}$.

Theorem 2.2. Let $\gamma_i \in \mathcal{R}$, $i \in \{1, 2, ..., n\}$, $\gamma_i > 0$ and $f_i \in A$. Suppose that $F_{\gamma_1,\ldots,\gamma_n}$ is locally univalent in U,

1) If

$$\|T_{f_i}\| \leq \frac{1}{\sum\limits_{i=1}^{n} \gamma_i} \tag{2.1}$$

then $F_{\gamma_1,\ldots,\gamma_n}$ is univalent.

2) If

$$\|T_{f_i}\| \leq \frac{2}{\sum\limits_{i=1}^{n} \gamma_i} \tag{2.2}$$

then $F_{\gamma_1,\ldots,\gamma_n}$ is bounded, where $F_{\gamma_1,\ldots,\gamma_n}$ is the integral operator defined as in (1.3).

Proof. Since

$$\| T_{F_{\gamma_1,\ldots,\gamma_n}} \| = \sup_{z\in U}(1 - |z|^2) \, | T_{F_{\gamma_1,\ldots,\gamma_n}} |$$

.

Then

$$\| T_{F_{\gamma_1,\ldots,\gamma_n}} \| = \sup_{z\in U}(1 - |z|^2) \left| \frac{\left(\int\limits_0^z [f_1'(t)]^{\gamma_1} ...[f_n'(t)]^{\gamma_n} dt\right)''}{\left(\int\limits_0^z [f_1'(t)]^{\gamma_1} ...[f_n'(t)]^{\gamma_n} dt\right)'} \right|$$

$$= \sup_{z\in U}(1 - |z|^2) \left| \sum_{i=1}^{n} \gamma_i \frac{f_i''}{f_i'} \right| .$$

Therefore

$$\| T_{F_{\gamma_1,...,\gamma_n}} \| \leq \sup_{z\in U}(1-|z|^2)\sum_{i=1}^{n}\gamma_i\left|\frac{f_i''}{f_i'}\right|.$$

Then

$$\| T_{F_{\gamma_1,...,\gamma_n}} \| \leq \sum_{i=1}^{n}\gamma_i \, \mathop{Sup}_{z\in U}(1-|z|^2)\left|\frac{f_i''}{f_i'}\right|.$$

Thus

$$\| T_{F_{\gamma_1,...,\gamma_n}} \| \leq \sum_{i=1}^{n}\gamma_i \| T_{f_i}\| . \tag{2.3}$$

From (2.1), (2.2) and (2.3) and applying Theorem 1.1, we obtain the assertions.

Theorem 2.3. Let $f_i,\ i\in\{1,2,...,n\}$ be a family of functions and $f_i\in S$.

1) If f_i are starlike of order $\beta_i,\ i\in\{1,2,...,n\}$, then

$$\| T_{F_{\gamma_1,...,\gamma_n}} \| \leq 2\sum_{i=1}^{n}\gamma_i(3-2\beta_i).$$

2) If f_i are convex of order $\beta_i,\ i\in\{1,2,...,n\}$, then

$$\| T_{F_{\gamma_1,...,\gamma_n}} \| \leq 4\sum_{i=1}^{n}\gamma_i(1-\beta_i).$$

Proof. The results follow from (2.3) and by using Theorem 1.2.

Corollary 2.2. Let $f_i,\ i\in\{1,2,...,n\}$ be a family of functions and $f_i\in S$.

1) If f_i, $i\in\{1,2,...,n\}$, are starlike of order β, then

$$\| T_{F_{\gamma_1,...,\gamma_n}} \| \leq 2(3-2\beta)\sum_{i=1}^{n}\gamma_i$$

2) If f_i, $i\in\{1,2,...,n\}$, are convex of order β, then

$$\| T_{F_{\gamma_1,...,\gamma_n}} \| \leq 4(1-\beta)\sum_{i=1}^{n}\gamma_i.$$

Proof. We consider in Theorem 2.2 such that $\beta_1 = \beta_2 = ... = \beta_n$.

Letting $n = 1$ in theorems 2.2 and 2.3 respectively, we have the following:

Corollary 2.3. Let $\gamma_1 \in \mathcal{R}, \ \gamma_1 > 0$ and $f_1 \in A$. Suppose that F_{γ_1} is locally univalent in U:

1) If

$$\| T_{f_1} \| \leq \frac{1}{\gamma_1}.$$

Then F_{γ_1} is univalent.

2) If

$$\| T_{f_1} \| \leq \frac{2}{\gamma_1}.$$

Then F_{γ_1} is bounded.

Corollary 2.4. Let $f_1 \in S$.

1) If f_1 are starlike of order β_1, then

$$\left\| T_{F_{\gamma_1}} \right\| \leq 2\gamma_1(3 - 2\beta_1).$$

2) If f_1 are convex of order β_1, then

$$\left\| T_{F_{\gamma_1}} \right\| \leq 4\gamma_1(1 - \beta_1).$$

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