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Hölder, Chebyshev and Minkowski Type Inequalities for Stolarsky Means

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Abstract. In this paper, the Hölder, Chebyshev and Minkowski type inequalities for Stolarsky means are established. In addition, another new and concise proof of Minkowski type inequality for Stolarsky means is presented.

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1. Introduction

For one-parameter means of two positive numbers x and y denoted by

$$J(\alpha;x,y)=\begin{cases}\frac{\alpha(x^{\alpha+1}-y^{\alpha+1})}{(\alpha+1)(x^{\alpha}-y^{\alpha})}, & \alpha\neq 0,-1, x\neq y,\\ \frac{x-y}{\ln x-\ln y}, & \alpha=0, x\neq y,\\ \frac{xy(\ln x-\ln y)}{x-y}, & \alpha=-1, x\neq y,\\ y, & x=y,\end{cases} \tag{1.1}$$

H. Alzer presented Chebyshev and Minkowski type inequalities in 1988 (see [1, 2].

For generalized logarithmic means of two positive numbers x and y denoted by

$$S(\alpha;x,y)=\begin{cases}\left(\frac{x^{\alpha}-y^{\alpha}}{\alpha(x-y)}\right)^{\frac{1}{\alpha-1}} & \alpha\neq 0,1, x\neq y,\\ L(x,y) & \alpha=0, x\neq y,\\ I(x,y) & \alpha=1, x\neq y,\\ y & x=y,\end{cases} \tag{1.2}$$

H.-E. Lou proved that the Hölder type inequality is valid in 1996 (see [4]). Zh.-H. Yang proved that its Hölder, Chebyshev and Minkowski type inequalities are also true by using classical integral inequalities in 2005 (see [6])

The Stolarsky means (extended means) of positive numbers x and y is defined by

$$E(r,s;x,y)=\begin{cases}\left(\dfrac{s}{r}\dfrac{x^r-y^r}{x^s-y^s}\right)^{\frac{1}{r-s}} & r\neq s, rs\neq 0, x\neq y,\\ L^{\frac{1}{r}}(x^r,y^r) & r\neq 0, s=0, x\neq y,\\ L^{\frac{1}{s}}(x^s,y^s) & r=0, s\neq 0, x\neq y,\\ I^{\frac{1}{r}}(x^r,y^r) & r=s\neq 0, x\neq y,\\ G(x,y) & r=s=0, x\neq y,\\ y & x=y,\end{cases} \tag{1.3}$$

where

$$L(x,y)=(x-y)\,/\ln(x/y), I(x,y)=e^{-1}(x^x/y^y)^{\frac{1}{x-y}}, G(x,y)=\sqrt{xy}.$$

In 1998, L. Losonczi and Zs. Páles showed that Minkowski type inequality hold for Stolarsky means (extended means) holds, which is read as follows:

Theorem 1 (Minkowski Type Inequality for Stolarsky Mean (see [3, Theorem 1, 3]))**.** *For arbitrary pairs $(x_1,y_1),(x_2,y_2)\in\mathbb{R}_+$, the following inequality*

$$E(r,s;x_1+x_2,y_1+y_2)\leq(\geq)E(r,s;x_1,y_1)+E(r,s;x_2,y_2) \tag{1.4}$$

holds iff $r+s\geq(\leq)3$ and $\min(r,s)\geq(\leq)1$ with $(r,s)\neq(2,1),(1,2)$. With equality iff $x_1:y_1=x_2:y_2$.

Zh.-H. Yang studied the AA, GG, AG and GA convexity of homogeneous functions of two variables and presented simplified decisions. As applications, new simple proofs of Hölder, Chebyshev and Minkowski type inequalities for homogeneous means such as $J(\alpha;x,y)$ and $S(\alpha;x,y)$ were given (see [7, 8]).

The main purpose of this paper is to investigate the GG-convexity of the Stolarsky means (extended means) by applying theorems or corollaries in [7, 8], and then Hölder and Chebyshev type inequalities for Stolarsky means (extended means) are given. Our main results are as follows:

Theorem 2 (Hölder Type Inequality for Stolarsky Mean)**.** *For given $p,q>0$ with $p+q=1$ and arbitrary pairs $(x_1,y_1),(x_2,y_2)\in\mathbb{R}_+$, inequality*

$$E(r,s;x_1^px_2^q,y_1^py_2^q)\leq(\geq)E^p(r,s;x_1,y_1)E^q(r,s;x_2,y_2) \tag{1.5}$$

holds iff $r+s>(<)0$. With equality iff $x_1:y_1=x_2:y_2$.

Theorem 3 (Chebyshev Type Inequality for Stolarsky Mean)**.** *If (x_1,y_1) and (x_2,y_2) are oppositely (similarly) ordered, then*

$$E(r,s;x_1x_2,y_1y_2)\leq(\geq)E(r,s;x_1,y_1)E(r,s;x_2,y_2) \tag{1.6}$$

if $r+s>(<)0$.

In addition, as a result of AA-convexity of the Stolarsky means (extended means) proved by L. Losonczi and Zs. Páles, another new proof of Minkowski type inequality is presented.

To prove the above theorems, we need corollaries in [7] as preparations. For the sake of convenience, we read them as follows:

Lemma 1 ([7, Corollary 3.2.1])**.** *Let* $f : \mathbb{D}(\subseteqq \mathbb{R}_+ \times \mathbb{R}_+) \to \mathbb{R}_+$ *is an* n*-order homogeneous and two-time differentiable function. Then for given* $p, q > 0$ *with* $p + q = 1$ *and arbitrary pairs* $(x_1, y_1), (x_2, y_2), (x_1^p x_2^q, y_1^p y_2^q) \in \mathbb{D}$*, inequality*

$$f(x_1^p x_2^q, y_1^p y_2^q) \le f^p(x_1, y_1) f^q(x_2, y_2) \tag{1.7}$$

holds iff $\mathcal{I} = (\ln f)_{xy} < 0$*. With equality iff* $x_1 : y_1 = x_2 : y_2$*.*

Lemma 2 ([7, Corollay 3.2.2])**.** *Let* $f : \mathbb{D}(\subseteqq \mathbb{R}_+ \times \mathbb{R}_+) \to \mathbb{R}_+$ *is an* n*-order homogeneous and two-time differentiable function. If* $\mathcal{I} = (\ln f(x, y))_{xy} < (>)0$*, then for arbitrary pairs* $(x_1, y_1), (x_2, y_2)$ *and* $(1, 1) \in \mathbb{D}$ *we have*

$$f(1, 1) f(x_1 x_2, y_1 y_2) < (>) f(x_1, y_1) f(x_2, y_2) \tag{1.8}$$

if (x_1, y_1) *and* (x_2, y_2) *are oppositely ordered. It is reversed if* (x_1, y_1) *and* (x_2, y_2) *are similarly ordered.*

Lemma 3 ([7, Corollary 3.1.1])**.** *Let* $f : \mathbb{D} \to \mathbb{R}$ *is a one-order homogeneous function and is two-time differentiable. Then for given* $p, q > 0$ *with* $p + q = 1$ *and arbitrary pairs* $(x_1, y_1), (x_2, y_2), (px_1 + qx_2, py_1 + qy_2) \in \mathbb{D}$*, the following inequality*

$$f(px_1 + qx_2, py_1 + qy_2) \le pf(x_1, y_1) + qf(x_2, y_2) \tag{1.9}$$

holds iff $xyf_{xy} < 0$*. With equality iff* $x_1 : y_1 = x_2 : y_2$*.*

2. Proofs of Theorem 2 and 3

Proof of Theorem 2. By Lemma 1, to prove Theorem 2, it is enough to prove that $\mathcal{I} = \mathcal{I}(r, s) = \frac{\partial^2 \ln E}{\partial x \partial y} < (>)0$ for all $(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+$ iff $r + s > (<)0$.

By partial derivative calculations, for $rs(r - s) \neq 0$ we have

$$\ln E = \frac{1}{r - s} \ln \left(\frac{s}{r} \frac{x^r - y^r}{x^s - y^s} \right), \tag{2.1}$$

$$\frac{\partial \ln E}{\partial x} = \frac{1}{E} \frac{\partial E}{\partial x} = \frac{1}{x(r - s)} \left(\frac{rx^r}{x^r - y^r} - \frac{sx^s}{x^s - y^s} \right), \tag{2.2}$$

$$\frac{\partial \ln E}{\partial y} = \frac{1}{E} \frac{\partial E}{\partial y} = \frac{1}{y(r - s)} \left(\frac{-ry^r}{x^r - y^r} + \frac{sy^s}{x^s - y^s} \right), \tag{2.3}$$

$$\frac{\partial^2 \ln E}{\partial x \partial y} = \frac{1}{xy(r - s)} \left(\frac{r^2 x^r y^r}{(x^r - y^r)^2} - \frac{s^2 x^s y^s}{(x^s - y^s)^2} \right). \tag{2.4}$$

1) For $rs(r-s)\neq 0$. Put $\ln\sqrt{x/y}=t, t\in\mathbb{R}$ and use $\sinh x=\frac{1}{2}(e^x-e^{-x}), \cosh x=\frac{1}{2}(e^x+e^{-x})$, then $\mathcal{I}$ can be written as

$$\begin{aligned}\mathcal{I}(r,s) &= \frac{1}{xy(r-s)}\left(\frac{r^2}{4\sinh^2 rt}-\frac{s^2}{4\sinh^2 st}\right)\\ &= \frac{1}{4xyt}\left(\frac{rt}{\sinh rt}+\frac{st}{\sinh st}\right)\frac{\frac{rt}{\sinh rt}-\frac{st}{\sinh st}}{rt-st}\\ &= \frac{1}{4xyt}\left(g(rt)+g(st)\right)\frac{g(rt)-g(st)}{rt-st}, \qquad (2.5)\end{aligned}$$

where $g(u)=\frac{u}{\sinh u}$if $u\neq 0$, $g(0)=1$.

It is easy to verify that $g(u)>0$ and is even on $(-\infty,\infty)$ and

$$g'(u)=\frac{\sinh u-u\cosh u}{\sinh^2 u}=\frac{h(u)}{\sinh^2 u}.$$

Since $h'(u)=-u\sinh u\leq 0$, then $h(u)<0$ for $u\in(0,\infty)$, it follows that $g'(u)<0$ for $u\in(0,\infty)$. Hence,

$$\frac{g(rt)-g(st)}{rt-st}=\frac{rt+st}{|rt|+|st|}\frac{g(|rt|)-g(|st|)}{|rt|-|st|},$$

and then

$$\operatorname{sgn}\frac{g(rt)-g(st)}{rt-st}=\operatorname{sgn}\frac{rt+st}{|rt|+|st|}\operatorname{sgn}\frac{g(|rt|)-g(|st|)}{|rt|-|st|}=-\operatorname{sgn}(r+s)t.$$

Thus

$$\operatorname{sgn}\mathcal{I}(r,s)=\operatorname{sgn}\frac{1}{4xyt}\operatorname{sgn}\left(g(rt)+g(st)\right)\operatorname{sgn}\frac{g(rt)-g(st)}{rt-st}=-\operatorname{sgn}(r+s).$$

This shows (1.5) holds iff $r+s>(<)0$.

2) For $r=0, s(r-s)\neq 0$. By (2.5) we have

$$\mathcal{I}(0,s):=\lim_{r\to 0}\mathcal{I}(r,s)=\frac{1}{4xyt}\left(1+g(st)\right)\frac{1-g(st)}{0-st}.$$

From 1) it follows that

$$\operatorname{sgn}\mathcal{I}(0,s)=-\operatorname{sgn}s.$$

3) For $s=0, r(r-s)\neq 0$. Similarly, we have

$$\operatorname{sgn}\mathcal{I}(r,0)=-\operatorname{sgn}r.$$

4) For $r=s\neq 0$. By (2.5) we have

$$\mathcal{I}(s,s):=\lim_{r\to s}\mathcal{I}(r,s)=\frac{1}{4xyt}\left(g(st)+g(st)\right)g'(st),$$

it follows that

$$\operatorname{sgn}\mathcal{I}(s,s)=\operatorname{sgn}\frac{1}{4xyt}\operatorname{sgn}\left(g(st)+g(st)\right)\operatorname{sgn}g'(st)=-\operatorname{sgn}s.$$

5) For $r=s\neq 0$. It is clear $\mathcal{I}(0,0)=0$.

Summarizing all cases above, we see that $\mathcal{I}(r,s)<(>)0$ iff $s+r>(<)0$.

It is obvious equality holds iff $x_1:y_1=x_2:y_2$.

This completes the proof. ∎

Proof of Theorem 3. By Lemma 2 and Theorem 2, note $E(r,s;1,1)=1$, we immediately obtain the result required.

This proof is completed. ∎

3. A new Proof of Theorem 1

Since $E(r,s;x,y)$ is a one-order homogeneous function of variables x and y, to prove Theorem 1, it is enough to prove that $E_{xy}=\frac{\partial^2 E(r,s;x,y)}{\partial x\partial y}<(>)0$ for all $(x,y)\in\mathbb{R}_+\times\mathbb{R}_+$ iff $r+s\geq(\leq)3$ and $\min(r,s)\geq(\leq)1$ with $(r,s)\neq(2,1),(1,2)$.

By (2.2)-(2.4) and note $\frac{\partial^2\ln E}{\partial x\partial y}=\frac{1}{E^2}\left(E\frac{\partial^2 E}{\partial x\partial y}-\frac{\partial E}{\partial x}\frac{\partial E}{\partial y}\right)$, for $rs(r-s)\neq 0$ we have

$$\begin{aligned}\frac{1}{E}\frac{\partial^2 E}{\partial x\partial y} &= \frac{\partial^2\ln E}{\partial x\partial y}+\left(\frac{1}{E}\frac{\partial E}{\partial x}\right)\left(\frac{1}{E}\frac{\partial E}{\partial y}\right)\\ &= \frac{1}{xy(r-s)}\left[\frac{r^2x^ry^r}{(x^r-y^r)^2}-\frac{s^2x^sy^s}{(x^s-y^s)^2}\right]\\ &\quad+\frac{1}{xy(r-s)^2}\left[\frac{-r^2x^ry^r}{(x^r-y^r)^2}+\frac{rs(x^ry^s+x^sy^r)}{(x^r-y^r)(x^s-y^s)}+\frac{-s^2x^sy^s}{(x^s-y^s)^2}\right]\\ &= \frac{1}{xy(r-s)^2}\left[\frac{r^2(r-s-1)x^ry^r}{(x^r-y^r)^2}+\frac{rs(xy)^s(x^{r-s}+y^{r-s})}{(x^r-y^r)(x^s-y^s)}-\frac{s^2(r-s+1)x^sy^s}{(x^s-y^s)^2}\right].\end{aligned}\tag{3.1}$$

Substituting $\ln(x/y)=t, t\in\mathbb{R}$ and using $\sinh x=\frac{1}{2}(e^x-e^{-x}),\cosh x=\frac{1}{2}(e^x+e^{-x})$, then $\frac{1}{E}\frac{\partial^2 E}{\partial x\partial y}$ can be expressed as

$$\begin{aligned}\frac{1}{E}\frac{\partial^2 E}{\partial x\partial y} &= \frac{1}{xy(r-s)^2}\left[\frac{r^2(r-s-1)}{4\sinh^2\frac{r}{2}t}+\frac{2rs\cosh\frac{r-s}{2}t}{4\sinh\frac{r}{2}t\sinh\frac{s}{2}t}-\frac{s^2(r-s+1)}{4\sinh^2\frac{s}{2}t}\right]\\ &= \frac{\sinh^{-2}\frac{r}{2}t\sinh^{-2}\frac{s}{2}t}{4xy(r-s)^2}[r^2(r-s-1)\sinh^2\tfrac{s}{2}t\\ &\quad+2rs\cosh\tfrac{r-s}{2}t\sinh\tfrac{r}{2}t\sinh\tfrac{s}{2}t-s^2(r-s+1)\sinh^2\tfrac{r}{2}t].\end{aligned}$$

Using the half-angle formula for $\sinh^2$ and "product into sum" formula for hyperbolic functions yield

$$\frac{1}{E}\frac{\partial^2 E}{\partial x\partial y}=c_0[A\cosh rt+B\cosh st+C\cosh(r-s)t+D],\tag{3.2}$$

where

$$\begin{aligned} c_0 &= -\frac{1}{8xy}(r-s)^{-2}\sinh^{-2}\frac{r}{2}t\sinh^{-2}\frac{s}{2}t, \\ A &= s(s-1)(r-s), \\ B &= -r(r-1)(r-s), \\ C &= rs, \\ D &= (r-s)^2(r+s)-(r^2-rs+s^2). \end{aligned}$$

Next let us prove stepwise.

Step 1. *Let*

$$g(t) := A\cosh rt + B\cosh st + C\cosh(r-s)t + D. \tag{3.3}$$

Then $\frac{\partial^2 E}{\partial x \partial y} < (>)0$ *for all* $(x,y) \in R_+ \times R_+$ *iff* $g''(t) > (<)0$.

Proof. By simple calculations we get $g(0) = g'(0) = 0$. Using Taylor formula, for arbitrary $t \in \mathbb{R}$ we have

$$g(t) = g''(\xi)\frac{t^2}{2}, \xi \text{ lie in } 0 \text{ and } t.$$

This shows $g(t) > (<)0$ for all $t \in \mathbb{R}$ iff $g''(t) > (<)0$ for all $t \in \mathbb{R}$. From $E, c_0 > 0$ and $\frac{1}{E}\frac{\partial^2 E}{\partial x \partial y} = c_0 g(t)$ it follows that $\frac{\partial^2 E}{\partial x \partial y} < (>)0$ for all $(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+$ iff $g''(t) > (<)0$.

Step 1 is proved. ∎

Step 2. *If* $(r+s)(r-s) \neq 0$, *then* $g''(t)$ *can be written as*

$$g''(t) = rs(r-s)p_1(t)p_2(t), \tag{3.4}$$

where

$$p_1(t) \ : \ = \cosh rt - \cosh st, \tag{3.5}$$

$$p_2(t) \ : \ = \begin{cases} r(s-1) + (r-s)\frac{\cosh(r-s)t - \cosh st}{\cosh rt - \cosh st}, & t \neq 0; \\ \frac{rs}{r+s}(r+s-3), & t = 0. \end{cases} \tag{3.6}$$

Proof. It is easy to check that

$$Ar^2 + Bs^2 + C(r-s)^2 = 0. \tag{3.7}$$

A simple derivative calculations yields

$$\begin{aligned} g''(t) &= Ar^2\cosh rt + Bs^2\cosh st + C(r-s)^2\cosh(r-s)t \\ &= Ar^2(\cosh rt - \cosh st) + C(r-s)^2[\cosh(r-s)t - \cosh st] \qquad (3.8) \\ &= (\cosh rt - \cosh st)\left[Ar^2 + C(r-s)^2\frac{\cosh(r-s)t - \cosh st}{\cosh rt - \cosh st}\right] \\ &= rs(r-s)(\cosh rt - \cosh st)\left[r(s-1) + (r-s)\frac{\cosh(r-s)t - \cosh st}{\cosh rt - \cosh st}\right] \\ &= rs(r-s)p_1(t)p_2(t) \text{ for } t \neq 0, \\ g''(0) &= Ar^2 + Bs^2 + C(r-s)^2 = 0 = rs(r-s)p_1(0)p_2(0). \end{aligned}$$

Step 2 is completed. ∎

Step 3. *$p_2(t) \geq (\leq) 0$ for all $t \in \mathbb{R}$ iff $p_2(0) \geq (\leq) 0$ and $p_2(\infty) \geq (\leq) 0$.*

Proof. 1) we first consider the monotone of $p_2(t)$. Derivative calculations yield

$$p_2'(t) = (r-s)\tfrac{\cosh(r-s)t-\cosh st}{\cosh rt-\cosh st}\left[\tfrac{(r-s)\sinh(r-s)t-s\sinh st}{\cosh(r-s)t-\cosh st} - \tfrac{r\sinh rt-s\sinh st}{\cosh rt-\cosh st}\right]. \tag{3.9}$$

Since $x\sinh x = |x|\sinh|x|, \cosh x = \cosh|x|$, so $p_2'(t)$ can be expressed as (3.10)

$$p_2'(t) = \frac{r-s}{|t|}\tfrac{\cosh(r-s)t-\cosh st}{\cosh rt-\cosh st}\left[\tfrac{|(r-s)t|\sinh|(r-s)t|-|st|\sinh|st|}{\cosh|(r-s)t|-\cosh|st|} - \tfrac{|rt|\sinh|rt|-|st|\sinh|st|}{\cosh|rt|-\cosh|st|}\right].$$

It is easy to prove that

$$\operatorname{sgn}\left(\tfrac{t_1\sinh t_1-t_2\sinh t_2}{\cosh t_1-\cosh t_2} - \tfrac{t_2\sinh t_2-t_3\sinh t_3}{\cosh t_2-\cosh t_3}\right) = \operatorname{sgn}(\cosh t_1 - \cosh t_3), \tag{3.11}$$

where $t_1, t_2, t_3 > 0$ and are pairwise unequal.

In fact, denote by $f(x) = \sqrt{x^2-1}\ln(x+\sqrt{x^2-1})$. Simple derivative calculations yield

$$\begin{aligned} f'(x) &= 1 + \frac{x}{\sqrt{x^2-1}}\ln(x+\sqrt{x^2-1}), \\ f''(x) &= \frac{x\sqrt{x^2-1}-\ln(x+\sqrt{x^2-1})}{(\sqrt{x^2-1})^3} = \frac{h(x)}{(\sqrt{x^2-1})^3}, \\ h'(x) &= \frac{2}{\sqrt{x^2-1}} > 0, \end{aligned}$$

and then $h(x) > h(1) = 0$. It follows that $f''(x) > 0$ on $(1, +\infty)$.

The property of convex functions implies

$$\frac{1}{x_1-x_3}\left[\frac{f(x_1)-f(x_2)}{x_1-x_2} - \frac{f(x_2)-f(x_3)}{x_2-x_3}\right] > 0, \tag{3.12}$$

in other words,

$$\operatorname{sgn}\left[\frac{f(x_1)-f(x_2)}{x_1-x_2} - \frac{f(x_2)-f(x_3)}{x_2-x_3}\right] = \operatorname{sgn}(x_1-x_3), \tag{3.13}$$

where x_1, x_2 and x_3 are pairwise unequal.

Put $x_i = \cosh t_i$ with $t_i > 0, i = 1, 2, 3$, then $\sqrt{x_i^2-1} = \sinh t_i, \ln(x_i + \sqrt{x_i^2-1}) = t_i$, consequently, (3.13) implies (3.11).

By (3.11) and (3.10) we have

$$\begin{aligned} \operatorname{sgn} p_2'(t) &= \operatorname{sgn}\left(\frac{r-s}{|t|}\right)\operatorname{sgn}\left[\tfrac{\cosh(r-s)t-\cosh st}{\cosh rt-\cosh st}\right]\operatorname{sgn}[\cosh|(r-s)t| - \cosh|rt|] \\ &= \operatorname{sgn}\left(\frac{r-s}{|t|}\right)\operatorname{sgn}\left[\tfrac{r(r-2s)}{(r-s)(r+s)}\right]\operatorname{sgn}[-s(2r-s)] \\ &= -\operatorname{sgn}(rs)\operatorname{sgn}(r+s)\operatorname{sgn}(r-2s)\operatorname{sgn}(2r-s). \end{aligned} \tag{3.14}$$

This shows $p_2(t)$ is always monotone in $t \in \mathbb{R}$ for fixed r, s.

2) Next let us observe that $p_2(0)$ and $p_2(\infty)$. It is easy to verify that

$$p_2(0) \ :=\lim_{t\to+0} p_2(t)=\frac{rs}{r+s}(r+s-3), \tag{3.15}$$

$$p_2(\infty) \ :=\lim_{t\to\infty} p_2(t)=\begin{cases} r(s-1), & r>s>0; \\ \infty, & r>0>s, r+s>0; \\ -\infty, & r>0>s, r+s<0; \\ s(r-1), & 0>r>s. \end{cases} \tag{3.16}$$

3) It is easy to verify that $p_2(t)$ is an even function on $\mathbb{R}$. And then because $p_2(t)$ is always monotone in $t\in\mathbb{R}$ we obtain immediately that $p_2(t)\geq 0$ for all $t\in\mathbb{R}$ iff $p_2(0)\geq 0$ and $p_2(\infty)\geq 0$, and $p_2(t)\leq 0$ for all $t\in\mathbb{R}$ iff $p_2(0)\leq 0$ and $p_2(\infty)\leq 0$.

Thus we complete step 3. ∎

Step 4. *For $rs(r-s)\neq 0$ we have*

1) $g''(t)\geq 0$ for all $t\in\mathbb{R}$ iff

$$r+s\geq 3 \text{ and } \min(r,s)\geq 1. \tag{3.17}$$

2) $g''(t)\leq 0$ for all $t\in\mathbb{R}$ iff

$$r+s\leq 3 \text{ and } \min(r,s)\leq 1. \tag{3.18}$$

Proof. Without loss of generality, we assume $r>s$. Note

$$\begin{aligned} \operatorname{sgn}(p_1(t)) &= \operatorname{sgn}(\cosh rt-\cosh st) \\ &= \operatorname{sgn}(rt+st)\operatorname{sgn}(rt-st) \\ &= \operatorname{sgn}(r-s)\operatorname{sgn}(r+s). \end{aligned} \tag{3.19}$$

Next let us consider three cases:

1) For $r+s>0$. By our assumption and (3.19) we obtain $p_1(t)>0$. From (3.4) and Step 3, we see that $g''(t)\geq 0, t\in\mathbb{R}$ iff $rsp_2(t)\geq 0$ for all $t\in\mathbb{R}$ iff $rsp_2(0)\geq 0$ and $rsp_2(\infty)\geq 0$. It follows from (3.15) and (3.16) that $g''(t)\geq 0, t\in\mathbb{R}$ iff

$$r+s\geq 3 \text{ and } r>s\geq 1.$$

Likewise $g''(t)\leq 0, t\in\mathbb{R}$ iff

$$r+s\leq 3 \text{ and } r>s>0, s\leq 1 \text{ or } r>0>s. \tag{3.20}$$

2) For $r+s<0$. By our assumption and (3.19) we have $p_1(t)<0$. From (3.4) and Step 3, we see that $g''(t)\geq 0, t\in\mathbb{R}$ iff $rsp_2(t)\leq 0$ for all $t\in\mathbb{R}$ iff $rsp_2(0)\leq 0$ and $rsp_2(\infty)\leq 0$. From (3.15) and (3.16) we see that is impossible.

That $g''(t)\leq 0, t\in\mathbb{R}$ iff $rsp_2(t)\geq 0$ for all $t\in\mathbb{R}$ iff $rsp_2(0)\geq 0$ and $rsp_2(\infty)\geq 0$. It follows from (3.15) and (3.16) that

$$r>0>s \text{ or } 0>r>s. \tag{3.21}$$

3) For $r+s=0$. By (3.8) we have,

$$g''(t) = C(r-s)^2[\cosh(r-s)t - \cosh st] = -s^2(r-s)^2[\cosh(-2st) - \cosh(st)], \tag{3.22}$$

and then

$$\operatorname{sgn}(g''(t)) = -\operatorname{sgn}(s^2(r-s)^2)\operatorname{sgn}(3s)\operatorname{sgn}(s) = -1.$$

Combining 1) and 2) with 3), we immediately get the desired results.

Step 4 is completed. ∎

Step 5. *For $rs(r-s) \neq 0$ we have $g''(t) = 0$ for all $t \in \mathbb{R}$ iff $(r,s) = (2,1),(1,2)$.*

Proof. We still assume $r > s$. It is obvious that $g''(t)$ is not always zero for all $t \in \mathbb{R}$ if $r+s=0$ by (3.22), and need only consider the case of $r+s \neq 0$.

If $(r,s) = (2,1)$, then by a direct calculation we have $g''(t) = rs(r-s)p_1(t)p_2(t) = 0$ for all $t \in \mathbb{R}$.

If $g''(t) = rs(r-s)p_1(t)p_2(t) = 0$ for all $t \in \mathbb{R}$, then $p_2(t) = 0$ for all $t \in \mathbb{R}$ since $rs(r-s) \neq 0$ and $p_1(t) = \cosh rt - \cosh st \neq 0$. From $p_2(t) = p_2(0) = p_2(+\infty) = 0$ we have $r+s-3=0$ and $r(s-1)=0$. Solving the equalities yields $(r,s) = (2,1)$.

Step 5 is completed. ∎

Step 6. *For $rs(r-s) = 0$, we still have $\frac{\partial^2 E(r,s;x,y)}{\partial x \partial y} \leq (\geq) 0$ for all $(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+$ iff*

$$r+s \geq (\leq) 3 \text{ and } \min(r,s) \geq (\leq) 1.$$

Proof. If $r=0, s(r-s) \neq 0$, then

$$E_{xy}(0,s;x,y) = \frac{\partial^2[\lim\limits_{r\to 0} E(r,s;x,y)]}{\partial x \partial y} = \lim_{r\to 0} \frac{\partial^2 E(r,s;x,y)}{\partial x \partial y} = \lim_{r\to 0}\left(E \cdot c_0 g(t)\right).$$

For r close to 0 $\operatorname{sgn}(c_0) = 1$, by Step 1 and 4 we have $\frac{\partial^2 E(r,s;x,y)}{\partial x \partial y} \leq (\geq) 0$ iff $g(t) \geq (\leq) 0$ iff $g''(t) \geq (\leq) 0$ iff both $r+s \geq (\leq) 3$ and $\min(r,s) \geq (\leq) 1$ hold. This shows that $\frac{\partial^2 E(0,s;x,y)}{\partial x \partial y} \leq (\geq) 0$ iff $0+s \geq (\leq) 3$ and $\min(0,s) \geq (\leq) 1$, i.e., this Lemma is valid.

Likewise, this Lemma is also true in the cases of $s=0, r(r-s) \neq 0$ and $rs \neq 0, r-s=0$ and $r=s=0$.

Step 6 is completed. ∎

Proof of Theorem 1. For $rs(r-s) \neq 0$ with $(r,s) \neq (2,1),(1,2)$, by Step1, 4 and 5, we have $E_{xy} = \frac{\partial^2 E(r,s;x,y)}{\partial x \partial y} < (>) 0$ iff $r+s \geq (\leq) 3$ and $\min(r,s) \geq (\leq) 1$; which is also valid for $rs(r-s) = 0$ by Step 6. Applying Lemma 3, we immediately obtain the desired results.

This completes the proof. ∎

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