The Construction of the Lemarié-Meyer Smooth Wavelets with Respect to a Riesz Basis

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Abstract

Auscher, Weiss, and Wickerhauser showed that the local orthonormal sine and cosine bases of Coifman and Meyer generate the Lemarié-Meyer smooth wavelets [1]. In this paper we show that more general Riesz basis of Coifman and Meyer type[3] also generate the same wavelets.

Mathematics Subject Classification: 42C40

Keywords: Riesz bases, Wavelets, Basis perturbation

1 Introduction

In [1], Auscher, Weiss, and Wickerhauser showed that the Lemarié-Meyer smooth wavelets is generated by the local sine and cosine orthonormal bases of Coifman and Meyer [2]. In this context, a natural question is whether one can obtain the same result with Riesz basis instead. We use the Riesz basis of Coifman and Meyer type [3], to provide conditions for a positive answer to this question(Theorem 3.4).

2 Riesz basis of Coifman and Meyer type

Throughout this paper, we will consider the usual inner product of functions $f, g$ of $L^2[\alpha, \beta]$ given by

$$< f, g > = \int_{\alpha}^{\beta} f \bar{g}.$$
Choose an even non-negative function $\psi$ with $\text{Supp } \psi = [-\varepsilon, \varepsilon]$, normalized so that $\|\psi\|_{L^1} = \frac{\pi}{2}$, and let

$$\theta(x) = \int_{-\infty}^{x} \psi(t) \, dt.$$ 

We put $c_\varepsilon(x) = \cos \theta(x)$ and $s_\varepsilon(x) = \sin \theta(x)$ and define the bell function $b_I$ by $b_I = s_\varepsilon(x - \alpha)c_\varepsilon(x - \beta)$. By taking the 'smoother' orthonormal projection $P_I$ associated with the bell function, Auscher, Weiss, and Wickerhauser [1] show that

$$L^2(\mathbb{R}) = \bigoplus_{k \in \mathbb{Z}} P_k L^2(\mathbb{R})$$

where $\mathbb{R} = \bigcup_{k \in \mathbb{Z}} I_k$ and provide four orthonormal bases. In our previous work [3], we showed how to obtain a Riesz basis from one of such orthonormal bases. More precisely,

**Theorem 2.1 ([3], Theorem 3.2)** For an interval $I = [0, 1]$ and for an orthonormal basis $\{\sqrt{2}b_I(x) \sin \left(\frac{2n+1}{2} \pi x\right)\}$ of $P_I L^2(\mathbb{R})$,

$$\left\| \sum c_n \sqrt{2} b_I(x) \left( \sin \left(\frac{2n+1}{2} \pi x\right) - \sin(\lambda_n x) \right) \right\| \leq 1 + \phi(L).$$

Thus $\{\sqrt{2}b_I(x) \sin \lambda_n x\}$ forms a Riesz basis for $P_I L^2(\mathbb{R})$, if $|\frac{2n+1}{2} \pi - \lambda_n| \leq L$ for some $L$ for which the function $\phi(L) < 0$, where

$$\phi = \phi_A + \phi_B + \phi_C + \phi_D + \phi_E + \phi_F$$

and

$$\phi_A(L) \equiv -\frac{\sin 2L}{3L} - \frac{\sin L}{3L},$$

$$\phi_B(L) \equiv \frac{1}{2} \left( \frac{\sin 2L \sin \frac{3L}{2}}{\cos \frac{3L}{2}} - \frac{\sin L \sin \frac{3L}{2}}{\cos \frac{3L}{2}} \right),$$

$$\phi_C(L) \equiv \frac{1}{2} \left( \frac{2 \sin 2L}{3L} + \frac{2 \sin L}{3L} - \frac{\sin 2L \cos \frac{3L}{2}}{\sin \frac{3L}{2}} - \frac{\sin L \cos \frac{3L}{2}}{\sin \frac{3L}{2}} \right),$$

$$\phi_D(L) \equiv \frac{\cos L - \cos 2L}{3L},$$

$$\phi_E(L) \equiv \frac{1}{2} \left( \frac{\cos 2L \sin \frac{3L}{2}}{\cos \frac{3L}{2}} + \frac{\cos L \sin \frac{3L}{2}}{\cos \frac{3L}{2}} \right),$$

$$\phi_F(L) \equiv \frac{1}{2} \left( \frac{2 \cos 2L}{3L} - \frac{2 \cos L}{3L} - \frac{\cos 2L \cos \frac{3L}{2}}{\sin \frac{3L}{2}} + \frac{\cos L \cos \frac{3L}{2}}{\sin \frac{3L}{2}} \right).$$

The largest $L$ is about 0.3788
We can extend this result to $P_I L^2(\mathbb{R})$ for any interval $I = [\alpha, \beta]$, that is, under the same condition given in Theorem 2.1, the family $\left\{ \sqrt{\frac{2}{|I|}} b_I(x) \sin \frac{\lambda_n}{|I|} (x - \alpha) \right\}$ forms a Riesz basis for $P_I L^2(\mathbb{R})$ which is ‘close’ to the orthonormal basis $\left\{ \sqrt{\frac{2}{|I|}} b_I(x) \sin \frac{2n+1}{2} \frac{\pi}{|I|} (x - \alpha) \right\}$. Similarly, under the same condition, we also have a local cosine type Riesz basis $\left\{ \sqrt{\frac{2}{|I|}} b_I(x) \cos \frac{\lambda_n}{|I|} (x - \alpha) \right\}$ regards to the orthonormal basis $\left\{ \sqrt{\frac{2}{|I|}} b_I(x) \cos \frac{2n+1}{2} \frac{\pi}{|I|} (x - \alpha) \right\}$ of $P_I L^2(\mathbb{R})$ for any interval $I = [\alpha, \beta]$.

3 The construction of Lemarié-Meyer smooth wavelets by using a Riesz basis

The Lemarié and Meyer wavelet basis [4] is given by

$$\{ w_{k,n}(x) \} = \{ 2^{-\frac{k}{2}} w(2^{-k} x - n) \}, \quad k, n \in \mathbb{Z},$$

where $w \in S(\mathbb{R})$ and $\text{Supp } \hat{w} = [-\frac{8\pi}{3}, -\frac{2\pi}{3}] \cup [\frac{8\pi}{3}, \frac{2\pi}{3}]$. Auscher, Weiss, and Wickerhauser [1] show that

**Theorem 3.1** ([1], Theorem 6) The collection of functions

$$\alpha_{k,n} \equiv \Psi_{k,n} + i \Phi_{k,n} \quad \text{and} \quad \beta_{k,n} \equiv \Phi_{k,n} + i \Psi_{k,n},$$

$k = 0, \pm 1, \pm 2, \cdots, n = 0, 1, 2, \cdots$, is an orthonormal basis for $L^2(\mathbb{R})$, where

$$\Psi_{k,n}(\xi) = \sqrt{\frac{1}{2\pi}} 2^{\frac{k}{2}} b(2^k |\xi|) \cos \frac{2n+1}{2} (2^k |\xi| - \pi)$$

and

$$\Phi_{k,n}(\xi) = \sqrt{\frac{1}{2\pi}} 2^{\frac{k}{2}} b(2^k |\xi|) (\text{sgn} \xi) \sin \frac{2n+1}{2} (2^k |\xi| - \pi).$$

Now we generate a Riesz basis which is close to the orthonormal basis given in the Theorem 3.1. For $I = [\pi, 2\pi]$, $\varepsilon = \frac{\pi}{3}$, and $\varepsilon' = \frac{2\pi}{3}$, by Theorem 2.1 and with condition on it, we obtain that

$$\psi^*(n; \xi) = \sqrt{\frac{2}{\pi}} b_I(x) \cos \frac{\lambda_n}{\pi} (\xi - \pi) (\xi - \pi), \quad n = 0, 1, 2, \cdots,$$

is a Riesz basis for $P_I L^2(\mathbb{R})$, consequently, with dilation by $2^k$,

$$\psi_k^*(n; \xi) \equiv 2^k \psi^*(n; 2^k \xi), \quad k \in \mathbb{Z}. \quad (1)$$
forms a Riesz basis of $L^2(0, \infty)$. A completely analogous construction based on the Riesz basis of local sine type

$$\varphi^*(n; \xi) = \sqrt{\frac{2}{\pi}} b_I(x) \sin \frac{\lambda_n}{\pi} (\xi - \pi), \quad n = 0, 1, 2, \cdots,$$

gives us a Riesz basis of $L^2(0, \infty)$:

$$\varphi_k^*(n; \xi) \equiv 2^k \varphi^*(n; 2^k \xi), \quad k \in \mathbb{Z}. \quad (2)$$

Then by using even extensions of the functions (1) and odd extensions of the functions (2), we obtain our first result:

**Theorem 3.2** For $k \in \mathbb{Z}$ and $n = 0, 1, 2, \cdots$, let

$$\Psi_{k,n}(\xi) = \sqrt{\frac{1}{2\pi}} 2^{k^2} b(2^k |\xi|) \cos \frac{\lambda_n}{\pi} (2^k |\xi| - \pi)$$

and

$$\Phi_{k,n}(\xi) = \sqrt{\frac{1}{2\pi}} 2^{k^2} b(2^k |\xi|) (\text{sgn} \xi) \sin \frac{\lambda_n}{\pi} (2^k |\xi| - \pi).$$

Then the collection of functions

$$\alpha_{k,n}^* \equiv \Psi_{k,n}^* + i\Phi_{k,n}^* \quad \text{and} \quad \beta_{k,n}^* \equiv \Phi_{k,n}^* + i\Psi_{k,n}^*$$

forms a Riesz basis on $L^2(\mathbb{R})$ if $|\frac{2k+1}{2} \pi - \lambda_k| \leq L$ for some $L$ which makes $1 - \frac{\sqrt{2}}{4} + \phi(L)$ negative ($\phi$ is from Theorem 2.1). The largest $L$ is about $0.1542$.

**Proof:** To use the Paley-Wiener Theorem [5, Chapter 1, Theorem 13], we need to find the condition for $L$ such that $|\frac{2k+1}{2} \pi - \lambda_k| \leq L$ and

$$\left\| \sum_{k \in \mathbb{Z}} \sum_{n=0}^{\infty} c_{k,n} (\alpha_{k,n} - \alpha_{k,n}^*) + d_{k,n} (\beta_{k,n} - \beta_{k,n}^*) \right\|_{L^2(\mathbb{R})} < 1, \quad (3)$$

whenever $\sum_{k \in \mathbb{Z}} \sum_{n=0}^{\infty} |c_{k,n}|^2 + \sum_{k \in \mathbb{Z}} \sum_{n=0}^{\infty} |d_{k,n}|^2 \leq 1$.

First we rewrite the left-hand side of (3) as $K_1 + K_2 + K_3 + K_4$ where

$$K_1 = \left\| \sum_{k \in \mathbb{Z}} \sum_{n=0}^{\infty} c_{k,n} (\Psi_{k,n} - \Psi_{k,n}^*) \right\|_{L^2(\mathbb{R})} = \left\| \sum_{k \in \mathbb{Z}} \sum_{n=0}^{\infty} c_{k,n} (\Phi_{k,n} - \Phi_{k,n}^*) \right\|_{L^2(\mathbb{R})},$$

$$K_2 = \left\| \sum_{k \in \mathbb{Z}} \sum_{n=0}^{\infty} c_{k,n} (\Phi_{k,n} - \Phi_{k,n}^*) \right\|_{L^2(\mathbb{R})} = \left\| \sum_{k \in \mathbb{Z}} \sum_{n=0}^{\infty} d_{k,n} (\Phi_{k,n} - \Phi_{k,n}^*) \right\|_{L^2(\mathbb{R})},$$

$$K_3 = \left\| \sum_{k \in \mathbb{Z}} \sum_{n=0}^{\infty} d_{k,n} (\Phi_{k,n} - \Phi_{k,n}^*) \right\|_{L^2(\mathbb{R})} = \left\| \sum_{k \in \mathbb{Z}} \sum_{n=0}^{\infty} d_{k,n} (\Psi_{k,n} - \Psi_{k,n}^*) \right\|_{L^2(\mathbb{R})},$$

$$K_4 = \left\| \sum_{k \in \mathbb{Z}} \sum_{n=0}^{\infty} d_{k,n} (\Psi_{k,n} - \Psi_{k,n}^*) \right\|_{L^2(\mathbb{R})}.$$
The construction of smooth wavelets

Since functions $\Psi_{k,n} - \Psi_{k,n}^*$ and $\Phi_{k,n} - \Phi_{k,n}^*$ in above norms are even, $K_i \leq \frac{1}{\sqrt{2}} J_i$, for $i = 1, 2, 3,$ and $4$, where

\begin{align*}
J_1 &= \left\| \sum_{k \in \mathbb{Z}} \sum_{n=0}^{\infty} c_{k,n} (\psi_{k,n}(n; \xi) - \psi_{k,n}^*(n; \xi)) \right\|_{L^2(0,\infty)}, \\
J_2 &= \left\| \sum_{k \in \mathbb{Z}} \sum_{n=0}^{\infty} c_{k,n} (\varphi_{k,n}(n; \xi) - \varphi_{k,n}^*(n; \xi)) \right\|_{L^2(0,\infty)}, \\
J_3 &= \left\| \sum_{k \in \mathbb{Z}} \sum_{n=0}^{\infty} d_{k,n} (\psi_{k,n}(n; \xi) - \psi_{k,n}^*(n; \xi)) \right\|_{L^2(0,\infty)}, \\
J_4 &= \left\| \sum_{k \in \mathbb{Z}} \sum_{n=0}^{\infty} d_{k,n} (\varphi_{k,n}(n; \xi) - \varphi_{k,n}^*(n; \xi)) \right\|_{L^2(0,\infty)}.
\end{align*}

By Theorem 2.1, we have that each $J_i \leq 1 + \phi(L)$ for $i = 1, 2, 3,$ and $4$. Then we obtain that

$$\frac{1}{\sqrt{2}} (J_1 + J_2 + J_3 + J_4) \leq \frac{4}{\sqrt{2}} (1 + \phi(L)).$$

Thus, by the Paley-Wiener Theorem, if $\frac{1}{\sqrt{2}} (1 + \phi(L)) < 1$, the collection of functions $\alpha_{k,n}^*$ and $\beta_{k,n}^*$ forms a Riesz basis for $L^2(\mathbb{R})$. □.

Thus if $\lambda_n$ is close to $n$ with certain condition, then the collection of $\alpha_{k,n}^*$ and $\beta_{k,n}^*$ forms a Riesz basis. We now modify this Riesz basis to get another Riesz basis for $L^2(\mathbb{R})$ which is generated by the single function $\gamma^*(\xi) \equiv i\alpha_{0,0}^*(\xi)$. In this construction, it requires certain restriction to $\lambda_n$. The trick is that we apply $\lambda_n$ differently by sign of $n$.

**Theorem 3.3** For $0 < \delta \leq L$ and $\lambda_n = \begin{cases} 
\frac{2n+1}{2} \pi - \delta & \text{if } n \geq 0 \\
\frac{2n+1}{2} \pi + \delta & \text{if } n < 0
\end{cases}$

The functions

$$\gamma_{k,n}^*(\xi) \equiv 2^k e^{-i2^k n \xi} \gamma^*(2^k \xi), \quad k, n \in \mathbb{Z},$$

form a Riesz basis of $L^2(\mathbb{R})$. ($L$ is from the Theorem 3.2 and the largest $L$ is about 0.1542).

**Proof:** We define $\alpha_{k,-n}^*(\xi) = \overline{\alpha_{k,-n-1}^*(\xi)} = -i \beta_{k,-n}^*(\xi)$ for $k \in \mathbb{Z}$ and $n > 0$. Then by Theorem 3.2, $\{\alpha_{k,n}^*\}_{k,n \in \mathbb{Z}}$ forms a Riesz basis for $L^2(\mathbb{R})$. We will be done once we show that

$$\alpha_{0,n}^*(\xi) = (-1)^n (-i)^n e^{in \xi} \gamma^*(\xi), \quad n \in \mathbb{Z}. \quad (4)$$
First if \( n = 0 \), then \( \lambda_0 \) is \( \frac{1}{2} \pi - \delta \), thus we have
\[
\gamma^*(\xi) = i\alpha^*_{0,0}(\xi) = \frac{b(|\xi|)}{\sqrt{2\pi}} e^{-i\frac{\xi}{2}\delta} e^{i\xi|\xi|}e^{i(sgn\xi)\delta}.
\]

Now assume \( n \) is nonzero, we define \( \lambda_n \) differently by sign of \( n \). For \( n \geq 0 \), we let \( \lambda_n = \frac{2n+1}{2} \pi - \delta \), then we have
\[
\alpha^*_{0,n}(\xi) = i(\Psi_{0,n}^* + i\Phi_{0,n}^*)(\xi)
\]
\[
= i \frac{b(|\xi|)}{\sqrt{2\pi}} \left( \cos \frac{\lambda_n}{\pi} (|\xi| - \pi) + i(sgn\xi) \sin \frac{\lambda_n}{\pi} (|\xi| - \pi) \right)
\]
\[
= \frac{b(|\xi|)}{\sqrt{2\pi}} e^{i\frac{2n+1}{2}\pi - \frac{\xi}{2}\delta} e^{-i\frac{\xi}{2}\delta} \left\{ e^{-i\left(\frac{2n+1}{2}\pi - \delta\right)} \quad \text{if} \quad \xi \geq 0 \right. \\
\left. e^{i\left(\frac{2n+1}{2}\pi - \delta\right)} \quad \text{if} \quad \xi < 0 \right. \\
= (-1)^n(sgn\xi)(-i)e^{ln\xi} \frac{b(|\xi|)}{\sqrt{2\pi}} e^{i\frac{\xi}{2}\pi - \frac{\xi}{2}\delta} e^{-i\frac{\xi}{2}\delta} e^{i(sgn\xi)\delta}
\]
\[
= (-1)^n(-i)e^{in\xi} \gamma^*(\xi).
\]

If \( n < 0 \), define \( \lambda_n = \frac{2n+1}{2} \pi + \delta \),
\[
\alpha^*_{0,n}(\xi) = \frac{\alpha^*_{0,-n-1}(\xi)}{b(|\xi|)}
\]
\[
= \frac{b(|\xi|)}{\sqrt{2\pi}} e^{i\left(-n-1\right)\pi} e^{-i\frac{\xi}{2}\delta} e^{i\xi|\xi|}e^{i\left(sgn\xi\right)(n+1)\pi} e^{-i\left(sgn\xi\right)\delta}
\]
\[
= \frac{b(|\xi|)}{\sqrt{2\pi}} e^{i\left(n+1\right)\pi} e^{-i\frac{\xi}{2}\delta} e^{-i\frac{\xi}{2}\delta} (-1)^{n+1}i(sgn\xi)e^{i(sgn\xi)\delta}
\]
\[
= (-1)^n(-i)e^{in\xi} \gamma^*(\xi).
\]

Thus (4) holds for all integers \( n \). \( \square \)

To get our main results, we define \( w^* \) by
\[
w^*(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} \gamma^*(\xi) d\xi,
\]
then we have \( \widehat{w^*} = \sqrt{2\pi} \gamma^* \) and for \( k, n \in \mathbb{Z} \),
\[
w_{k,n}^* = 2^{-\frac{k}{2}} w^*(2^{-k}x - n) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} \gamma_{k,n}^*(\xi) d\xi.
\]
Theorem 3.4 \{w^*_{k,n}\}, k, n ∈ \mathbb{Z}, is a Riesz basis for \(L^2(\mathbb{R})\), and generates a Lemarié - Meyer wavelet basis.

Proof: By the Plancherel theorem and Theorem 3.3,
\[
\frac{A}{2\pi} \|f\|^2 \leq \frac{1}{2\pi} \sum_k \|\hat{f}, \gamma^*_k,n\|^2 = \sum_k |<f, w^*_{k,n}>|^2 \leq \frac{B}{2\pi} \|f\|^2,
\]
where \(A\) and \(B\) are frame bounds for a Riesz basis \(\gamma^*_k,n(\xi)\). Thus \(\{w^*_{k,n}\}\) is a frame. To show that \(\{w^*_{k,n}\}\) is a Riesz basis in \(L^2(\mathbb{R})\), it remains to verify that it is complete. For any \(f\) in \(L^2(\mathbb{R})\), suppose that \(<f, w^*_{k,n}> = 0\) for all \(k, n \in \mathbb{Z}\), then
\[
0 = <f, w^*_{k,n}> = \frac{1}{2\pi} \hat{f}, w^*_{k,n}> = \frac{1}{\sqrt{2\pi}} <\hat{f}, \gamma^*_k,n >.
\]
Thus \(<\hat{f}, \gamma^*_k,n > = 0\) for all \(k, n \in \mathbb{Z}\). Since \(\{\gamma^*_k,n\}\) is a Riesz basis for \(L^2(\mathbb{R})\), \(\hat{f} = 0\), therefore \(f = 0\).

Since \(\text{Supp} \ w^* = \text{Supp} \ b(|\xi|) = [-\frac{8\pi}{3}, -\frac{2\pi}{3}] \cup [\frac{8\pi}{3}, \frac{2\pi}{3}]\), \(w^*\) is a mother function of the type that we mentioned at the beginning of the section. Thus \(w^*\) generates a Lemarié - Meyer wavelets. □

ACKNOWLEDGEMENTS. The author acknowledges the valuable suggestions of Professor Alberto Torchinsky which improved the presentation and contents of this paper.

References


Received: April, 2010