A Fixed Point Theorem for a Sequence of Self Maps in a Menger Space Using a Contractive Control Function

K. P. R. Sastry 1, G. A. Naidu 2, P. V. S. Prasad 3 and S. S. A. Sastri 4

1 8-28-8/1, Tamil Street, Chinna Waltair, Visakhapatnam-530 017, India
kprsastry@hotmail.com

2, 3 Department of Mathematics, Andhra University
Visakhapatnam-530 003, India
drgolivean@yahoo.com
pvsprasad10@yahoo.in

4 Department of Basic Science and Humanities, Coastal Institute of Technology and Management, Narapam, Vizianagaram- 535 183, India
sambharasas@yahoo.co.in

Abstract. In this paper we observe that a result of Pathak and verma is not valid (Example 2.2). A suitable modification is also suggested (Theorem 2.5).

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1. INTRODUCTION

Throughout this paper, $\mathbb{R}^+$ is the set of all non negative real numbers. $E$ denotes the family of all functions $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\varphi$ is non decreasing with $\lim_{n\rightarrow\infty} \varphi^n(t) = \infty \forall \ t > 0$. 
Definition 1.1: [4] A function $F: \mathbb{R} \to [0, 1]$ is called a distribution function if

(i) $F$ is non-decreasing,
(ii) $F$ is left continuous,
(iii) $\inf_{x \in \mathbb{R}} F(x) = 0$ and $\sup_{x \in \mathbb{R}} F(x) = 1$

Definition 1.2: [2] A triangular norm $*: [0,1] \times [0,1] \to [0,1]$ is a function satisfying the following conditions

(i) $\alpha * 1 = \alpha \quad \forall \alpha \in [0,1]$
(ii) $\alpha * \beta = \beta * \alpha \quad \forall \alpha, \beta \in [0,1]$
(iii) $\gamma * \delta \geq \alpha * \beta \quad \forall \alpha, \beta, \gamma, \delta \in [0,1]$ with $\gamma \geq \alpha$ and $\delta \geq \beta$
(iv) $(\alpha * \beta)^* \gamma = \alpha * (\beta^* \gamma) \quad \forall \alpha, \beta, \gamma \in [0,1]$

A triangular norm is also denoted by t-norm. For any $\alpha, \beta \in [0,1]$, if we define $\alpha * \beta = \min \{\alpha, \beta\}$, then * is a t-norm and is denoted by ‘min’.

Definition 1.3: [4] Let $X$ be a non-empty set and let $F: X \times X \to \mathbb{D}$ (The set of distribution functions). For $p, q \in X$, we denote the image of the pair $(p, q)$ by $F_{p, q}$ which is a distribution function so that $F_{p, q}(x) \in [0, 1]$, for every real $x$.

Suppose $F$ satisfies:

a) $F_{p, q}(x) = 1$ for all $x > 0$ if and only if $p = q$

b) $F_{p, q}(0) = 0$

c) $F_{p, q}(x) = F_{q, p}(x)$

d) If $F_{p, q}(x) = 1$ and $F_{q, r}(y) = 1$ then $F_{p, r}(x + y) = 1$ where $p, q, r \in X$.

Then $(X, F)$ is called a probabilistic metric space.

Definition 1.4: [2] Let $X$ be a non-empty set, * is t-norm and $F: X \times X \to \mathbb{D}$ is a function satisfying

(i) $F_{p, q}(0) = 0 \forall p, q \in X$

(ii) $F_{p, q}(x) = 1$ for all $x > 0$ if and only if $p = q$

(iii) $F_{p, q}(x) = F_{q, p}(x) \forall p, q \in X$

(iv) $F_{p, r}(x + y) \geq * (F_{p, q}(x), F_{q, r}(y))$ for all $x, y \geq 0$ and $p, q, r \in X$.

Then the triplet $(X, F, *)$ is called a Menger space.

Definition 1.5: [5]

(i) Let $(X, F, *)$ be a Menger space and $p \in X$.

For $\varepsilon > 0, 0 < \lambda < 1$, the $(\varepsilon, \lambda)$-neighborhood of $p$ is defined as

$U_p(\varepsilon, \lambda) = \{q \in X: F_{p, q}(\varepsilon) > 1 - \lambda\}$. It may be observed that, if * is continuous then the topology induced by the family

$\{U_p(\varepsilon, \lambda): p \in X, \varepsilon > 0, 0 < \lambda < 1\}$ is a Hausdorff topology on $X$ and is known as the $(\varepsilon, \lambda)$ - topology.
A sequence \( \{x_n\} \) in \( X \) is said to converge to \( p \in X \) in the \((\varepsilon, \lambda)\) -topology, if for any \( \varepsilon > 0 \) and \( 0 < \lambda < 1 \) there exists a positive integer \( N = N(\varepsilon, \lambda) \) such that \( F_{x_n, p}(\varepsilon) > 1 - \lambda \) where \( n > N \).

A sequence \( \{x_n\} \) in \( X \) is said to be a Cauchy sequence in the \((\varepsilon, \lambda)\) - topology, if for \( \varepsilon > 0 \) and \( 0 < \lambda < 1 \) there exists a positive integer \( N = N(\varepsilon, \lambda) \) such that \( F_{x_m, x_n}(\varepsilon) > 1 - \lambda \) for all \( m, n > N \).

A Menger space \((X, F, *)\), where \( * \) is continuous, is said to be complete if every Cauchy sequence in \( X \) is convergent in \((\varepsilon, \lambda)\) - topology.

Definition 1.6: [1] Let \( * \) be a t-norm. For any \( a \in [0,1] \), write \( *_0(a) = 1 \) and \( *_1(a) = *_n(a, a) = *_n(a, a) \) for \( n = 0, 1, 2, \ldots \).

In general define \( *_{n+1}(a) = *_n(a, a) \) for \( n = 0, 1, 2, \ldots \).

Suppose that given \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( x > 1 - \delta \) implies \( *_n(x) > 1 - \varepsilon \) \( \forall n \in N \)

Then the sequence \( \{*_n\} \) is equicontinuous at 1. If \( \{*_n\} \) is equicontinuous at 1, then we say that \( * \) is a Hadzic type t-norm.

Definition 1.7: [2] Let \( A \) and \( S \) be two self-maps of a Menger space \((X, F, *)\).

Suppose there exists a sequence \( \{x_n\} \) in \( X \) such that

(i) compatible if \( \lim_{n \to \infty} F_{Ax_n, SAx_n}(t) = l \ \forall \ t > 0 \)

(ii) compatible of type (A), if

\( \lim_{n \to \infty} F_{SAx_n, AAx_n}(t) = \lim_{n \to \infty} F_{ASx_n, SSx_n}(t) = l \ \forall \ t > 0 \)

(iii) 2-compatible if \( \lim_{n \to \infty} F_{AAx_n, SSx_n}(t) = l \ \forall \ t > 0 \)

(iv) weakly compatible, if they commute at their coincidence point

i.e. \( SAx = ASx \) whenever \( Ax = Sx \).

2. Main Results

In this section we prove our main result (Theorem 2.5), which is a modification of Pathak and Verma ([2], Theorem 3.2) which is shown to be invalid through an example (Example 2.2).

Theorem 2.1([2], Theorem 3.2): If \( A, B, S \) and \( T \) be self maps of a complete Menger space \((X, F, *)\) with \( * = \min \) satisfying

(i) \( A(X) \subseteq T(X) \) and \( B(X) \subseteq S(X) \)

(ii) \( [1 + \alpha F_{SX, TY}(t)] * F_{AX, BY}(t) \geq \)
$\alpha \min \{ F_{AX, SX}(t) * F_{BY, TY}(t), F_{AX, TY}(2t) * F_{BY, SX}(2t) \} + F_{SX, TY}(\varphi(t)) * F_{AX, SX}(\varphi(t)) * F_{BY, TY}(\varphi(t)) * F_{AX, TY}(2\varphi(t)) * F_{BY, SX}(2\varphi(t))$

for all $x, y \in X, t > 0$ and $\varphi \in E$

(i) \( (A, S) \) is compatible (2-compatible or compatible of type (A)) and \( (B, T) \) is weakly compatible or vice-versa,

(iv) One mapping of the compatible (2-compatible or compatible of type (A)) pair is continuous, then \( A, B, S, T \) have a unique common fixed point.

The following example shows that Theorem 3.1 of [2] is not valid.

**Example 2.2:** Let \( X = Z^+ \) and for \( x, y \in X, x \neq y \), Define

\[
F_{X,Y}(t) = \begin{cases} 
1 & \text{if } t \in (n, n+1] \\
0 & \text{if } t \leq 0 
\end{cases}
\]

and \( \varphi(t) = \begin{cases} 
n + 1 & \text{if } t \in [n, n + 1) \\
0 & \text{if } t < 0 
\end{cases} \)

If we take \( A = T = I \) and \( B(n) = S(n) = n + 1 \), then \( \varphi \) is non-decreasing with \( \lim_{n \to \infty} \varphi^n(t) = \infty \forall t > 0 \) and \( (X, F, *) \) is a complete Menger space. \( (A, S) \) is compatible (2-compatible or compatible of type (A)) and \( (B, T) \) is compatible.

Clearly condition (ii) holds with \( \alpha = 0 \), but \( A, B, S, T \) have no common fixed point.

**Definition 2.3:** If \( \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \) is such that \( \varphi \) is strictly increasing and \( (\alpha - 1)\varphi(t) > t \) for some \( \alpha \in (1, 2) \), then \( \varphi \) is called a contractive control function.

**Notation:** The class of all contractive control functions is denoted by $\Phi$.

**Lemma 2.4:** [3] Let \( (X, F, *) \) be a Menger space with continuous Hadzic-type t-norm \( * \) and \( 0 < a < 1 \). Suppose \( \{x_n\}_{n=0}^\infty \) is a sequence in \( M \) such that for any \( s > 0 \), \( F_{x_n x_{n+1}}(s) \geq F_{x_n x_1}(\frac{s}{a^n}) \). Then \( \{x_n\} \) is a Cauchy sequence.

Now we present our main result, which is a modification of Theorem 2.1.

**Theorem 2.5:** Let \( A, B, S, T \) be self mappings of a complete Menger space \( (X, F, *) \) with \( * = \min \), satisfying

(i) \( A(X) \subseteq T(X) \) and \( B(X) \subseteq S(X) \)

(ii) \( F_{AX, BY}(t) > F_{SX, TY}(\varphi(t)) * F_{AX, SX}(\varphi(t)) * F_{BY, TY}(\varphi(t)) \)

\[
* F_{AX, TY}(\alpha \varphi(t)) * F_{BY, SX}(\alpha \varphi(t)) \quad \text{for all } x, y \in X, t > 0, \varphi \in \Phi, Ax \neq By \]

and for some \( \alpha \in (1, 2) \).

(iii) \( (A, S) \) is compatible (2-compatible or compatible of type (A)) and \( (B, T) \) is weakly compatible or vice-versa,
(iv) One mapping of the compatible (2-compatible or compatible of type (A)) pair is continuous.

Then A, B, S and T have a unique common fixed point.

**Proof:** For any \( x_0 \in X \), by condition (i), there exists \( x_1, x_2 \in X \) such that

\[ A x_0 = T x_1 = y_0 \text{ and } B x_1 = S x_2 = y_1. \]

Inductively we construct sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \) such that

\[ A x_{2n} = x_{2n+1}, B x_{2n+1} = x_{2n+2} = y_{2n+1} \text{ for } n = 0, 1, 2, \ldots. \]

Suppose \( A x_{2n} \neq B x_{2n+1} \). Then, by (ii), we have

\[ F_{y_{2n}, y_{2n+1}}(t) = F_{A x_{2n}, B x_{2n+1}}(t) > F_{S x_{2n}, S x_{2n}}(\alpha(t)) * F_{A x_{2n}, S x_{2n}}(\alpha(t)) * F_{B x_{2n+1}, S x_{2n}}(\alpha(t)) \]

Inductively, we construct sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \) such that

\[ A x_{2n} = x_{2n+1}, B x_{2n+1} = x_{2n+2} = y_{2n+1} \text{ for } n = 0, 1, 2, \ldots. \]

Suppose \( A x_{2n} = B x_{2n+1} \). Then, by (ii), we have

\[ F_{y_{2n}, y_{2n+1}}(t) = F_{y_{2n-1}, y_{2n}}(\alpha(t)) * F_{y_{2n}, y_{2n-1}}(\alpha(t)) \]

\[ \geq F_{y_{2n-1}, y_{2n}}(\alpha(t)) * F_{y_{2n}, y_{2n-1}}(\alpha(t)) \]

Thus \( F_{y_{2n}, y_{2n+1}}(t) \) is a Cauchy sequence in the complete Menger space [3].

Hence \( \{y_n\} \) converges to some \( z \) in \( X \).

\[ \lim_{n \to \infty} y_n = \lim_{n \to \infty} A x_{2n} = \lim_{n \to \infty} B x_{2n+1} = \lim_{n \to \infty} T x_{2n+1} = \lim_{n \to \infty} S x_{2n} = z. \]

Suppose \( S \) is continuous and 2-compatible with \( A \).

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Thus we always have
\[ F_{AZ_{2n},BZ_{2n+1}}(t) \geq F_{AZ_{2n},T_{2n+1}}(\varphi(t)) * F_{AX_{2n}}(\alpha \varphi(t)) \]

Now on letting \( n \to \infty \), we get
\[ F_{AZ_{z},T_{z}}(t) \geq F_{AZ_{z},S_{z}}(\varphi(t)) * F_{Z_{z},S_{z}}(\alpha \varphi(t)) \]
\[ \geq F_{Z_{z},S_{z}}(\varphi(t)) \]
\[ \geq F_{z_{z},S_{z}}(\varphi^{2}(t)) \geq \ldots \geq F_{z_{z},S_{z}}(\varphi^{n}(t)) \to 1 \text{ as } n \to \infty \]

Therefore \( Z_{z} = z \).

Similarly, by taking \( x = z \) and \( y = x_{2n+1} \) in (ii), we get \( Az = z \).
\[ \therefore Az = Sz = z. \]

Since \( A(X) \subseteq T(X) \), there exists \( v \in X \) such that \( z = Az = Tz \).

By taking \( x = x_{2n} \) and \( y = v \) in (ii), we get \( Bv = z \).

Therefore \( Bv = Tv = z \).

Since \( (B, T) \) is weakly compatible, we get \( Bz = BTv = TBv = Tz \).

By taking \( x = x_{2n} \) and \( y = z \) in (ii), we get \( z = Bz = Tz \).

Hence \( z \) is a common fixed point of \( A, B, S \) and \( T \).

To prove uniqueness, suppose \( w \) is also a common fixed point of \( A, B, S \) and \( T \).

If \( w \neq z \), by taking \( x = z \) and \( y = w \) in (ii), we get
\[ F_{AZ_{z},Bz_{z}}(t) \geq F_{AZ_{z},S_{z}}(\varphi(t)) * F_{Bz_{z},T_{z}}(\varphi(t)) * F_{AZ_{z},S_{z}}(\alpha \varphi(t)) \]
\[ \Rightarrow F_{z_{z},T_{z}}(t) \geq F_{z_{z},S_{z}}(\varphi(t)) * F_{z_{z},S_{z}}(\alpha \varphi(t)) \]
\[ \geq F_{z_{z},S_{z}}(\varphi(t)) \]
\[ \geq F_{z_{z},S_{z}}(\varphi^{2}(t)) \geq \ldots \geq F_{z_{z},S_{z}}(\varphi^{n}(t)) \to 1 \text{ as } n \to \infty \]

Therefore \( z = w \). This completes the proof.

**Corollary 2.6:** Let \( A, B \) be self maps of a complete Menger space \( (X, F, *) \) with \( * = \min \) satisfying

(i) \( A(X) \subseteq B(X) \)

(ii) \( F_{AX,AY}(t) > F_{BX,By}(\varphi(t)) * F_{AX,Bx}(\varphi(t)) * F_{AY,By}(\varphi(t)) \)
\[ * F_{AX,By}(\alpha \varphi(t)) * F_{Ay,Bx}(\alpha \varphi(t)) \text{ for all } x, y \in X, t > 0, \varphi \in \Phi, Ax \neq By \]
and for some \( \alpha \in (1, 2) \).

(iii) \( (A, B) \) is compatible (2-compatible or compatible of type (A))

(iv) One mapping of the compatible (2-compatible or compatible of type (A)) pair is continuous.

Then \( A, B \) have a unique common fixed point in \( X \).
Corollary 2.7: Let A, B, S and T be self mappings of a complete Menger space $(X, F, \ast)$ with $\ast = \min$, satisfying

(i) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$

(ii) $F_{Ax,By}(kt) > F_{Sx,Ty}(\varphi(t)) \ast F_{Ax,Sx}(\varphi(t)) \ast F_{By,Ty}(\varphi(t))$

for all $x, y \in X, t > 0, \varphi \in \Phi, Ax \neq By$ and for some $\alpha \in (1, 2)$ and $k \in (0,1)$

(iii) One of the pairs $(A,S)$ and $(B,T)$ is compatible and the other is weakly compatible

(iv) One mapping of the compatible pair is continuous.

Then A, B, S and T have a unique common fixed point in X.

References


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