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A Fixed Point Theorem for a Sequence of Self Maps in a Menger Space Using a Contractive Control Function

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Abstract. In this paper we observe that a result of Pathak and verma is not valid (Example 2.2). A suitable modification is also suggested (Theorem 2.5).

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1. INTRODUCTION

Throughout this paper, $\mathbb{R}^+$ is the set of all non negative real numbers. E denotes the family of all functions $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that φ is non decreasing with $\lim_{n\to\infty} \varphi^n(t) = \infty \ \forall \ t > 0$.

Definition1.1: [4] A function F: $\mathbb{R} \rightarrow [0, 1]$ is called a distribution function if

(i) F is non-decreasing,
(ii) F is left continuous,
(iii) $\inf_{x\in\mathbb{R}} F(x) = 0$ *and* $\sup_{x\in\mathbb{R}} F(x) = 1$

Definition1.2: [2] *A* triangular norm $*: [0,1] \times [0,1] \rightarrow [0,1]$ is a function satisfying the following conditions

(i) $\alpha * 1 = \alpha \quad \forall \alpha \in [0,1]$
(ii) $\alpha * \beta = \beta * \alpha \quad \forall \alpha, \beta \in [0,1]$
(iii) $\gamma * \delta \geq \alpha * \beta \ \forall \alpha, \beta, \gamma, \delta \in [0,1]$ *with* $\gamma \geq \alpha$ *and* $\delta \geq \beta$
(iv) $(\alpha * \beta) * \gamma = \alpha * (\beta * \gamma) \ \forall \alpha, \beta, \gamma \in [0,1]$

A triangular norm is also denoted by t-norm. For any $\alpha, \beta \in [0,1]$, if we define $\alpha * \beta = \min\{\alpha, \beta\}$, then * is a t-norm and is denoted by 'min'.

Definition1.3: [4] Let X be a non-empty set and let F: X x X→ $\mathfrak{D}$ (The set of distribution functions). For $p, q \in$ X, we denote the image of the pair (p, q) by $F_{p,q}$ which is a distribution function so that $F_{p,q}(x) \in [0, 1]$, for every real x.
Suppose F satisfies:

a) $F_{p,q}(x) = 1$ for all $x > 0$ if and only if $p = q$
b) $F_{p,q}(0) = 0$
c) $F_{p,q}(x) = F_{q,p}(x)$
d) If $F_{p,q}(x) = 1$ and $F_{q,r}(y) = 1$ then $F_{p,r}(x + y) = 1$ where $p, q, r \in$ X.

Then (X, F) is called a probabilistic metric space.

Definition1.4: [2] Let X be a non empty set, * is t-norm and F: X x X → $\mathfrak{D}$ is a function satisfying

(i) $F_{p,q}(0) = 0 \ \forall p, q \in X$
(ii) $F_{p,q}(x) = 1$ for all $x > 0$ if and only if $p = q$
(iii) $F_{p,q}(x) = F_{q,p}(x) \ \forall p, q \in X$
(iv) $F_{p,r}(x + y) \geq *(F_{p,q}(x), F_{q,r}(y))$ for all $x, y \geq 0$ *and* $p, q, r \in X$.

Then the triplet (X, F,*) is called a Menger space.

Definition 1.5: [5]

(i) Let (X, F, *) be a Menger space and $p \in$ X.
For $\varepsilon > 0, 0 < \lambda < 1$, the (ε, λ) -neighborhood of p is defined as $U_p(\varepsilon, \lambda) = \{q \in X: F_{p,q}(\varepsilon) > 1 - \lambda\}$. It may be observed that, if * is continuous then the topology induced by the family $\{U_p(\varepsilon, \lambda): p \in X, \varepsilon > 0, 0 < \lambda < 1\}$ is a Hausdorff topology on X and is known as the (ε, λ) - topology.

(ii) A sequence $\{x_n\}$ in X is said to converge to $p \in$ X in the (ε, λ) -topology, if for any $\varepsilon > 0$ and $0 < \lambda < 1$ there exists a positive integer N = N (ε, λ) such that $F_{x_n,p}(\varepsilon) > 1 - \lambda$ where $n > N$.

(iii) A sequence $\{x_n\}$ in X is said to be a Cauchy sequence in the (ε, λ) - topology, if for $\varepsilon > 0$ and $0 < \lambda < 1$ there exists a positive integer N = N (ε, λ) such that $F_{x_m,x_n}(\varepsilon) > 1 - \lambda$ for all *m, n* > N.

(iv) A Menger space (X, F, *), where * is continuous, is said to be complete if every Cauchy sequence in X is convergent in (ε, λ) - topology.

Definition 1.6: [1] Let * be a t-norm. For any $a \in [0,1]$, write $*_0(a) = 1$ and
$*_1(a) = *(*_0(a), a) = *(1, a) = a$
In general define $*_{n+1}(a) = *(*_n(a), a)\ for\ n = 0, 1, 2 \ldots$
Suppose that given
$\varepsilon > 0\ there\ exists\ \delta > 0\ such\ that\ x > 1 - \delta\ implies\ *_n(x) > 1 - \varepsilon\ \forall n \in N$
Then the sequence $\{*_n\}$ is equicontinuous at 1. If $\{*_n\}$ is equicontinuous at 1, then we say that * is a Hadzic type t-norm.

Definition1.7: [2] Let *A* and *S* be two self-maps of a Menger space (*X, F,* *). Suppose there exists a sequence $\{x_n\}$ in *X* such that
$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = l$. Then A, S are said to be

(i) compatible if $\lim_{n\to\infty} F_{ASx_n,SAx_n}(t) = l\ \forall\ t > 0$

(ii) compatible of type (A), if $\lim_{n\to\infty} F_{SAx_n,AAx_n}(t) = \lim_{n\to\infty} F_{ASx_n,SSx_n}(t) = l\ \ \forall\ t > 0$

(iii) 2-compatible if $\lim_{n\to\infty} F_{AAx_n,SSx_n}(t) = l\ \forall\ t > 0$

(iv) weakly compatible, if they commute at their coincidence point i.e. SA*x* = AS*x* whenever A*x* = S*x*.

2. Main Results

In this section we prove our main result (Theorem 2.5), which is a modification of Pathak and Verma ([2], Theorem 3.2) which is shown to be invalid through an example (Example 2.2).

Theorem 2.1([2], Theorem 3.2): If A, B, S and T be self maps of a complete Menger space (X, F, *) with * = min satisfying

(i) A(X) ⊆ T(X) and B(X) ⊆ S(X)

(ii) $[\,1 + \alpha F_{Sx,Ty}(t)] * F_{Ax,By}(t) \ \geq$

$$\alpha.\min\{F_{Ax,Sx}(t) * F_{By,Ty}(t), F_{Ax,Ty}(2t) * F_{By,Sx}(2t)\}$$
$$+ F_{Sx,Ty}(\varphi(t)) * F_{Ax,Sx}(\varphi(t)) * F_{By,Ty}(\varphi(t)) * F_{Ax,Ty}(2\varphi(t)) * F_{By,Sx}(2\varphi(t))$$

for all $x, y \in X, t > 0$ *and* $\varphi \in E$

(iii) (A,S) is compatible (2-compatible or compatible of type (A)) and (B,T) is weakly compatible or vice-versa,

(iv) One mapping of the compatible (2-compatible or compatible of type (A)) pair is continuous, then A, B, S and T have a unique common fixed point.

The following example shows that Theorem 3.1 of [2] is not valid.

Example 2.2: Let X = Z^+ and for $x, y \in X, x \neq y$, Define

$$F_{x,y}(t) = \begin{cases} 1 - \frac{1}{n+2} & if\ t \in (n, n+1] \\ 0 & if\ t \leq 0 \end{cases} \quad \text{and} \quad \varphi(t) = \begin{cases} n+1 & if \quad t \in [n, n+1) \\ 0 & if \quad t < 0 \end{cases}$$

If we take A = T = I and B (n) = S (n) = n + 1, then φ is non decreasing with $\lim_{n\to\infty} \varphi^n(t) = \infty\ \forall\ t > 0$ and (X, F, *) is a complete Menger space. (A, S) is compatible (2-compatible or compatible of type (A)) and (B, T) is compatible.
Since (B, T) coincidence point, so that (B, T) is weakly compatible.
Clearly condition (ii) holds with $\alpha = 0$, but A, B, S, and T have no common fixed point.

Definition 2.3: If $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is such that φ is strictly increasing and $(\alpha - 1)\varphi(t) > t$ for some $\alpha \in (1, 2)$, then φ is called a contractive control function.

Notation: The class of all contractive control functions is denoted by Φ.

Lemma 2.4: [3] Let (X, F, *) be a Menger space with continuous Hadzic-type t-norm * and $0 < a < 1$. Suppose $\{x_n\}_{n=0}^{\infty}$ is a sequence in M such that for any $s > 0$, $F_{x_n,x_{n+1}}(s) \geq F_{x_0,x_1}(\frac{s}{a^n})$. Then $\{x_n\}$ is a Cauchy sequence.

Now we present our main result, which is a modification of Theorem 2.1.

Theorem 2.5: Let A, B, S and T be self mappings of a complete Menger space (X, F, *) with * = min , satisfying

(i) A(X) ⊆ T(X) and B(X) ⊆ S(X)

(ii) $F_{Ax,By}(t) > F_{Sx,Ty}(\varphi(t)) * F_{Ax,Sx}(\varphi(t)) * F_{By,Ty}(\varphi(t)) * F_{Ax,Ty}(\alpha\varphi(t)) * F_{By,Sx}(\alpha\varphi(t))$ for all $x, y \in X, t > 0, \varphi \in \Phi$, $Ax \neq By$ and for some $\alpha \in (1, 2)$.

(iii) (A,S) is compatible (2-compatible or compatible of type (A)) and (B,T) is weakly compatible or vice-versa,

(iv) One mapping of the compatible (2-compatible or compatible of type (A)) pair is continuous.

Then A, B, S and T have a unique common fixed point.

Proof: For any $x_0 \in X$, by condition (i), there exists $x_1, x_2 \in X$ such that
$Ax_0 = Tx_1 = y_0$ and $Bx_1 = Sx_2 = y_1$
Inductively we construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that
$Ax_{2n} = Tx_{2n+1} = y_{2n}$ and $Bx_{2n+1} = Sx_{2n+2} = y_{2n+1}$ for $n = 0, 1, 2 \ldots$.
Suppose $Ax_{2n} \neq Bx_{2n+1}$. Then, by (ii), we have

$$F_{y_{2n},y_{2n+1}}(t) = F_{Ax_{2n},Bx_{2n+1}}(t) > F_{Sx_{2n},Tx_{2n+1}}(\varphi(t)) * F_{Ax_{2n},Sx_{2n}}(\varphi(t)) * F_{Bx_{2n+1},Tx_{2n+1}}(\varphi(t)) * F_{Ax_{2n},Tx_{2n+1}}(\alpha\varphi(t)) * F_{Bx_{2n+1},Sx_{2n}}(\alpha\varphi(t))$$

$$= F_{y_{2n-1},y_{2n}}(\varphi(t)) * F_{y_{2n},y_{2n-1}}(\varphi(t)) * F_{y_{2n+1},y_{2n}}(\varphi(t)) * F_{y_{2n},y_{2n}}(\alpha\varphi(t)) * F_{y_{2n+1},y_{2n-1}}(\alpha\varphi(t))$$

$$\geq F_{y_{2n-1},y_{2n}}(\varphi(t)) * F_{y_{2n},y_{2n-1}}(\varphi(t)) * F_{y_{2n+1},y_{2n}}((\alpha-1)\varphi(t)) * F_{y_{2n+1},y_{2n-1}}(\alpha\varphi(t))$$

$$= F_{y_{2n-1},y_{2n}}(\varphi(t)) * F_{y_{2n},y_{2n+1}}((\alpha-1)\varphi(t))$$

$$\geq F_{y_{2n-1},y_{2n}}(\varphi(t)) * F_{y_{2n},y_{2n+1}}(t)$$

Thus $F_{y_{2n},y_{2n+1}}(t) > F_{y_{2n-1},y_{2n}}(\varphi(t)) * F_{y_{2n},y_{2n+1}}(t)$

$\therefore F_{y_{2n},y_{2n+1}}(t) \geq F_{y_{2n-1},y_{2n}}(\varphi(t))$

If $Ax_{2n} = Bx_{2n+1}$, then $y_{2n} = y_{2n+1}$ so that $F_{y_{2n},y_{2n+1}}(t) = 1$
Consequently $F_{y_{2n},y_{2n+1}}(t) \geq F_{y_{2n-1},y_{2n}}(\varphi(t))$
Thus we always have $F_{y_{2n},y_{2n+1}}(t) \geq F_{y_{2n-1},y_{2n}}(\varphi(t))$
Similarly $F_{y_{2n+1},y_{2n+2}}(t) \geq F_{y_{2n+1},y_{2n}}(\varphi(t))\ \forall\, t > 0$ and $n = 0, 1, 2 \ldots$.
Hence $F_{y_n,y_{n+1}}(t) \geq F_{y_{n-1},y_n}(\varphi(t)) \geq F_{y_{n-1},y_n}\left(\frac{t}{\alpha-1}\right)\ \forall\, t > 0$ and $n = 0, 1, 2 \ldots$.
So $\{y_n\}$ is a Cauchy sequence in the complete Menger space [3].
Thus $\{y_n\}$ is converges to some z in X.

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} A\,x_{2n} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Sx_{2n} = z$$

Suppose S is continuous and 2-compatible with A.
$SSx_{2n} \to Sz, SAx_{2n} \to Sz$ and $\lim_{n\to\infty} F_{AAx_{2n}SSx_{2n}}(t) = 1$
We have $AAx_{2n} \to Sz$
By taking $x = Ax_{2n}$ and $y = x_{2n+1}$ in (ii), if $AAx_{2n} \neq Bx_{2n+1}$, we get

$$F_{AAx_{2n},Bx_{2n+1}}(t) > F_{SAx_{2n},Tx_{2n+1}}(\varphi(t)) * F_{AAx_{2n},SAx_{2n}}(\varphi(t)) * F_{Bx_{2n+1},Tx_{2n+1}}(\varphi(t)) * F_{AAx_{2n},Tx_{2n+1}}(\alpha\varphi(t)) * F_{Bx_{2n+1},SAx_{2n}}(\alpha\varphi(t))$$

If $AAx_{2n} = Bx_{2n+1}$, then

$$F_{AAx_{2n},Bx_{2n+1}}(t) = 1 \geq F_{SAx_{2n},Tx_{2n+1}}(\varphi(t)) * F_{AAx_{2n},SAx_{2n}}(\varphi(t)) * F_{Bx_{2n+1},Tx_{2n+1}}(\varphi(t))$$

$$* F_{AAx_{2n},Tx_{2n+1}}\big(\alpha\varphi(t)\big) * F_{Bx_{2n+1},SAx_{2n}}(\alpha\varphi(t))$$

Thus we always have

$$F_{AAx_{2n},Bx_{2n+1}}(t) \geq F_{SAx_{2n},Tx_{2n+1}}\big(\varphi(t)\big) * F_{AAx_{2n},SAx_{2n}}(\varphi(t)) * F_{Bx_{2n+1},Tx_{2n+1}}(\varphi(t))$$
$$* F_{AAx_{2n},Tx_{2n+1}}\big(\alpha\varphi(t)\big) * F_{Bx_{2n+1},SAx_{2n}}(\alpha\varphi(t))$$

Now on letting $n \to \infty$, we get

$$F_{Sz,z}(t) \geq F_{Sz,z}\ \big(\varphi(t)\big) * F_{Sz,Sz}(\varphi(t)) * F_{z,z}(\varphi(t))\ *\ F_{Sz,Sz}\big(\alpha\varphi(t)\big) * F_{z,Sz}(\alpha\varphi(t))$$
$$\geq F_{Sz,z}\ \big(\varphi(t)\big) * F_{z,Sz}(\alpha\varphi(t))$$
$$\geq F_{Sz,z}\ \big(\varphi(t)\big) \geq F_{Sz,z}\big(\varphi^2(t)\big) \geq \ldots\ldots \geq F_{Sz,z}\big(\varphi^n(t)\big)\ \ \to 1\ as\ n \to \infty$$

Therefore S$z = z$.
Similarly, by taking $x = z$ and $y = x_{2n+1}$ in (ii), we get A$z = z$.
$\therefore Az = Sz = z.$
Since A(X) ⊆ T(X), there exists $v \in X$ such that $z = Az = Tv$.
By taking $x = x_{2n}$ and $y = v$ in (ii), we get B$v = z$.
Therefore $Bv = Tv = z.$
Since (B, T) is weakly compatible, we get $Bz = BTv = TBv = Tz$.
By taking $x = x_{2n}$ and $y = z$ in (ii), we get $z = Bz = Tz$.
Hence z is a common fixed point of A, B, S and T.
To prove uniqueness, suppose w is also a common fixed point of A, B, S and T.
If $w \neq z$, by taking $x = z$ and $y = w$ in (ii), we get

$$F_{Az,Bw}(t) \geq F_{Sz,Tw}\ \big(\varphi(t)\big) * F_{Az,Sz}(\varphi(t)) * F_{Bw,Tw}(\varphi(t)\ * F_{Az,Tw}\big(\alpha\varphi(t)\big) * F_{Bw,Sz}(\alpha\varphi(t))$$
$$\Rightarrow F_{z,w}(t) \geq F_{z,w}\ \big(\varphi(t)\big) * F_{z,z}(\varphi(t)) * F_{w,w}(\varphi(t) * F_{z,w}\big(\alpha\varphi(t)\big) *\ F_{w,z}(\alpha\varphi(t))$$
$$= F_{z,w}\ \big(\varphi(t)\big) * F_{z,w}\big(\alpha\varphi(t)\big)$$
$$\geq F_{z,w}\ \big(\varphi(t)\big) \geq F_{z,w}\big(\varphi^2(t)\big) \geq \cdots \geq F_{z,w}\big(\varphi^n(t)\big) \to 1\ as\ n \to \infty$$

Therefore $z = w$. This completes the proof.

Corollary 2.6: Let A, B be self maps of a complete Menger space (X, F, *) with * = min satisfying

(i) A(X) ⊆ B(X)

(ii) $F_{Ax,Ay}(t) > F_{Bx,By}\big(\varphi(t)\big) * F_{Ax,Bx}\big(\varphi(t)\big) * F_{Ay,By}\big(\varphi(t)\big)$
$* F_{Ax,By}\big(\alpha\varphi(t)\big) * F_{Ay,Bx}\big(\alpha\varphi(t)\big)$ for all $x,y \in X, t > 0, \varphi \in \Phi$, $Ax \neq By$ and for some $\alpha \in (1, 2)$.

(iii) (A,B) is compatible (2-compatible or compatible of type (A)) and

(iv) One mapping of the compatible (2-compatible or compatible of type (A)) pair is continuous.

Then A, B have a unique common fixed point in X.

Corollary 2.7: Let A, B, S and T be self mappings of a complete Menger space $(X, F, *)$ with $* = \min$, satisfying

(i) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$

(ii) $F_{Ax,By}(kt) > F_{Sx,Ty}(\varphi(t)) * F_{Ax,Sx}(\varphi(t)) * F_{By,Ty}(\varphi(t))$
$* F_{Ax,Ty}(\alpha\varphi(t)) * F_{By,Sx}(\alpha\varphi(t))$ for all $x,y \in X, t > 0, \varphi \in \Phi$, $Ax \neq By$
and for some $\alpha \in (1,2)$ and $k \in (0,1)$

(iii) One of the pairs (A,S) and (B,T) is compatible and the other is weakly compatible

(iv) One mapping of the compatible pair is continuous.

Then A, B, S and T have a unique common fixed point in X.

References

1. **O. Hadzic:** A Generalization of the contraction principle in probabilistic metric spaces, Univ. u. Nvom Sadu Zb. Road, Prirod-Mat. Fak. 10(1980), 13-21 (1981).
2. **H.K. Pathak and R.K. Verma:** Common fixed point Theorems for weakly compatible mappings on Menger spaces and application, Int. Journal of Math. Analysis, Vol. 3, 2009, no.24, 1199-1206.
3. **K.P.R. Sastry, G.V.R. Babu and M.L. Sandhya:** Weak contractions in Menger spaces – II, Journal of Adv. Rersearch in Pure Mathematics, Vol. 2, issue 1, 2010, pp.65-73.
4. **B. Schweizer and A. Sklar:** Probabilistic metric spaces, North Holland Series in Probability and applied Mathematics, 1983; MR0790314 (86g: 54045).
5. **B. Schweizer and A. Sklar:** Statistical metric spaces, North Holland Amsterdam, (1983).

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