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Fractionalized Groups

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Abstract

The object of this paper is to introduce a new type of operation like a binary operation on a set. We know binary operations on a set; here we compose two elements of the set and we get a unique element of the set by a rule called binary operation. We can think that if the composition, not completed completely then the composition of the elements have been done partially, like we are mixing two things in a mixture grinder but the power connection cuts off before mixing been completed. This idea motivated us to think of binary operations fractionally. But to study such type of binary operation we have taken the base set with a group structure. Next we will define a real valued function on a fractionalized group named norm on the fractionalized group and we will see that this will generate a number of topologies on the fractionalized group indexed by the unit interval [0,1] and lower indexed topology is weaker than the upper indexed topology.

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1 Introduction and Preliminaries

We know binary operations on a non-empty set, here we compose two elements of the set by a rule or a mapping called a binary operation on the set. We can think that we are composing two elements of the set, but due to an interruption in the middle of the process of completing composing them, the composition process have not done completely, like we are going to mix two things in a mixture grinder, but the power connection cuts off before mixing them completely - this idea motivated us to think of binary operations fractionally.

Here we have taken the set say G equipped with a binary operation "o" ; we are thinking to compose a, b∈G fractionally with respect to the operation "o".

So fractional composition can be done from zero fractional level to fractional level 1, fractional level 1 mean complete composition i.e. aob. Now what will be the value if we compose a and b with respect to "o" at fractional level zero, so we think to equiped G with a group structure with respect to "o" and assign the value of fractional composition a and b with respect to "o" at zero level be the identity element e of the group (G , o). We call this group as fractionalized group.

Usually norm or length concept is defined on a linear space, but likewise we can define the norm or length or magnitude of a element in a group or fractionalized group. We will see in our study that this norm on a fractionalized group will generate a quasi-metric or non-symmetric metric on the set and that will generate a topology on the normed fractionaliez group - named as norm induce topology on the normed partial groupoid.

The symbol ■ will be use to indicate the end of any definition, example, theorem and its proof etc.

Before going to the main topic let us first recall some basic definitions:

Definition1.1.(Steen and Seebach[6]) A metric on a set X is a mapping: d:X×X→$\mathbf{R}^+$ where $\mathbf{R}^+$
is the set of non-negative real numbers satisfying the following properties:
∀ x,y,z∈X
M_1:d(x,x)=0
M_2:d(x,z)≤d(x,y)+d(y,z)
M_3:d(x,y)=d(y,x)
M_4:if x≠y, d(x,y)>0 .

We call d(x,y)the distance between x and y.

If d satisfies only M_1, M_2 and M_4 it is called quasi-metric.

If d satisfies only M_1, M_2 and M_3 it is called pseudo-metric.

If d satisfies only M_1 and M_2 it is called quasi-pseudo-metric. ■

2 Fractionalized Group and Norm on a Fractionalized Group

Definition2.1. A fractionalized group is an order triplet (G, o, $\odot$), where (G, o) is a group and $\odot$ is a mapping

$\odot$:G×G×[0,1]⟶G

(a, b, t)↦$a\odot_t b$

satisfying the following conditions:

(i) $a\odot_0 b=e$, the identity element of the group (G, o) and

(ii) $a\odot_1 b=aob$. ■

Definition2.2. Let (G, o, $\odot$) be a fractionalized group, A mapping

$\|.\|$:G⟶**R**

a ↦ $\|a\|$

is said to be norm on the fractionalized group if it satisfies the following conditions:

∀a, b ∈G and t, s∈[0, 1]

(i) $\|a\|$ iff a=e

(ii) $\|a\odot_t b\| \leq \|a\odot_s b\|$, for t≤s

(iii)$\|a\odot_t b\| \leq \|a\odot_t e\|+\|e\odot_t b\|$

The fractionalized group with the norm $\|.\|$ on G will be denoted by [(G, o, $\odot$), $\|.\|$] and this will be called a normed fractionalized group.

It is easy to see that if we put t = 1 in the third condition we have $\|aob\| \leq\|a\|+\|b\|$.

The norm is said to be symmetric if ∀a∈G, $\|a^{-1}\|=\|a\|$. ■

Theorem2.3.*For a normed fractionalized group [(G, o, $\odot$), $\|.\|$] (i)$\|a\| \geq 0$, ∀a∈G. (ii) $e\odot_t e=e$, for any t∈[0, 1]. (iii) $a\odot_t a^{-1}=e=a^{-1}\odot_t a$, for any a∈G and t∈[0, 1].*

Proof. (i) It is since $0=\|e\|=\|e\odot_0 a\| \leq \|e\odot_1 a\|=\|eoa\|=\|a\|$.

(ii) $0\leq \|e\odot_t e\| \leq \|e\odot_1 e\|=\|eoe\|=\|e\|=0$, so $\|e\odot_t e\|=0$, hence $e\odot_t e=e$.

(iii) $0=\|e\|=\|a\odot_0 a^{-1}\| \leq \|a\odot_t a^{-1}\| \leq \|\|a\odot_1 a^{-1}\| =\|aoa^{-1}\|=\|e\|=0$, so $\|a\odot_t a^{-1}\|=0$, hence $a\odot_t a^{-1}=e$. Similarly $a^{-1}\odot_t a=e$.■

Example2.4. Let us consider the group (**R** , +). We define the mapping

$\odot$: **R**×**R**×[0, 1] ⟶**R**

(a, b, t) ↦$a\odot_t b=t(a+b)$.

Then (**R** , +, $\odot$) is a fractionalized group.

Now we define the mapping:

$\|.\|$: **R** ⟶ **R**, defined as $\|a\|=|a|$

then $\|.\|$ is a norm on the fractionalized group (**R** , +, $\odot$). ■

§For a normed fractionalized group [(G, o, $\odot$), $\|.\|$] we define the mapping

d : G×G $\longrightarrow$**R**
as ∀a, b∈G, d(a, b)=$\|aob^{-1}\|$.

Now we have the following theorem:
Theorem2.5.*d is a quasi-metric on G, i.e. (G, d) is a quasi-metric space.*
Proof. ∀a, b, c∈G, we see that:
(i) as $\|aob^{-1}\| \geq 0$, so d(a, b)≥0.
(ii) d(a, b)=0 iff $\|aob^{-1}\|=0$ iff $aob^{-1}=e$ i.e. iff a=b.
(iii) d(a, b)= $\|aob^{-1}\| = \|(aoc^{-1})o(cob^{-1})\|$
$\leq \|aoc^{-1}\|+\|cob^{-1}\|$ = d(a, c)+d(c, b).

Hence d is a quasi-metric on G, i.e. (G, d) is a quasi-metric space. ■

Corollary2.6. *If ‖.‖ is symmetric then d is a metric.*
Proof.It is since, ∀a, b∈G, d(a, b)=$\|aob^{-1}\|=\|(aob^{-1})^{-1}\|=\|boa^{-1}\|$=d(b, a).■

Definition2.7.Let [(G, o, $\bigodot$), ‖.‖] be a normed fractionalized group. We define a ball with centre a∈G and radius r>0 by:
$B_r(a)$={ x∈G : d(x, a)=$\|xoa^{-1}\|$ <r }. ■

Theorem2.8.*IThe collection $\sum$={ $B_r(a)$: a∈G, r>0 } forms a base for a topology on G.*
Proof.We see that :
d(a, a)=0,
so for any r>0 and a∈G, a∈$B_r(a)$.
Hence, G=$\cup_{a\in G}B_r(a)$, so,

(i) There is a sub collection of $\sum$ whose union is G.

(ii) Let a, b∈G and r, l>0 and $B_r(a)\cap B_l(b)\neq \varnothing$.
Let c∈$B_r(a)\cap B_l(b)$.
So c∈$B_r(a)$ and c∈$B_l(b)$, i.e. d(c, a)<r and d(c, b)<l.
Let p=min{r-d(c, a), l-d(c, b)}, then p>0.
c∈$B_p(c)\subseteq B_r(a)\cap B_l(b)$.

(i) and (ii) shows that forms base for a topology $\tau(\Sigma)$ (say) on G, we will say this topology as the norm induced topology on G. Let us denote this topology by $\tau(\|.\|)$.■

Theorem2.9.*In the normed fractionalized group [(G, o, $\bigodot$), ‖.‖], where (G, o) is a commutative group and the norm is symmetric, then G is a topo-*

logical group with respect to the group (G, o) and the topology induced from the norm.

Proof.(i) Let a, b∈G. For any r>0 we consider the ball B_r(aob) about aob in G. Again we consider the open balls $B_{r/2}$(a) and $B_{r/2}$(b) about a and b respectively in G, so $B_{r/2}(a)\times B_{r/2}(b)$ is a open set in G×G.

For (x, y)∈$B_{r/2}(a)\times B_{r/2}(b)$, x∈$B_{r/2}$(a) and y∈$B_{r/2}$(b).

i.e. d(x, a)=$\|xoa^{-1}\|$ <r/2 and d(y, b)=$\|yob^{-1}\|$ <r/2.

Now, xoy∈G.

And d(xoy, aob)=$\|(xoy)o(aob)^{-1}$=$\|(xoy)o(b^{-1}oa^{-1})\|$

=$\|(xoa^{-1})o(yob^{-1})\|$, (Since (G, o) is a commutative group.)

≤ $\|xoa^{-1}\|+\|yob^{-1}\|$ <r/2+r/2=r.

So, xoy∈B_r(aob),

i.e. (x, y)∈$B_{r/2}(a)\times B_{r/2}(b)\Rightarrow$ xoy∈B_r(aob)

i.e. the mapping o : G×G ⟶ G, defined as (x, y) ↦ xoy is continuous.

(ii) Let a∈G. For any r>0 we consider the ball $B_r(a^{-1})$ about a^{-1} in G. Again we consider the open ball B_r(a) about a in G.

For x∈B_r(a), i.e. d(x, a)=$\|xoa^{-1}\|$ <r.

Now, d(x^{-1}, a^{-1})=$\|x^{-1}o(a^{-1})^{-1}\|=\|x^{-1}oa\|$

=$\|(x^{-1}oa)^{-1}\|$, (Since the norm is symmetric)

=$\|a^{-1}ox\|$

=$\|xoa^{-1}\|$ <r, (Since (G, o) is a commutative groups)

Hence, x∈B_r(a) ⇒ x^{-1} ∈$B_r(a^{-1})$,

i.e. the mapping G ⟶ G defined by x ↦ x^{-1} is continuous.

(i) and (ii) shows that G is a topological group.■

Theorem2.10.*In the normed fractionalized group [(G, o, ⨀), ‖.‖], where the norm is symmetric, then the mapping ‖.‖ : G ⟶* ***R*** *is continuous.*

Proof.Let a, b∈G. Now, $\|a\|=\|(aob^{-1})ob\| \leq \|aob^{-1}\|+\|b\|$, i.e. $\|a\|$-$\|b\| \leq \|aob^{-1}\|$. Similarly $\|b\|$-$\|a\| \leq \|boa^{-1}\| =\|(boa^{-1})^{-1}\|=\|aob^{-1}\|$. Hence, $|\|a\|-\|b\|| \leq \|aob^{-1}\|$. For any given ε >0 we consider the ball B_ε(a). So x∈B_ε(a) ⇒ $\|xoa^{-1}\| < \varepsilon \Rightarrow |\|x\|-\|a\|| \leq \varepsilon$ and the theorem follows. ■

Definition2.11.A fractionalized group (G, o, ⨀) is said to be an associative fractionalized group if ∀a, b, c∈G and t∈[0, 1], a$\bigodot_t$(boc)=(aob)$\bigodot_t$c.

If the corresponding fractionalized group of a normed fractionalized group [(G, o, ⨀), ‖.‖] is associative the normed fractionalized group is said to be associative normed fractionalized group. ■

Theorem2.12.*In a associative fractionalized group (G, o, ⨀) (i) e$\bigodot_t$a=a$\bigodot_t$e, ∀a∈G and t∈[0, 1]. (ii)If the group (G, o) is commutative than a$\bigodot_t$b=b$\bigodot_t$a,*

$\forall$a, b$\in$G and t$\in$[0, 1]. And conversely if this condition hold then the group (G, o) obviously a commutative group. (iii) (aob)$\bigodot_t$e=a$\bigodot_t$b=e$\bigodot_t$(aob), $\forall$a, b$\in$G and t$\in$[0, 1].

Proof. (i) e$\bigodot_t$a=e$\bigodot_t$(aoe)=(eoa)$\bigodot_t$e=a$\bigodot_t$e.

(ii)Let the group G be commutative. Now,

a$\bigodot_t$b=a$\bigodot_t$(boe)=(aob)$\bigodot_t$e=(boa)$\bigodot_t$e=b$\bigodot_t$(aoe)=b$\bigodot_t$a.

(iii) (aob)$\bigodot_t$e=a$\bigodot_t$(boe)=a$\bigodot_t$b=(eoa)$\bigodot_t$b=e$\bigodot_t$(aob). ■

§For a associative normed fractionalized group [(G, o, $\bigodot$), $\|.\|$], for each t$\in$[0, 1] we define the mapping d_t : G$\times$G$\longrightarrow$**R** as $\forall$a, b$\in$G, d_t(a, b)=$\|a\bigodot_t b^{-1}\|$.

With the above definition we have the following theorem:

Theorem2.13.*d_t is a quasi-psudo-metric on G, i.e. (G, d_t) is a quasi-metric space.*

Proof. $\forall$a, b, c$\in$G we see that:

(i) as $\|a\bigodot_t b\| \geq 0$, so d_t(a, b)$\geq$0

(ii) d_t(a, a)=$\|a\bigodot_t a^{-1}\|$=$\|e\|$=0.

(iii)d_t(a, b)=$\|a\bigodot_t b^{-1}\|$=$\|a\bigodot_t(eob^{-1})\|$ $\|a\bigodot_t(c^{-1}o(cob^{-1}))\|$

=$\|(aoc^{-1})\bigodot_t(cob^{-1})\|$

$\leq$ $\|(aoc^{-1})\bigodot_t e\|$+$\|e\bigodot_t(cob^{-1})\|$

=$\|a\bigodot_t c^{-1}\|$+$\|c\bigodot_t b^{-1}\|$ =d_t(a, c)+d_t(c, b)

Hence d_t is a quasi-metric on G, i.e. (G, d_t) is a quasi-psudo-metric space.■

We will say that d_t(a, b) is the t-level distance from a to b.

It is easy to see that d_1 = d and d_t(a, b)$\leq$$d_s(a, b)\leq$d(a, b) for any t, s$\in$[0, 1] with t$\leq$s. Also, $\forall$ a, b$\in$G, d_0=$\|a\bigodot_0 b^{-1}\|$=$\|e\|$=0, i.e. in the zero level distance it is seems that all points of the space collapsed to a single point.■

Definition2.14. Let [(G, o, $\bigodot$), $\|.\|$] be an associative normed fractionalized group. For each t$\in$[0, 1] we define a t-ball with centre a$\in$G and radius r>0 by:

$B_r{}^t$(a)={ x$\in$G : d_t(x, a)=$\|x\bigodot_t a^{-1}\|$ $\leq$r }.

It is easy to see that each such t-ball is non empty since at least a$\in$$B_r{}^t$(a).

Theorem2.15.*The collection Σ_t={$B_r{}^t$(a) : a$\in$G, r>0 } forms a basis for a topology on G.*

Proof. The proof is similar to that of the Theorem2.8.

Let us denote the topology generated by the base Σ_t by $\tau_t(\|.\|)$, we will say this topology as the t-level topology on the associated normed fractionalized group. It is obvious that Σ_1=Σ and so $\tau_1(\|.\|)$=$\tau(\|.\|)$. It is also to be noted that for any r>0 and t, s∈[0, 1] with t≤s, $B_r^s(a) \subseteq B_r^t(a)$, since $d_t(a, b) \leq d_s(a, b)$ for any a, b∈G. The zero-level topology $\tau_0(\|.\|) = \{ \varnothing, G \}$ is the indiscrete topology on G, i.e. the weakest topology on G. ■

Theorem2.16.*For t, s∈[0, 1] with t≤s, $\tau_t(\|.\|) \subseteq \tau_s(\|.\|)$, i.e. the s-level topology is the stronger topology than the t-level topology on the associative normed fractionalized group [(G, o, ⊙), ‖.‖].*

Proof. For r>0 and a∈G, since $B_r^t(a)$is open with respect to the topology $\tau_t(\|.\|)$, so for $b \in B_r^t(a)$, there exists l>0 such that $b \in B_l^t(a) \subseteq B_r^t(a)$. Since $b \in B_l^s(a) \subseteq B_l^t(a)$, so $b \in B_l^s(a) \subseteq B_r^t(a)$, i.e. s-level topology is the stronger topology than the t-level topology■

In the above theorem it seems that, at the zero-level topology the space G is looked like collapsed at a single point as there is no non-zero distance at the zero-level from any point to any other point, and as the level increases the space G started to expand gradually with more stronger topological structure and at the final level, i.e. at 1-level we get the maximum distance from any point to any other point and that gives the last structure as the topological space (G, $\tau_1(\|.\|)$), where $\tau_1(\|.\|) = \tau(\|.\|)$.

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