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# Large Solution of Quasilinear Elliptic Equations under the Keller-Osserman Condition

**G. A. Afrouzi and S. Shokooh**

Department of Mathematics, Faculty of Basic Sciences
University of Mazandaran, Babolsar, Iran
afrouzi@umz.ac.ir, sd.shokooh@gmail.com

**Abstract**

In this paper, our main purpose is to consider the quasilinear equation

$$div(|\nabla u|^{p-2}\nabla u) = a(x)f(u)$$

on a domain $\Omega \subseteq R^N, N \geq 3$, where $a$ is a nonnegative nontrivial continuous function and $f$ is continuous and nondecreasing on $[0, \infty]$, satisfies $f(0) = 0, f(s) > 0$ for $s > 0$ and the Keller-Osserman condition $\int_1^\infty (F(s))^{-1/p} ds = \infty$ where $F(s) = \int_0^s f(t)dt$. We establish condition on the function $a$ that are necessary and sufficient for the existence of positive solutions, bounded and unbounded, of the given equation.

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**Keywords:** Entire solution; Large solution; Elliptic equation; Sublinear

## 1 Introduction

We consider the existence of explosive solutions of the problem

$$div(|\nabla u|^{p-2}\nabla u) = a(x)f(u), \quad x \in \Omega, \tag{1}$$

$$u|_{\partial\Omega} = \infty, \tag{2}$$

where $p > 1$, and $\Omega \subset R^N$ is a bounded domain with a smooth boundary or an un-bounded domain or the whole $R^N$. We call these explosive solutions of (1) on $\Omega$. If $\Omega$ is bounded, we require $u(x) \to \infty$ as $x \to \partial\Omega$. within $\Omega$. If $\Omega$, is un-bounded, we also require $u(x) \to \infty$ as $|x| \to \infty$ within $\Omega$. If $\Omega = R^N$, we call these entire explosive solutions. This problem appears in the study of non-Newtonian fluids [2,19] and non-Newtonian filtration [8,9]. Such problems also arise in the study of the subsonic motion of a gas [21], the electric potential in some bodies [15], and Riemannian geometry [4].

Explosive solutions of the problem

$\Delta u(x) = f(u(x)), \quad x \in \Omega$

$u|_{\partial\Omega} = \infty,$

where $\Omega$ is a bounded domain in $R^N (N \geq 1)$ have been extensively studied, see [1-10,11,17]. A problem with $f(u) = -e^u$ and $N = 2$ was first considered by Bieberbach [3] in 1916. Bieberbach showed that if $\Omega$ is a bounded domain in $R^2$ such that $\partial\Omega$ is a $C^2$ submanifold of $R^2$, then there exists a unique $u \in C^2(\Omega)$ such that $\Delta u = -e^u$in $\Omega$ and $|u(x) - ln(d(x))^{-2}|$ is bounded on $\Omega$. Here $d(x)$ denotes the distance from a point $x$ to $\partial\Omega$. Rademacher [23], using the idea of Bieberbach, extended the above result to a smooth bounded domain in $R^3$. In this case the problem plays an important role, when $N = 2$, in the theory of Riemann surfaces of constant negative curvature and in the theory of automorphic functions, and when $N = 3$, according to [23], in the study of the electric potential in a glowing hollow metal body. Lazer and McKenna [15] extended the results for a bounded domain $\Omega$ in $R^N (N \geq 1)$ satisfying a uniform external sphere condition and the nonlinearity $f = f(x, u) = p(x)e^u$, where $p(x)$ is continuous and strictly negative on $\overline{\Omega}$. Very recently, Lazer and McKenna [16] obtained similar results when D is replaced by the Monge-Ampere operator and X is a smooth, strictly convex, bounded domain. Similar results were also obtained for $f = p(x)u^a$ with $a > 1$. Posteraro [22], for $f(u) = -e^u$ and $N \geq 2$, proved the estimates for the solution $u(x)$ of Eq. (1.3) and for the measure of $\Omega$ comparing with a problem of the same type defined in a ball. In particular, when $N = 2$, Posteraro [22] obtained an explicit estimate of the minimum of $u(x)$ in terms of the measure of $\Omega$:

$minu_(x) \geq ln(8\pi/|\Omega|).$

The existence, but not uniqueness, of solutions of above equation with $f$ monotone was studied by Keller [10]. For $f(u) = -u^a$ with $a > 1$, the problem above is of interest in the study of the subsonic motion of a gas when $a = 2$, and is related to a problem involving super-diffusion, particularly for $1 < a \leq 2$ (see [6,7]). Pohozaev [21] proved the existence, but not uniqueness, for the above problem when $f(u) = -u^2$. For the case where $f(u) = -u^{(N+2)/(N-2)} (N > 2)$, Loewener and Nirenberg [17] proved that if $\partial\Omega$ consists of a disjoint union of finitely compact $C^\infty$ manifolds, each having co-dimension less than $N/2 + 1$, then there exists a unique solution of the problem (2). The uniqueness was established for $f(u) = u^a$ with $a > 3$, when $\partial\Omega$ is a $C^2$-submanifold and $\Delta$ is replaced by a more general second-order elliptic operator, by Kondrat'ev and Nikishken [11]. Marcus and Veron [18] proved the uniqueness for $f(u) = -u^a$ with $a > 1$, when $\partial\Omega$ is compact and is locally the graph of a continuous function defined on an $(N-1)$-dimensional space. Explosive solutions of the problem (1) and (2), Diaz and Letelier [5] have proved the existence and uniqueness of explosive solutions to (1) and (2) both for $f(u) = u^\gamma, \gamma > p-1$(super-linear case) and $\partial\Omega$ is of the class $C^2$. Very recently, Lu et al. [20] proved the existence of explosive solutions to (1) and (2) both for $f(u) = u^\gamma, \gamma > p-1$, $\Omega = R^N$ or $\Omega$ being a bounded domain (super-linear case) and $\gamma \leq p-1$, $\Omega = R^N$ (sub-linear case) respectively. In [24], which considered the existence of explosive positive solutions to quasilinear elliptic systems. In this paper, we obtain some existence results under following conditions :

$(H_1) f$ is continuous and nondecreasing on $[0, \infty)$, satisfies $f(0) = 0, \quad f(s) > 0$ for $s > 0$

$(H_2) f$ satisfy keller-Osserman condition

$$\int_1^\infty [F(s)]^{-1/p} ds = \infty \qquad (F(s) \equiv \int_0^s f(t)\, dt). \tag{3}$$

## 2 Main results

**Theorem 2.1**: Let $\Omega = R^N$ in (1). Suppose $f$ satisfies (3) and there exists a positive number $\varepsilon$ such that $a$ satisfies

$$\int_0^\infty t^{1+\varepsilon} \phi(t) dt < \infty, \quad where \;\; \phi(t) = \max_{|x|=t} a(x), \tag{4}$$

and $r^{p(N-1)}\phi(r)$is nondecreasing for large $r$. Then Eq. (1) has a nonnegative nontrivial entire bounded solution on $R^N$.

**Theorem 2.2**. Suppose that $a$ is spherically symmetric (i.e., $a(x) = a(|x|)$) and $\Omega = R^N$. If $f$ satisfies (3), then Eq. (1) has a nonnegative nontrivial entire solution. Suppose furthermore that $r^{p(N-1}a(r)$ is nondecreasing for large $r$. If $a$ satisfies $\int_0^\infty r^{\frac{1}{p-1}} a(r)^{\frac{1}{p-1}} dr = \infty$, then any nonnegative nontrivial entire solution $u$ of (1) is large. Conversely, if (1) has a nonnegative entire solution, then $a$ satisfies

$$\int_0^\infty r^{1+\varepsilon} a(r) dr = \infty \tag{5}$$

for every $\varepsilon > 0$.

# 3 Proof of Theorems

**Proof of theorem 2.1**: Suppose (4) holds. We will show that (1) has a solution by finding an upper solution, $v$, and a lower solution , $w$, for which $w \leq v$. To do this, we first prove the existence of $w$ to the equation

$$\Delta_p w = \phi(r) f(w). \tag{6}$$

We note that this equation becomes in this case

$$(r^{N-1}|w'(r)|^{p-2}w'(r))' = r^{N-1}\phi(r)f(w(r))$$

and that any solution $w$ to the integral equation

$$w(r) = 1 + \int_0^r t^{\frac{1-N}{p-1}} (\int_0^t s^{N-1}\phi(s)f(w(s))ds)^{\frac{1}{p-1}} dt, \quad r > 0,$$

is a solution to (6). To establish a solution to this equation, we use successive approximation. Let $w_0 = 1$ and define the sequence $\{w_k\}$ by

$w_k(r) = 1 + \int_0^r t^{\frac{1-N}{p-1}} (\int_0^t s^{N-1}\phi(s)f(w_{k-1}(s))ds)^{\frac{1}{p-1}} dt$ , for all $k$ and all $r \geq 0$.

Clearly $w_0 \leq w_1$, which, in turn, yields $w_1 \leq w_2$ since $f$ is nondecreasing. Hence the sequence $\{w_k\}$ is an increasing sequence, nondecreasing functions. We now show that the sequence $\{w_k\}$ is bounded above and hence convergence. We note that $w_k$ satisfies

$$(r^{N-1}|w_k'|^{p-2}w_k')' = r^{N-1}\phi(r)f(w_{k-1}), \quad k \geq 1, \tag{7}$$

and the monotonicity of $\{w_k\}$ yields

$$(r^{N-1}|w_k'|^{p-2}w_k')' \leq r^{N-1}\phi(r)f(w_k) \tag{8}$$

Choose $R > 0$ so that $r^{p(N-1)}\phi(r)$ is nondecreasing for $r \geq R$. We first show that $w_k(R)$ and $w_k'(R)$, both of which are nonnegative, are bounded above independent of k. To do this, let $\Phi_R = \max\{\phi(r) : 0 \leq r \leq R\}$. Using this and the fact that $w_k' \geq 0$, we note that (8) yields

$$w_k'^{\,p-2}\, w_k'' \leq \Phi_R f(w_k), \quad 0 \leq r \leq R.$$

Multiply this by $w_k'$ and integrate to get

$$(w_k')^p \leq p\,\Phi_R \int_1^{w_k(r)} f(s)\,ds, \quad 0 \leq r \leq R, \tag{9}$$

which yields

$$\int_1^{w_k(R)} [\int_1^t f(s)\,ds]^{-1/p}\ dt \leq \sqrt{p\,\Phi_R}\,R.$$

From (3), we now conclude that $w_k(R)$ is bounded above independent of $k$ and using this fact in (9) shows that the same is true of $w_k'(R)$. We now show that $w_k$ is bounded for all $r \geq 0$ and all $k$.

We have from (8)

$$(N-1)r^{N-2}{w_k'}^{p-1} + (p-1)r^{N-1}w_k''\,{w_k'}^{p-2} \leq r^{N-1}\phi(r)f(w_k),$$

hence

$$(p-1)r^{N-1}w_k''\,{w_k'}^{p-2} \leq r^{N-1}\phi(r)f(w_k).$$

Multiplying by $r^{(p-1)(N-1)}w_k'$ and integrating gives

$$(r^{N-1})w_k'(r))^p \leq \frac{(R^{N-1}w_k'(R))^p}{p-1} + \frac{p}{p-1}\int_R^r t^{p(N-1)}\phi(t)\frac{d}{dt}\int_1^{w_k(t)} f(s)\,ds\,dt \quad (r \geq R).$$

Using the monotonicity of $t^{p(N-1)}\phi(t)$ for $t \geq r$, we get $C \equiv (\frac{(R^{N-1}w_k'(R))^p}{p-1})$

$$(r^{N-1})w_k'(r))^p \leq C + \frac{p}{p-1}r^{p(N-1)}\phi(t)F(w_k(r)),$$

which yields

$$w_k'(r) \leq \sqrt[p]{C}r^{1-N} + \sqrt[p]{\frac{p}{p-1}\phi(r)}[F(w_k(r))]^{1/p} \tag{10}$$

and hence

$$\frac{d}{dr}\int_1^{w_k(r)} [F(t)]^{-1/p}dt \leq \sqrt[p]{C}r^{1-N}[F(w_k(r))]^{-1/p} + \sqrt[p]{\frac{p}{p-1}\phi(r)}.$$

Integrating this and using the fact that

$$\sqrt[p]{\tfrac{p}{p-1}\phi(r)} = \sqrt[p]{\tfrac{p}{p-1}r^{(1+\varepsilon)/p}\phi(r)r^{(-1-\varepsilon)/p}} \leq r^{1+\varepsilon}\phi(r) + r^{-1-\varepsilon}$$

for every $\varepsilon > 0$, we have

$$\int_{w_k(R)}^{w_k(r)} [F(t)]^{-1/p} dt \leq \sqrt[p]{C} \int_R^r t^{1-N}[F(w_k(t))]^{-1/p}\, dt + \int_R^r t^{1+\varepsilon}\phi(t)\, dt + \int_R^r t^{-1-\varepsilon}\, dt$$

$$\leq \sqrt[p]{C}[F(w_k(r))]^{-1/p} \int_R^r t^{1-N} dt + \int_R^r t^{1+\varepsilon}\phi(t)\, dt + \frac{1}{\varepsilon R^{\varepsilon}}. \qquad (11)$$

Since for $\varepsilon > 0$ the right side of this inequality is bounded independent of $k$ (note that $w_k(R) \geq 1)$), so is the left side and hence, in light of (3), the sequence $\{w_k\}$ is a bounded sequence. Thus $w_k \uparrow w$ as $k \to \infty$ and hence $w$ is a solution to (6). Furthermore, $w' \geq 0$ and since the sequence $\{w_k\}$ is bounded above, so is $w$. We let $M$ be the least upper bound of $w$ and note that $M = \lim_{r\to\infty} w(r)$. Now let $v$ be positive increasing bounded solution of

$$v(r) = M + \int_0^r t^{\frac{1-N}{p-1}} \left(\int_0^t s^{N-1}\psi(s) f(v(s))\, ds\right)^{\frac{1}{p-1}} dt \quad r \geq 0,$$

which, of course, satisfies (6) with $w$ replace with $v$ and $\psi(t) = \min\limits_{|x|=t} a(x)$. It is also clear that $v \geq M$. (the proof of the existence of $v$ and that it has the properties mentioned is virtually identical to the proof for $w$ above and is therefore omitted.) Thus we have that $w$ and $v$ satisfy, respectively,

$$\Delta_p w \geq a(x) f(w), \qquad \Delta_p v \leq a(x) f(v)$$

on $R^N$ and $w \leq v$. Hence the standard upper-lower solution principle implies that(1) has a solution $u$, such that $w \leq u \leq v$ on $R^N$, which is the desired solution.

**Proof of theorem 2.2**: For any $a_1 > 0$ a solution of

$$v(r) = a_1 + \int_0^r t^{\frac{1-N}{p-1}} \left(\int_0^t s^{N-1} a(s) f(s)\, ds\right)^{\frac{1}{p-1}} dt, \qquad (12)$$

exists, at least, small r. Since $v' \geq 0$, the only way that the solution can become singular at $R$ is for $v(r) \to \infty$ as $r \uparrow R$. Thus, we can show that, for each $R > 0$, there exists $C_R > 0$ so that $v(R) \leq C_R$, we have existence. To this end, let $M_R = \max\{a(r) : 0 \leq r \leq R\}$ and consider the equation

$$w(r) = a_2 + M_R^{\frac{1}{p-1}} \int_0^r t^{\frac{1-N}{p-1}} (\int_0^t s^{N-1} f(s)\, ds)^{\frac{1}{p-1}} dt,$$

where $a_2 > a_1$. The solution to this equation exists for all $r \geq 0$ and of course, it is a solution to $\Delta_p w = M_R^{\frac{1}{p-1}} f(w)$ on $R^N$. We now show that $v(r) \leq w(r)$ for all $0 \leq r \leq R$ and hence complete the proof of existence. Clearly $v(0) < w(0)$ so that $v(r) < w(r)$ for at least all $r$ near zero. Let $r_0 = \sup\{r : v(s) < w(s) \text{ for all } s \in [0, 1]\}$. If $r_0 = R$, then we are done. Thus assume that $r_0 < R$. Then we have

$$\begin{aligned} v(r_0) &= a_1 + \int_0^{r_0} t^{\frac{1-N}{p-1}} (\int_0^t s^{N-1} a(s) f(s)\, ds)^{\frac{1}{p-1}} dt \\ &< a_2 + M_R^{\frac{1}{p-1}} \int_0^{r_0} t^{\frac{1-N}{p-1}} (\int_0^t s^{N-1} f(s)\, ds)^{\frac{1}{p-1}} dt = w(r_0). \end{aligned}$$

Thus there exists $\varepsilon > 0$ so that $v(r) < w(r)$ for all $[0, r_0 + \varepsilon)$, contradicting the definition or $r_0$. Thus we conclude that $v < w$ on $[0, R]$ for all $R > 0$ and hence $v$ is a nontrivial entire solution of (1). Now let $u$ be any nonnegative nontrivial entire solution of (1) and suppose $a$ satisfies $\int_0^\infty r^{\frac{1}{p-1}} a(r)^{\frac{1}{p-1}} dr = \infty$. Then the proof that $u(r) \to \infty$ as $r \to \infty$ is very similar to part of the proof of theorem 1 of [14] so we provide details here only of the difference between the two proofs. Indeed since $u$ is nontrivial and nonnegative, there exists $R > 0$ so that $u(R) > 0$. Since $u' \geq 0$, we get $u(r) \geq u(R)$ for $r \geq R$, and thus from (14)(since $u$ will satisfy that equation for all $r \geq 0$) we get

$$\begin{aligned} u(r) &= u(0) + \int_0^r t^{\frac{1-N}{p-1}} (\int_0^t s^{N-1} a(s) f(s)\, ds)^{\frac{1}{p-1}} dt \\ &\geq u(R) + f(u(R))^{\frac{1}{p-1}} \int_R^r t^{\frac{1-N}{p-1}} (\int_R^t s^{N-1} a(s)\, ds)^{\frac{1}{p-1}} dt \to \infty \quad as \quad r \to \infty, \end{aligned}$$

where we have applied Eq.(8) in [13] to establish the limit.

To prove the converse we use an argument similar to part of the proof theorem 1 above. Indeed, if $f$ satisfies (3) and $w$ is a nonnegative entire large solution of (1), then $w$ satisfies

$$(r^{N-1} |w'|^{p-2} w')' = r^{N-1} \phi(r) f(w).$$

Now, as in proof of theorem 2.1, we multiply by $r^{p-1}N - 1w'(r)$, integrate, and use the monotonicity of $r^{p(N-1)} a(r)$ for $r \geq R$to get (see inequality and its derivation)

$$w'(r) \leq \sqrt[p]{C} r^{1-N} + \sqrt[p]{\frac{p}{p-1} a(r)} [F(w(r))]^{1/p}$$

and hence, as with (11), we get

$$\begin{aligned}\int_{w(R)}^{w(r)}[F(t)]^{-1/p}dt &\leq \sqrt[p]{C}[F(w(R))]^{-1/p}\int_R^r t^{1-N}\,dt+\int_R^r t^{1+\varepsilon}a(t)\,dt+\frac{1}{\varepsilon R^{\varepsilon}}\\ &\leq C_R+\int_R^r t^{1+\varepsilon}a(t)\,dt,\end{aligned}$$

where $C_R = \sqrt[p]{C}[F(w(R))]^{-1/p}R^{N-2}/(N-2)+1/(\varepsilon R^{\varepsilon})$. Letting $r \to \infty$, we find that $a$ satisfies (5) since $w$ is large and$f$ satisfies (3). This completes the proof.

# References

[1] V. Anuradha, C. Brown, R. Shivaji, Explosive nonnegative solutions to two point boundary value problems, Nonlinear Anal. 26(1996) 613-630.

[2] G. Astrita, G. Marrucci, Principles of non-Newtonian fluid mechanics, McGraw-Hill, 1974.

[3] L. Bieberbach, Du = eu und die automorphen Funktionen, Math. Ann. 77 (1916) 173-212.

[4] K.-S. Cheng, W.-M. Ni, On the structure of the conformal scalar curvature equation on RN, Indiana Univ. Math. J. 41 (1992) 261-278.

[5] G. Diaz, R. Letelier, Explosive solutions of quasilinear elliptic equations: existence and uniqueness, Nonlinear Anal. 20 (1993) 97125.

[6] E.B. Dynkin, Superprocesses and partial differential equations, Ann. Probab. 21 (1993) 1185-1262.

[7] E.B. Dynkin, S.E. Kuznetsov, Superdiffusions and removable singularities for quasilinear partial differential equations, Commun. Pure Appl. Math. 49 (1996) 125-176.

[8] J.R. Esteban, J.L. Vazquez, On the equation of turbulent filteration in one-dimensional porous media, Nonlinear Anal. 10 (1982) 1303-1325.

[9] A.S. Kalashnikov, On a nonlinear equation appearing in the theory of non-stationary filtration, Trud. Sem. I.G. Petrovski (1978) (in Russian).

[10] J.B. Keller, On solutions of Du = f(u), Commun. Pure Appl. Math. 10 (1957) 503-510.

[11] V.A. Kondrat'ev, V.A. Nikishken, Asymptotics, near the boundary, of a singular boundary-value problem for a semilinear elliptic equation, Diff. Urav. 26 (1990) 465-468 (English translation: Differen. Equat. 26 (1990) 345-348).

[12] A. V. Lair, Large solution of semilinear elliptic equations under the Keller-Osserman condition, j. math. Anal. Appl. 328(2007)1247-1254.

[13] A. V. Lair, A. W. Wood, Large solution of semilinear elliptic problems, Nonlinear Anal. 37(1999)805-812.

[14] A. V. Lair, A. W. Wood, Large solution of semilinear elliptic equations, Nonlinear Anal. 39(2000)745-753.

[15] A.C. Lazer, P.J. McKenna, On a problem of Bieberbach and Rademacher, Nonlinear Anal. 21 (1993) 327-335.

[16] A.C. Lazer, P.J. McKenna, On singular boundary value problems for the Monge-Ampere operator, J. Math. Anal. Appl. 197 (1996)341-362.

[17] C. Loewner, L. Nirenberg, Partial differential equations invariant under conformal or projective transformations, in: Contributions to Analysis (A Collection of Paper Dedicated to Lipman Bers), Academic Press, New York, 1974, pp. 245-272.

[18] M. Marcus, L. Veron, Uniqueness of solutions with blow-up at the boundary for a class of nonlinear elliptic equation, C.R. Acad. Sci. Paris 317 (1993) 559-563.

[19] L.K. Martinson, K.B. Pavlov, Unsteady shear flows of a conducting fluid with a rheological power law, Magnitnaya Gidrodinamika 2 (1971) 5058.

[20] Q. Lu, Z. Yang, E.H. Twizell, Existence of entire explosive positive solutions of quasi-linear elliptic equations, Appl. Math. Comput. 148 (2004) 359-372.

[21] S. L. Pohzaev, The Dirichlet problem for the equation $\Delta u = u^2$, Soviet Math.Dokl. 1(1960)1143-1146

[22] M.R. Posteraro, On the solutions of the equation Du = eu blowing up on the boundary, C.R. Acad. Sci. Paris 322 (1996) 445-450.

[23] H. Rademacher, Einige besondere problem partieller Differentialgleichungen, in: Die Differential- und Integralgleichungen, der Mechanik und Physikl, second ed., Rosenberg, New York, 1943, pp. 838-845.

[24] Z. Yang, Existence of entire explosive positive radial solutions for a class of quasilinear elliptic systems, J. Math. Anal. Appl. 288 (2003) 768-783.

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