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Entire Functions of Certain Singular Distributions and Interaction of Delta-Waves in Nonlinear Conservation Laws

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Abstract

With the help of our theory of multiplication of distributions it is possible to give a meaning in $\mathcal{D}'$ to the composition $\phi \circ T$, where ϕ is an entire function and T belongs to a certain class of strongly singular distributions. As an application we are able to prove that, in our setting, the nonlinear conservation law $u_t + [\phi(u)]_x = 0$ has solutions which propagate like solitary delta-waves with constant speed. The interaction of two delta-waves is also studied and the speed functions of these waves are evaluated. We conclude that such speed functions are always bounded and the collision of two such delta waves is impossible. These results are obtained with the help of a rigorous and consistent concept of solution we have already introduced in previous works.

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1 Introduction

Let us consider the conservation law

$$u_t + [\phi(u)]_x = 0, \tag{1}$$

where x stands for the space variable (one-dimensional in this text), t is the time variable, $\phi(u)$ is the flux and $u(x,t)$ represents the physical state.

In the present paper we will prove that if ϕ is an entire function then, in a sense to be defined later, delta-waves with the form

$$u(x,t) = m_1\delta(x - \gamma_1(t))$$

can be solutions of (1) and the speed of propagation $\gamma_1'(t)$ of such delta-waves is necessarily constant (here δ stands for the Dirac measure supported at the origin, m_1 is a complex number $\neq 0$ and γ_1 is a C^1-real function).

We also prove that the propagation of the superposition of two such delta-waves is possible but such waves can never collide. Thus,

$$u(x,t) = m_1\delta(x-\gamma_1(t)) + m_2\delta(x-\gamma_2(t))$$

can be a solution of (1), for convenient C^1-real functions γ_1, γ_2, but we always have $\gamma_1(t) \neq \gamma_2(t)$ at any instant t; in this case, we will see that the speeds $\gamma_1'(t)$ and $\gamma_2'(t)$ are always bounded functions.

For this purpose we use our theory of distributional products which affords in a simple way physically significant results, as we will explain now (the reader can obtain a general view of the ideas of this theory in [11] and the details in [12,13]).

There exist two kinds of approaches to the multiplication of distributions: those in which the outcome of the products are distributions and those in which the final result is not a distributional entity. This second type of approach is strongly related with the framework of Rosinger [7,8,9,10] which brought into light the algebraic structures involved in imbedding the space of distributions $\mathcal{D}'$ into factor algebras. Undoubtedly, within this frame, the most popular approach is Colombeau's multiplication of distributions [2,3]. The book of Oberguggenberger [6] is an excellent guide for a quick review of this topic.

Some distributional products do not succeed in multiplying distributions with a strong singularity at the same point such as, for example, the product $\delta\delta$ of two Dirac-delta measures. Others obtain such products at the price of leaving out the space of distributions. For example, $\delta\delta$ is an element of the Colombeau algebra G, but this element has not an associated distribution. Consequently, from the mathematical point of view, $\delta\delta$ is well defined but difficult to interpret at a physical theoretical level; also some indeterminacies arise.

Our approach is a general theory that provides a distribution as the outcome of any product of distributions, once we fix a certain function α. Such a function quantifies the indeterminacy inherent to the products, and once fixed, its physical interpretation becomes clear. We stress that this indeterminacy is not, in general, avoidable and in many questions it plays an essential role. Concerning this point let us mention [1,4,5], and also Section 6 of this paper, to see the necessity of the indeterminacy which underlies the distributional product in a simple physical context. Within our framework we have for instance exhibited explicitly [11,14] Dirac-delta wave solutions (and also solutions which are not measures) for the turbulent model ruled by Burgers nonconservative equation; also phenomena such as "narrow soliton solutions" (in the sense of Maslov, Omel'yanov and Tsupin) can be rigorously obtained

[14]. Let us summarize the contents of the present paper.

In Section 2, we give formulas for computing the products that will be needed and define natural powers for certain distributions T. In Section 3, we will consider power series of distributions to give a meaning to $\phi \circ T$, with ϕ entire, and we compute $\phi \circ (m\delta_a)$, where $m \in \mathbb{C}$ and δ_a stands for the Dirac measure at $a \in \mathbb{R}$. In Section 4 we compute $\phi \circ (m_a\delta_a + m_b\delta_b)$ with $m_a, m_b \in \mathbb{C}$. In Section 5 we define the notion of α-solution for (1) and we prove that this notion is a consistent extension of the notion of a classical solution for this equation. In Sections 6 and 7 we analyze the propagation of the delta-waves $u(x,t) = m_1\delta(x - \gamma_1(t))$ and $u(x,t) = m_1\delta(x - \gamma_1(t)) + m_2\delta(x - \gamma_2(t))$. Lastly, in Section 8, we prove the impossibility of collision of two delta-waves in any model ruled by the conservation law (1) with ϕ entire; the boundedness of the speeds $\gamma_1'(t)$ and $\gamma_2'(t)$ is also established.

2 Products and powers of certain distributions

Let $\mathcal{D}$ be the space of indefinitely differentiable complex-valued functions defined on $\mathbb{R}$ with compact support, $\mathcal{D}'$ the space of Schwartz distributions and $\alpha \in \mathcal{D}$ even with $\int_{-\infty}^{+\infty} \alpha = 1$. In our theory of products we can compute the α-product $T \dot{\alpha} S$ by the formula

$$T \dot{\alpha} S = T\beta + (T * \alpha) f \tag{2}$$

for $T \in \mathcal{D}'^p$ and $S = \beta + f \in C^p \oplus \mathcal{D}'_\mu$, where $p \in \{0, 1, 2, ..., \infty\}$, $\mathcal{D}'^p$ is the space of distributions of order $\leq p$ in the sense of Schwartz, ($\mathcal{D}'^\infty$ means $\mathcal{D}'$), $\mathcal{D}'_\mu$ is the space of complex-valued distributions defined on $\mathbb{R}$, the support of which has Lebesgue measure zero, and $T\beta$ is the usual Schwartz product of a $\mathcal{D}'^p$-distribution by a C^p-function[1]. We stress that the convolution $T * \alpha$ is not to be understood as an approximation of T. Formula (2) is to be considered as an exact one.

This α-product is bilinear, has unit element (the constant function taking the value 1 seen as a distribution), is invariant for translations and also for the action of the transformation $t \longmapsto -t$ from $\mathbb{R}$ onto $\mathbb{R}$. In general it is not associative[2] nor commutative but we have

$$\int_{\mathbb{R}} T \dot{\alpha} S = \int_{\mathbb{R}} S \dot{\alpha} T,$$

[1] In [11] we also have defined α-products of distributions $T \in \mathcal{D}'^{-1}$ by distributions $S \in L^1_{loc} \oplus \mathcal{D}'_\mu$ (where $\mathcal{D}'^{-1}$ is the space of functions that, locally, are of bounded variation, seen as distributions) which are consistent with products (2). In the present paper we don't need this framework.

[2] Recall that the usual Schwartz product of distributions by C^∞-functions is not associative.

for all α, if $T, S \in \mathcal{D}'_\mu$ and one of them has compact support[3].

In general, the α-products cannot be completely localized. This will be clear noting that $\mathrm{supp}(T\dot{_\alpha}S) \subset \mathrm{supp}\, S$ as for usual functions, but it may happen that $\mathrm{supp}(T\dot{_\alpha}S) \not\subset \mathrm{supp}\, T$. Thus, in the following, the α-products are considered as global products and when we apply them to differential equations the solutions are naturally viewed as global solutions.

The product (2) is consistent with the Schwartz products of $\mathcal{D}'^p$-distributions by C^p-functions (if these ones are placed on the right-hand side) and satisfies the usual differential rules. Leibniz formula must be written in the form

$$D(T\dot{_\alpha}S) = (DT)\dot{_\alpha}S + T\dot{_\alpha}(DS),$$

where D is the derivative operator in distributional sense. For instance, from (2) we have, $\delta\dot{_\alpha}\delta = \alpha(0)\delta$, $\delta\dot{_\alpha}(D\delta) = \alpha(0)D\delta$, $(D\delta)\dot{_\alpha}\delta = 0$ and $H\dot{_\alpha}\delta = \frac{1}{2}\delta$, for each α; here H stands for the Heaviside function.

We can use (2) to define powers of certain distributions. Thus, if $T = \beta + f \in C^p \oplus (\mathcal{D}'^p \cap \mathcal{D}'_\mu)$, we can compute

$$T\dot{_\alpha}T = \beta^2 + [\beta + (\beta * \alpha) + (f * \alpha)]f,$$

because $T \in \mathcal{D}'^p \cap (C^p \oplus \mathcal{D}'_\mu)$. Since $T\dot{_\alpha}T \in C^p \oplus (\mathcal{D}'^p \cap \mathcal{D}'_\mu)$, we can define α-powers T^n_α ($n \geq 0$ is an integer), by the recurrence formula

$$T^0_\alpha = 1, \tag{3}$$

$$T^n_\alpha = (T^{n-1}_\alpha)\dot{_\alpha}T. \tag{4}$$

Since distributional products (2) are consistent with the Schwartz products of distributions by functions (when these ones are placed at the right-hand side), we have $\beta^n_\alpha = \beta^n$ for all $\beta \in C^p$ and the consistence of this definition with the usual powers of C^p-functions is proved. For instance, if $m \in \mathbb{C}$, we have $(m\delta)^0_\alpha = 1$, $(m\delta)^1_\alpha = m\delta$ and for $n \geq 2$ we have $(m\delta)^n_\alpha = m^n[\alpha(0)]^{n-1}\delta$ as we can easily see by induction.

We also have $(\tau_a T)^n_\alpha = \tau_a (T)^n_\alpha$, where τ_a is the translation operator determined by $a \in \mathbb{R}$, in distributional sense. So, in what follows, and to simplify the notation, we will write T^n instead of T^n_α, assuming that α is fixed.

3 The composition of an entire function with a distribution

Let $\phi : \mathbb{C} \to \mathbb{C}$ be an entire function. Then we have

$$\phi(u) = a_0 + a_1 u + a_2 u^2 + \cdots \tag{5}$$

[3] As usual, $\int_{\mathbb{R}} U = < U, 1 >$ for distributions U with compact support.

for the sequence $a_n = \frac{\phi^{(n)}(0)}{n!}$ of complex numbers and all $u \in \mathbb{C}$. If $T \in C^p \oplus (\mathcal{D}'^p \cap \mathcal{D}'_\mu)$, we define the composition $\phi \circ T$ by the formula

$$\phi \circ T = a_0 + a_1 T + a_2 T^2 + \cdots \tag{6}$$

if this series converges in $\mathcal{D}'$.

This definition is consistent, because if $T = \beta \in C^p$ then $\phi \circ T$ is the usual composition $\phi \circ \beta$. Remember that, in general, $\phi \circ T$ depends on α. As an example, we will show that for $\delta_a = \tau_a \delta$, then $\phi \circ (m\delta_a)$ is a distribution for all $m \in \mathbb{C}$. We have $(m\delta_a)^0 = 1$, $(m\delta_a)^1 = m\delta_a$, and for $n \geq 2$,

$$(m\delta_a)^n = m^n [\alpha(0)]^{n-1} \delta_a. \tag{7}$$

Then, according to (6) we have

$$\phi \circ (m\delta_a) = a_0 + a_1(m\delta_a) + a_2(m\delta_a)^2 + \cdots$$

because, as we will see, this series is convergent in $\mathcal{D}'$. Indeed, we have by (7),

$$\phi \circ (m\delta_a) = a_0 + a_1 m\delta_a + a_2 m^2 \alpha(0)\delta_a + a_3 m^3 [\alpha(0)]^2 \delta_a + \cdots$$

so that, if $\alpha(0) = 0$ then $\phi \circ (m\delta_a) = a_0 + a_1 m\delta_a$, while if $\alpha(0) \neq 0$ we have

$$\alpha(0)[\phi \circ (m\delta_a) - a_0] = a_1 m\alpha(0)\delta_a + a_2 m^2 [\alpha(0)]^2 \delta_a + a_3 m^3 [\alpha(0)]^3 \delta_a + \cdots$$

which is equivalent to

$$\alpha(0)[\phi \circ (m\delta_a) - a_0] = \{a_1 m\alpha(0) + a_2 m^2 [\alpha(0)]^2 + a_3 m^3 [\alpha(0)]^3 + \cdots\}\delta_a,$$

because, by (5), the series $\{...\}$ converges to $\phi[m\alpha(0)] - a_0$. Then

$$\alpha(0)[\phi \circ (m\delta_a) - a_0] = \{\phi[m\alpha(0)] - a_0\}\delta_a$$

and we can write, since $a_0 = \phi(0)$ and $a_1 = \phi'(0)$

$$\phi \circ (m\delta_a) = \begin{cases} \phi(0) + \phi'(0) m\delta_a & \text{if } \alpha(0) = 0 \\ \phi(0) + \frac{\phi[m\alpha(0)] - \phi(0)}{\alpha(0)}\delta_a & \text{if } \alpha(0) \neq 0 \end{cases} . \tag{8}$$

We will use this result in section 6.

As for another example, take $\phi(u) = e^u$. We have from (8),

$$e^{m\delta_a} = 1 + A_{\alpha,m}\delta_a,$$

where the numbers $A_{\alpha,m}$ are defined by

$$A_{\alpha,m} = \begin{cases} m & \text{if } \alpha(0) = 0 \\ \frac{e^{m\alpha(0)} - 1}{\alpha(0)} & \text{if } \alpha(0) \neq 0 \end{cases} .$$

Our approach here seems more natural than Colombeau's. In fact, $e^{m\delta_a}$ is an element of the Colombeau algebra G but it has no associated distribution. Also it is not true that

$$e^{m\delta_a} = \sum_{n=0}^{\infty} \frac{(m\delta_a)^n}{n!},$$

in the sense of Colombeau (as required by physicists), because this series does not converges in G.

It is interesting to observe moreover that the following formal property of the exponential function

$$e^{m_1\delta_a + m_2\delta_a} = (e^{m_1\delta_a})_{\dot{\alpha}}(e^{m_2\delta_a})$$

is still conserved in our setting. This follows easily from the relations between the numbers $A_{\alpha,m}$:

$$A_{\alpha,m_1+m_2} = A_{\alpha,m_1} + A_{\alpha,m_2} + A_{\alpha,m_1}A_{\alpha,m_2}\alpha(0) \qquad !$$

In next section we will see that $\phi \circ (m_a\delta_a + m_b\delta_b)$ also has a meaning in $\mathcal{D}'$.

4 The distribution $\phi \circ (m_a\delta_a + m_b\delta_b)$

First let us establish two lemmas.

Lemma 1 *If $n \geq 1$ is an integer and $a \in \mathbb{R}$, $b \in \mathbb{R}$, $m_a \in \mathbb{C}$, $m_b \in \mathbb{C}$, then*

$$(m_a\delta_a + m_b\delta_b)^n = A_{n-1}^{a,b} m_a\delta_a + A_{n-1}^{b,a} m_b\delta_b \tag{9}$$

where each sequence $n \longmapsto A_n^{a,b}$, $n \geq 0$, is defined by the recurrence formula

$$\begin{cases} A_0^{a,b} = 1 \\ A_n^{a,b} = A_{n-1}^{a,b} m_a\alpha(0) + A_{n-1}^{b,a} m_b\alpha(a-b) \end{cases}.$$

Proof. By induction. We have for $n = 1$, $(m_a\delta_a + m_b\delta_b)^1 = A_0^{a,b} m_a\delta_a + A_0^{b,a} m_b\delta_b$, which is true. Also, if by assumption we have (9), then

$$(m_a\delta_a + m_b\delta_b)^{n+1} = [(m_a\delta_a + m_b\delta_b)^n]_{\dot{\alpha}}(m_a\delta_a + m_b\delta_b) =$$

$$(A_{n-1}^{a,b} m_a\delta_a + A_{n-1}^{b,a} m_b\delta_b)_{\dot{\alpha}}(m_a\delta_a + m_b\delta_b) =$$

$$A_{n-1}^{a,b} m_a^2\alpha(0)\delta_a + A_{n-1}^{a,b} m_a m_b\alpha(a-b)\delta_b + A_{n-1}^{b,a} m_b m_a\alpha(a-b)\delta_a + A_{n-1}^{b,a} m_b^2\alpha(0)\delta_b =$$

$$[A_{n-1}^{a,b} m_a\alpha(0) + A_{n-1}^{b,a} m_b\alpha(a-b)]m_a\delta_a + [A_{n-1}^{a,b} m_a\alpha(a-b) + A_{n-1}^{b,a} m_b\alpha(0)]m_b\delta_b =$$

$$A_n^{a,b} m_a\delta_a + A_n^{b,a} m_b\delta_b.$$

■

Lemma 2 *Let $A_n^{a,b}$ be the sequence defined in the previous lemma. Then we have the following estimates*

$$\left|A_{n-1}^{a,b}\right| \leq k^{n-1},$$

for all $n \geq 2$, where $k = (|m_a| + |m_b|) \max_{x\in\mathbb{R}} |\alpha(x)|$.

Proof. By induction. We have, for $n = 2$,

$$\left|A_1^{a,b}\right| = |m_a\alpha(0) + m_b\alpha(a-b)| \leq |m_a|\,|\alpha(0)| + |m_b|\,|\alpha(a-b)| \leq$$

$$\leq (|m_a| + |m_b|) \max_{x\in\mathbb{R}} |\alpha(x)| = k.$$

Also, if by assumption we have $\left|A_{n-1}^{a,b}\right| \leq k^{n-1}$, then

$$\left|A_n^{a,b}\right| = \left|A_{n-1}^{a,b} m_a\alpha(0) + A_{n-1}^{b,a} m_b\alpha(a-b)\right| \leq |A_{n-1}^{a,b}||m_a||\alpha(0)|+$$

$$+|A_{n-1}^{b,a}||m_b||\alpha(a-b)| \leq k^{n-1}\,|m_a|\,|\alpha(0)| + k^{n-1}\,|m_b|\,|\alpha(a-b)| \leq$$

$$\leq (k^{n-1}\,|m_a| + k^{n-1}\,|m_b|) \max_{x\in\mathbb{R}} |\alpha(x)| = k^{n-1}(|m_a| + |m_b|) \max_{x\in\mathbb{R}} |\alpha(x)| = k^n.$$

■

Now, let $\phi : \mathbb{C} \to \mathbb{C}$ be an entire function. We have $\phi(u) = a_0 + \sum_{n=1}^{\infty} a_n u^n$ and according to (6) the distribution $\phi \circ (m_a\delta_a + m_b\delta_b)$ ought to be defined by the formula

$$\phi \circ (m_a\delta_a + m_b\delta_b) = a_0 + \sum_{n=1}^{\infty} a_n(m_a\delta_a + m_b\delta_b)^n =$$

$$= a_0 + \sum_{n=1}^{\infty} a_n(A_{n-1}^{a,b} m_a\delta_a + A_{n-1}^{b,a} m_b\delta_b).$$

This makes sense because, as we will see, this series is always convergent in $\mathcal{D}'$. Indeed, by the previous lemma we have

$$\left|a_n A_{n-1}^{a,b}\right| \leq |a_n|\,k^{n-1} \text{ for } n \geq 2, \sum_{n=2}^{\infty} |a_n|\,k^{n-1} = \sum_{n=2}^{\infty} \frac{|a_n|}{k} k^n,$$

and $\sum_{n=2}^{\infty} \frac{|a_n|}{k} k^n$ is a convergent series. Thus, $\sum_{n=1}^{\infty} a_n A_{n-1}^{a,b}$ is convergent and the same happens to $\sum_{n=1}^{\infty} a_n A_{n-1}^{b,a}$. As a consequence, we have

$$\phi \circ (m_a\delta_a + m_b\delta_b) = a_0 + \left(\sum_{n=1}^{\infty} a_n A_{n-1}^{a,b}\right) m_a\delta_a + \left(\sum_{n=1}^{\infty} a_n A_{n-1}^{b,a}\right) m_b\delta_b,$$

and since $a_n = \frac{\phi^{(n)}(0)}{n!}$ for all $n \geq 0$ we can write

$$\phi \circ (m_a\delta_a + m_b\delta_b) = \phi(0) + \left(\sum_{n=1}^{\infty} \frac{\phi^{(n)}(0)}{n!} A_{n-1}^{a,b}\right) m_a\delta_a + \left(\sum_{n=1}^{\infty} \frac{\phi^{(n)}(0)}{n!}{}_n A_{n-1}^{b,a}\right) m_b\delta_b, \tag{10}$$

where the sequences $A_{n-1}^{a,b}$ are defined in lemma 1. We will use this result in section 7.

5 Classical solutions and α-solutions

Let us consider equation (1). By a classical solution of (1) we mean a continuously differentiable complex function $(x,t) \mapsto u(x,t)$ which satisfies (1) at every point of its domain. Let I be an interval of $\mathbb{R}$ with non-empty interior and $\mathcal{F}(I)$ the space of continuously differentiable maps $\tilde{u} : I \to \mathcal{D}'$ in the sense of the topology of $\mathcal{D}'$. For $t \in I$ the notation $[\tilde{u}(t)](x)$ is sometimes used to emphasize that the distribution $\tilde{u}(t)$ acts on functions $\xi \in \mathcal{D}$ which depend on x.

Definition 3 *The map $\tilde{u} \in \mathcal{F}(I)$ is said to be an α-solution of (1) if and only if there exists α such that for all $t \in I$,*

(a) $\phi \circ \tilde{u}(t)$ is well defined,

$$\text{(b) } \frac{d\tilde{u}}{dt}(t) + D[\phi \circ \tilde{u}(t)] = 0. \tag{11}$$

This definition sees equation (1) as an evolution equation and we have the following results:

Theorem 4 *If u is a classical solution of (1) on $\mathbb{R}\times I$ then, for any α, the map $\tilde{u} \in \mathcal{F}(I)$ defined by $[\tilde{u}(t)](x) = u(x,t)$ is an α-solution of (1).*

Theorem 5 *If $u : \mathbb{R}\times I \to \mathbb{C}$ is a C^1-function and $\tilde{u} \in \mathcal{F}(I)$ defined by $[\tilde{u}(t)](x) = u(x,t)$ is an α-solution of (1), then u is a classical solution of (1).*

For the proof, it is enough to observe that a C^1-function $u(x,t)$ can be read as a continuously differentiable function $\tilde{u} \in \mathcal{F}(I)$ defined by $[\tilde{u}(t)](x) = u(x,t)$ and use the consistency of the α-products with the classical products.

Let $\Sigma(I)$ be the space of functions $u : \mathbb{R}\times I \to \mathbb{C}$ such that $\tilde{u} : I \to \mathcal{D}'$ defined by $[\tilde{u}(t)](x) = u(x,t)$ is in $\mathcal{F}(I)$, and for each $t \in I$, $u(x,t) \in L^1_{loc}(\mathbb{R})$. The natural injection $u \mapsto \tilde{u}$ of $\Sigma(I)$ into $\mathcal{F}(I)$ allows us to identify any function in $\Sigma(I)$ with a certain map in $\mathcal{F}(I)$. Since $C^1(\mathbb{R}\times I) \subset \Sigma(I)$ we can write the inclusions

$$C^1(\mathbb{R}\times I) \subset \Sigma(I) \subset \mathcal{F}(I).$$

Thus, we have a consistent extension of the concept of classical solution for equation (1).

6 The propagation of a Dirac-delta wave

Here we will consider the propagation, according to (1), of a travelling wave with the shape of a Dirac-delta.

Theorem 6 *Let $\tilde{u} : I \to \mathcal{D}'$ be defined by*

$$\tilde{u}(t) = m_1 \tau_{\gamma_1(t)} \delta,$$

where $m_1 \in \mathbb{C}\backslash\{0\}$, and $\gamma_1 : I \to \mathbb{R}$ is a C^1-function. Then, for any α and all $t \in I$, $\phi \circ \tilde{u}(t) \in \mathcal{D}'$ and the wave $\tilde{u}$, α-propagates according to (1), if and only if,

$$\gamma'(t) = \phi'(0), \tag{12}$$

in the case $\alpha(0) = 0$, or

$$\gamma'(t) = \frac{\phi[m_1\alpha(0)] - \phi(0)}{m_1\alpha(0)}, \tag{13}$$

in the case $\alpha(0) \neq 0$.

Proof. For all $t \in I$, $\phi[\tilde{u}(t)] \in \mathcal{D}'$ is a direct consequence of (8). Now, let us suppose that $\tilde{u}$, α-propagates according to (1). Then, for all $t \in I$ we have

$$\frac{d\tilde{u}}{dt}(t) = m_1(\tau_{\gamma_1(t)} D\delta)[-\gamma_1'(t)],$$

and by (8),

$$D[\phi \circ \tilde{u}(t)] = \begin{cases} \phi'(0) m_1(\tau_{\gamma_1(t)} D\delta) & \text{if } \ \alpha(0) = 0 \\ \frac{\phi[m_1\alpha(0)] - \phi(0)}{\alpha(0)}(\tau_{\gamma_1(t)} D\delta) & \text{if } \ \alpha(0) \neq 0 \end{cases}.$$

Suppose $\alpha(0)=0$. Then by (11) we have

$$m_1(\tau_{\gamma_1(t)}D\delta)[-\gamma_1'(t)]+\phi'(0)m_1(\tau_{\gamma_1(t)}D\delta)=0, \tag{14}$$

and (12) follows immediately. Suppose $\alpha(0)\neq 0$. Then by (11) we have

$$m_1(\tau_{\gamma_1(t)}D\delta)[-\gamma_1'(t)]+\frac{\phi[m_1\alpha(0)]-\phi(0)}{\alpha(0)}(\tau_{\gamma_1(t)}D\delta)=0, \tag{15}$$

and (13) follows. Conversely, let us suppose that we have (12). Then we have (14) which means that (11) is satisfied. If we suppose that we have (13), then we have (15), which means that (11) is also satisfied. Thus, the wave $\tilde{u}$, α-propagates according to (1). ■

From this theorem we conclude that in a model ruled by the conservation law (1), a delta wave propagates always with constant speed. As a particular case, for $\phi(u)=cu$, where $c\in\mathbb{R}$ is a constant, we obtain (from (12) or (13) and for any α) the speed $\gamma'(t)=c$, which is the well-known speed of the travelling waves in models ruled by the linear transport equation $u_t+cu_x=0$.

For another particular case, consider $\phi(u)=\frac{u^2}{2}$; we have $\gamma'(t)=\frac{m_1\alpha(0)}{2}$, and this shows the necessity of the indeterminacy which underlies the distributional product, in a simple physical context. In fact, if this speed was perfectly determined, then the speed of a mass-point in a model ruled by the conservative[4] Burgers equation $u_t+(\frac{u^2}{2})_x=0$ would be perfectly determined, which is physically unacceptable. Recall that we can interpret the state variable $u(x,t)$ of the referred equation as the density of matter at x at time t. Then the travelling wave $u(x,t)=m\delta(x-\gamma(t))$ corresponds to a distribution of mass on the real line reduced to a point of mass m located at $x=\gamma(t)$ and the absence of source term means that there are no forces acting on the system; so, by Newton's first law, the speed can take any constant value but not a preassigned determined value.

7 The interaction of two Dirac-delta waves

This section is dedicated to the propagation of the superposition of two Dirac-delta waves in a model ruled by equation (1).

Theorem 7 *Let $\tilde{u}:I\to\mathcal{D}'$ be defined by*

$$\tilde{u}(t)=m_1\tau_{\gamma_1(t)}\delta+m_2\tau_{\gamma_2(t)}\delta,$$

[4]Remember that the conservative Burgers equation $u_t+\left(\frac{u^2}{2}\right)_x=0$ and the nonconservative Burgers equation $u_t+uu_x=0$ have the same classical solutions, but not the same distributional solutions. See [11,14] for details.

where $m_1, m_2 \in \mathbb{C}\backslash\{0\}$ *and* $\gamma_1, \gamma_2 : I \to \mathbb{R}$ *are* C^1*-functions. Then the wave* $\tilde{u}$*,* α*-propagates according to (1) if and only if*

$$\gamma_1'(t) = \sum_{n=1}^{\infty} \frac{\phi^{(n)}(0)}{n!} A_{n-1}^{1,2}(t), \tag{16}$$

$$\gamma_2'(t) = \sum_{n=1}^{\infty} \frac{\phi^{(n)}(0)}{n!} A_{n-1}^{2,1}(t), \tag{17}$$

for all $t \in I$ *such that* $\gamma_1(t) \neq \gamma_2(t)$ *and*

$$m_1\gamma_1'(t) + m_2\gamma_2'(t) = m_1 \sum_{n=1}^{\infty} \frac{\phi^{(n)}(0)}{n!} A_{n-1}^{1,2}(t) + m_2 \sum_{n=1}^{\infty} \frac{\phi^{(n)}(0)}{n!} A_{n-1}^{2,1}(t), \tag{18}$$

for all $t \in I$ *such that* $\gamma_1(t) = \gamma_2(t)$*, where the sequence* $A_n^{1,2}(t) = A_n^{\gamma_1(t),\gamma_2(t)}$ *is defined with the help of Lemma 1, i.e.*

$$\begin{cases} A_0^{1,2}(t) = 1 \\ A_n^{1,2}(t) = A_{n-1}^{1,2}(t) m_1 \alpha(0) + A_{n-1}^{2,1}(t) m_2 \alpha[\gamma_1(t) - \gamma_2(t)] \end{cases},$$

and the sequence $A_n^{2,1}(t)$ *can be obtained from* $A_n^{1,2}(t)$ *by exchanging the indices 1 and 2.*

Proof. First of all let us compute $\frac{d\tilde{u}}{dt}(t)$ and $D\phi[\tilde{u}(t)]$:

$$\frac{d\tilde{u}}{dt}(t) = m_1(\tau_{\gamma_1(t)} D\delta)[-\gamma_1'(t)] + m_2(\tau_{\gamma_2(t)} D\delta)[-\gamma_2'(t)],$$

and by (10),

$$D[\phi \circ (m_1\tau_{\gamma_1(t)}\delta + m_2\tau_{\gamma_2(t)}\delta)] = \left(\sum_{n=1}^{\infty} \frac{\phi^{(n)}(0)}{n!} A_{n-1}^{1,2}(t) \right) m_1\tau_{\gamma_1(t)} D\delta +$$

$$+ \left(\sum_{n=1}^{\infty} \frac{\phi^{(n)}(0)}{n!} A_{n-1}^{2,1}(t) \right) m_2\tau_{\gamma_2(t)} D\delta.$$

Now, let us suppose that $\tilde{u}$, α-propagates according to (1). Then, by (11) we have

$$\left[m_1(-\gamma_1'(t)) + m_1 \sum_{n=1}^{\infty} \frac{\phi^{(n)}(0)}{n!} A_{n-1}^{1,2}(t) \right] \tau_{\gamma_1(t)} D\delta +$$

$$+\left[m_2(-\gamma_2'(t)) + m_2 \sum_{n=1}^{\infty} \frac{\phi^{(n)}(0)}{n!} A_{n-1}^{2,1}(t)\right] \tau_{\gamma_2(t)} D\delta = 0. \qquad (19)$$

If $\gamma_1(t) \neq \gamma_2(t)$ we have (16) and (17); if $\gamma_1(t) = \gamma_2(t)$ we have (18). Conversely, if we have (16), (17) when $\gamma_1(t) \neq \gamma_2(t)$, we have also (18), so that the equation (19) is satisfied for all $t \in I$, which means that equation (11) is also satisfied for all $t \in I$. ■

As a particular case, if $\phi(u) = cu$, we obtain (from (16) and (17) and for any α) the speeds

$$\gamma_1'(t) = \phi'(0) A_0^{1,2}(t) = c,$$

$$\gamma_2'(t) = \phi'(0) A_0^{2,1}(t) = c,$$

in the case $\gamma_1(t) \neq \gamma_2(t)$, and from (18), the speed $\gamma_1'(t) = \gamma_2'(t) = c$, in the case $\gamma_1(t) = \gamma_2(t)$, which is also in agreement with the speed c of travelling waves in models ruled by the transport equation $u_t + cu_x = 0$.

8 The impossibility of collision of two Dirac-delta waves

Now, we will see that in a physical setting, it is impossible to obtain a collision of delta waves when the model is ruled by the conservation law $u_t + [\phi(u)]_x = 0$. This impossibility is a direct consequence of theorem 7 and of the following result

Theorem 8 *Let $\xi_1, \xi_2, m_1, m_2 \in \mathbb{R}$ be such that $\xi_1 < \xi_2$, $m_1 \neq 0$ and $m_2 \neq 0$. Let $\gamma_1, \gamma_2 : [0, +\infty[\to \mathbb{R}$ be the solution of the initial value problem:*

$$\gamma_1'(t) = \sum_{n=1}^{\infty} \frac{\phi^{(n)}(0)}{n!} A_{n-1}^{1,2}(t), \qquad (20)$$

$$\gamma_2'(t) = \sum_{n=1}^{\infty} \frac{\phi^{(n)}(0)}{n!} A_{n-1}^{2,1}(t), \qquad (21)$$

$$\gamma_1(0) = \xi_1, \qquad (22)$$

$$\gamma_2(0) = \xi_2. \qquad (23)$$

Then, $\gamma_1(t) - \gamma_2(t) \neq 0$ for all $t \in [0, +\infty[$.

Proof. It is clear that this initial value problem has an unique global solution (γ_1, γ_2) defined on $[0, +\infty[$. Putting $y(t) = \gamma_1(t) - \gamma_2(t)$ we have, by subtraction of (20) and (21), and also (22) and (23),

$$y'(t) = \sum_{n=1}^{\infty} \frac{\phi^{(n)}(0)}{n!}[A^{1,2}_{n-1}(t) - A^{2,1}_{n-1}(t)], \tag{24}$$

$$y(0) = \xi_1 - \xi_2. \tag{25}$$

Here, in the expression $A^{1,2}_{n-1}(t) - A^{2,1}_{n-1}(t)$, $y(t)$ instead of $\gamma_1(t) - \gamma_2(t)$ must be introduced. Now, suppose by contradiction that there exists $t^* \in [0, +\infty[$ such that

$$y(t^*) = 0. \tag{26}$$

Then, since problem (24),(25) has an unique solution $y(t)$, this problem is equivalent to problem (24),(26). However, problem (24),(26) has as unique solution $y(t) = 0$ for all $t \in [0, +\infty[$ because otherwise, for $n \geq 2$, we would have for $y = 0$,

$$A^{1,2}_{n-1}(t) - A^{2,1}_{n-1}(t) = A^{1,2}_{n-2}(t)m_1\alpha(0) + A^{2,1}_{n-2}(t)m_2\alpha(0) -$$

$$-A^{1,2}_{n-2}(t)m_1\alpha(0) - A^{2,1}_{n-2}(t)m_2\alpha(0) = 0,$$

and for $n = 1$ we have $A^{1,2}_0(t) - A^{2,1}_0(t) = 1 - 1 = 0$, which is in contradiction with (25). Thus, for all $t \in [0, +\infty[$ it must be $y(t) = \gamma_1(t) - \gamma_2(t) \neq 0$ and the theorem is proved. ■

In a physical setting, it is also important to note that the speeds $\gamma_1'(t)$ and $\gamma_2'(t)$ are always bounded on $[0, +\infty[$ since, by applying Lemma 2, we have, for $n \geq 2$

$$\left|A^{1,2}_{n-1}(t)\right| \leq k^{n-1},$$

where $k = (|m_1| + |m_2|) \max_{x\in\mathbb{R}} |\alpha(x)|$. Thus, for all $t \in [0, +\infty[$, we have

$$|\gamma_1'(t)| \leq |\phi'(0)| + \sum_{n=2}^{\infty} \frac{\left|\phi^{(n)}(0)\right|}{n!} k^n,$$

and this series converges. The same hods for $|\gamma_2'(t)|$.

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