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Integral Type Contractive Condition for Converse Commuting Mappings in Uniform Space

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Abstract: In this paper we use the concept of converse p-commuting mappings to prove some common fixed point results satisfying an integral type contractive condition in uniform space. Our theorem generalized the theorem of Amri and Matakawaski [1] for quadruple of mappings satisfying an integral type inequality for converse p-commuting mappings.

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1. Introduction

In 1968 Kannan [7] proved a fixed point theorem for a map satisfying a contractive condition that did not require continuity at each point. Rhoades [14] find a comparison of various definition of contraction. Sessa [18] coined the notion of weakly commuting mappings. Jungck [5] generalized this idea to compatible mapping and then weakly compatible mapping in [3]. During this time a number of authors established fixed point theorems for pair of maps (see [17, 18] and reference therein).

Kada, Suzuki and Takahashi [6] have introduced the concept of a W-distance on metric spaces and have generalized some important results in non- convex minimizations and in fixed point theorems to uniform spaces. Following ideas in [6] Montes and Charris [9] established, some results on fixed and coincidence points of maps by means of appropriate W-contractive or W-expansive assumptions in uniform spaces..

A uniform space $(X, \Im)$ a nonempty set X endowed of an uniformity $\Im$, the latter being a special kind of filter on $X \times X$, all whose elements contain the diagonal $\Delta = \{(x, x)/ x$

$\in$ X}. If $V \in \Im$ and $(x, y) \in V$, $(y, x) \in V$, x and y are said to be V-close, and a sequence (x_n) in X is a Cauchy sequence for $\Im$ if for any $V \in \Im$, there exists $N \geq 1$ such that x_n and x_m are V-close for $n, m \geq N$. An uniformity $\Im$ defines a unique topology $T(\Im)$ on X for which the neighborhoods of $x \in X$ are the sets $V(x) = \{y \in X/(x, y) \in V\}$. Since each $V \in \Im$ contains a symmetrical $W \in \Im$ and if $(x, y) \in W$ then x and y are both W and V-close.

On the other hand Branciari [2] obtained a fixed point theorem for single valued mappings satisfying an analogue of Banach's contraction principle for an integral type inequality. Rhoades [15] proved two fixed point theorems involving more general contractive conditions. For more results on integral type inequality please see([4],[12], [13], [15], [16], [19]).

2. Preliminaries

Definition 2.1: Let $(X, \Im)$ be a uniform space. A function $p: X\times X\to R^+$ is said to be an A-distance if for any $V \in \Im$ there exists $\delta > 0$ such that if $p(z, x) \leq \delta$, and $p(z, y) \leq \delta$, for some $z \in X$, then $p(x, y) \in V$.

Definition 2.2: Let $(X, \Im)$ be a uniform space. A function $p: X\times X\to R^+$ is said to be an E-distance, if

(p_1) p is an A-distance,

(p_2) $p(x, y) \leq p(x, z) + p(z, y), \quad \forall x, y, z \in X$.

Example 2.1: Let $(X, \Im)$ be a uniform space and let d be a distance on X. Clearly $(X, \Im_d)$ is a uniform space where $\Im_d$ is the set of all subsets of $X \times X$ containing a "band" $B_\varepsilon = \{(x, y) \in X^2 / d(x, y) < \varepsilon\}$ for some $\varepsilon > 0$. Moreover, if $\Im \subseteq \Im_d$, then d is an E-distance on $(X, \Im)$.

Recently, Montes and Charris [9] introduced the concept of W-distance on uniform spaces. Every W-distance p is an E-distance, since it satisfies (p_1), (p_2) and the following condition:

For all $x \in X$, the function $p(x, .)$ is lower semi-continuous.

Example 2.2: Let $X = [0, \infty)$ and $d(x, y) = |x - y|$ the usual metric. Consider the function p defined as follows

$$p(x,y) = \begin{cases} y, & y \in [0, 1) \\ 2y, & y \in [1, \infty) \end{cases}$$

It is easy to see that the function p is an E-distance on $(X, \Im_d)$ but it is not an W-distance on $(X, \Im_d)$ since the function $p(x, .) : X \times X \to R^+$ is not lower semi-continuous at 1.

We have the following lemmas:

Lemma 2.1: Let $(X, \Im)$ be a Hausdorff uniform space and p be an A-distance on X. Let (x_n), (y_n) be arbitrary sequences in X and (α_n), (β_n) be sequences in R^+ converging to 0. Then, for x, y, z $\in$ X, the following holds:

- If $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for all $n \in N$, then y = z. In particular, if $p(x, y) = 0$ and $p(x, z) = 0$, then y = z.
- If $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for all $n \in N$, then (y_n) converges to z.
- If $p(x_n, x_m) \leq \alpha_n$ for all m> n, then (x_n) is a Cauchy sequence in $(X, \Im)$.

Definition 2.3: Let $(X, \Im)$ be a uniform space with an A-distance p. A sequence in X is p-Cauchy if it satisfies the usual metric condition. There are several concepts of completeness in this setting as follows:

(i) X is S-complete if for every p-Cauchy sequence (x_n), there exists x in X with $\lim_{n\to\infty} p(x_n, x) = 0$.

(ii) X is p-Cauchy complete if for every p-Cauchy sequence (x_n), there exists x in X with $\lim_{n\to\infty} x_n = x$ with respect to $T(\Im)$.

(iii) $f : X \to X$ is p-continuous if $\lim_{n\to\infty} p(x_n, x) = 0$ implies $\lim_{n\to\infty} p(f(x_n), f(x)) = 0$.

(iv) $f : X \to X$ is $T(\Im)$-continuous if $\lim_{n\to\infty} x_n = x$ with respect to $T(\Im)$ implies $\lim_{n\to\infty} f(x_n) = f(x)$ with respect to $T(\Im)$.

(v) X is said to be p-bounded if $\delta_p(X) = \sup\{p(x, y)/x, y \in X\} < \infty$.

Remark 2.1: Let $(X, \Im)$ be a Hausdorff uniform space and let (x_n) be a p-Cauchy sequence. Suppose that X is p-complete, then there exists $x \in X$ such that $\lim_{n\to\infty} p(x_n, x) = 0$. Lemma 2.1 then gives $\lim_{n\to\infty} x_n = x$ with respect to the topology $T(\Im)$. Therefore p-completeness implies p-Cauchy completeness.

Definition 2.4: Let $(X, \Im)$ be a Hausdorff uniform space and p be an A-distance on X. Two self mappings f and g of X are said to be point wise converse p-commuting if for all $x \in X$,

$$p(f(g(x)), g(f(x))) \leq p(f(x), g(x))$$

Amri and Matakawaski [1] prove the following theorem for weakly compatible maps:

Theorem 2.4[1]: Let $(X, \Im)$ be a Hausdorff uniform space and p be an A-distance on X such that X is p-bounded. Let f and g be two weakly compatible self mappings of X such that

(a) $f(X) \subseteq g(X)$

(b) $p(f(x), f(y)) \leq \psi\{p(g(x), g(y))\}$

for all $x, y \in X$, and $\psi : R^+ \to R^+$. If the range of f or g is a p-complete space of X, then f and g have a common fixed point.

Throughout this paper, let F* ={ϕ: $R_+ \to R_+$; ϕ is a Lebesgue integral mappings which is summable, non-negative and satisfies $\int_0^{\varepsilon} \phi(t)dt > 0$, for each $\varepsilon > 0$}. Let $C_p(A,S)$ denotes the set of converse p-commuting points of A and S.

Let us define a non-decreasing function $\Psi : R^+ \to R^+$ satisfying
(Ψ_1) For each $t \in (0, \infty)$, $\Psi(t) > 0$.
(Ψ_2) $\lim_{n\to\infty} \Psi^n(t) = 0$, $\forall\ t \in (0, \infty)$
It is easy to see that under the above properties, Ψ satisfies also $\Psi(t) < t$, for each $t > 0$.

3. Main Result

We generalize the above theorem of Amri and Matakawaski [1] for quadruple of mappings satisfying an integral type inequality for converse commuting mappings.

Theorem 3.1: Let (X, $\Im$) be a Housdroff uniform space and let p be an A-distance on X such that p-bounded. Let f, g, S and T be four self mappings such that
(1) the pair (f, S) and (g, T) are point wise conversely p-commuting
(2) the contractive condition

$$\int_0^{p(S(x),\,T(y))} \phi(t)dt < \int_0^{M(x,y)} \phi(t)dt,$$

holds for all x, y $\in$ X, where $\phi \in F^*$ and $\Psi : R^+ \to R^+$,

$$M(x, y) = \Psi[p(S(x), f(x)), p(f(x), g(y)), p(T(y), g(y)), \{p(T(y), f(x)) + p(S(x), g(y))\}/2]$$

then there exists a unique common fixed point of f, g, S and T.

Proof: Suppose the pair (f, S) have a p-commuting point z and the pair (g, T) have a commuting point w, then from condition (1)

$$z \in p(f, S) \text{ and } w \in p(g, T)$$

i.e. $p(f(S(z)), S(f(z))) \le p(f(z), S(z))$
i.e. $f(S(z)) = S(f(z)) \Rightarrow f(z) = S(z)$,
and similarly $g(T(w)) = T(g(w)) \Rightarrow g(w) = T(w)$.
Hence $f(S(z)) = S(f(z)) = f(f(z))$, $p(f(z), S(z)) = 0$,
and $g(T(w)) = T(g(w)) = g(g(w))$, $p(g(w), T(w)) = 0$
Let us now show that $f(f(z)) = g(T(w))$.
If it is not, so we take x = f(z) and y = g(w), in (2), we have

$$\int_0^{p(f(f(z),\,T(g(w)))} \phi(t)dt < \int_0^{M(f(z),\,g(w))} \phi(t)dt$$

where

$$\begin{aligned} M(f(z), g(w)) &= \Psi[p(S(f(z)), f(f(z))), p(f(f(z)), g(g(w))), p(T(g(w)), g(g(w))), \\ &\qquad \{p(T(g(w)), f(f(z))) + p(S(f(z)), g(g(w)))\}/2] \\ &= \Psi[p(f(f(z)), f(f(z))), p(f(f(z)), T(g(w))), p(T(g(w)), T(g(w))), \\ &\qquad \{p(T(g(w)), f(f(z))) + p(f(f(w)), T(g(w)))\}/2] \end{aligned}$$

$= \Psi[0, p(f(f(z)), T(g(w))), 0, p(f(f(z)), T(g(w)))]$

$\leq \Psi[p(f(f(z)), T(g(w)))] < p(f(f(z)), T(g(w)))$

i.e. $$\int_0^{p(f(f(z), T(g(w)))} \phi(t)dt < \int_0^{M(f(z), g(w))} \phi(t)dt = \int_0^{p(f(f(z)), T(g(w)))} \phi(t)dt$$

which is a contradiction, so that f(f(z)) = T(g(w)).

Next we claim that S(z) = T(w), then from (2), we have

$$p(S(z), T(w)) = \Psi[p(S(z), f(z)), p(f(z), g(w)), p(T(w), g(w)), \{p(T(w), f(z)) + p(S(z), g(w))\}/2]$$
$$= \Psi[p(S(z), S(z)), p(S(z), T(w)), p(T(w), T(w)), \{p(T(w), S(z)) + p(S(z), T(w))\}/2]$$
$$= \Psi[0, p(S(z), T(w)), 0, p(S(z), T(w))]$$
$$\leq \Psi[p(S(z), T(w))] < p(S(z), T(w))$$

i.e. $$\int_0^{p(S(z), T(w))} \phi(t)dt < \int_0^{p(S(z), T(w))} \phi(t)dt$$

which is a contradiction, so that S(z) = T(w). Therefore

$$f(z) = S(z) = g(w) = T(w).$$

Now to show that S(z) = S(S(z)), for which we take x = S(z) and y = w, in (2), we have

$$\int_0^{p(S(S(z)), T(w))} \phi(t)dt < \int_0^{M(S(z), w))} \phi(t)dt$$

where,

$$M(S(z)), w)) = \Psi[p(S(S(z)), f(S(z))), p(f(S(z), g(w))), p(T(w), g(w)), \{p(T(w), f(S(z)) + p(S(S(z), g(w))\}/2]$$
$$= \Psi[p(S(S(z), S(S(z)), p(S(S(z), S(z)))), p(S(z), S(z))), \{p(S(z), S(S(z))) + p(S(S(z), S(z))\}/2]$$
$$= \Psi[0, p(S(S(z), S(z)), 0, p(S(S(z), S(z))] < p(S(S(z), S(z)),$$

i.e. $$\int_0^{p(S(S(z)), S(z))} \phi(t)dt < \int_0^{p(S(S(z), S(z))} \phi(t)dt$$

which is a contradiction, so that S(S(z) = S(z), and so S(z) = S(S(z)) = f(S(z)). Thus S(z) is a common fixed point of S and f. Similarly we can prove that T(w) is a common fixed point of T and g. Since T(w) = S(z), hence S(z) is a common fixed point of f, g, S and T. Thus S(z) is a common fixed point of f, g, S and T. For uniqueness suppose that S(u) (≠ S(z)) is a common fixed point of f, g, S and T. Then from (2), we have

$$\int_0^{p(S(S(z)), T(S(u))} \phi(t)dt$$
$$< \int_0^{\Psi[p(S(S(z), f(S(z))), p(f(S(z), g(S(u)), p(T(S(u), g(S(u))), \{p(T(S(u), f(S(z)))+p(S(S(z), g(S(u)))\}/2]} \phi(t)dt$$
$$= \int_0^{\Psi[p(S(z), S(z)), p(S(z), S(u)), p(S(u), S(u)), \{p(S(u), S(z))) + p(S(z), S(u))\}/2]} \phi(t)dt$$
$$= \int_0^{\Psi[0, p(S(z), S(w)), 0, p(S(z), S(u)]} \phi(t)dt$$
$$\leq \int_0^{\Psi[p(S(z), S(u)]} \phi(t)dt < \int_0^{p(S(z), S(u))} \phi(t)dt$$

implies that p(S(S(z), T(S(u))) = 0, or p(S(z), S(u)) = 0 $\Rightarrow$ S(z) = S(u). Since an E-distance function p is an A-distance, then if f, g, S and T have a unique common fixed point S (z), then there exists z $\in$ X such that S(z) = z. Thus z is a unique common fixed point of f, g, S and T. □

Corollary 3.1: Let (X, $\Im$) be a Hausdroff uniform space and p be an A-distance on X such that X is p-bounded. Let S and T be two conversely p-commuting self mappings of X and

(1)o the pair (f, S) and (g, T) are point wise conversely p-commuting

(2)o $\int_0^{p(S(x), T(y)} \phi(t)dt < \int_0^{\Psi[p(S(x), x), p(x, y), p(T(y), y), \{p(T(y), x) + p(S(x), y)\}/2]} \phi(t)dt$

for all x, y $\in$ X, where $\phi \in F^*$. Then S and T have a unique common fixed point.

Proof: Put f = g = I in theorem 3.1, we get proof easily. Thus our theorem generalize theorem of Amri and Matakawaski [1] to integral type.
□

The following example shows that the above conditions do not hold good for metric space:

Example 3.1: X = [0, 1] and d (x, y) = $|x - y|$, the usual metric. Let f, g, S and T defined as

$$fx = \begin{cases} x, & 0 \le x < \frac{1}{2} \\ 1, & \frac{1}{2} \le x \le 1 \end{cases}, \qquad gx = \begin{cases} x, & 0 \le x < \frac{1}{2} \\ 1, & \frac{1}{2} \le x \le 1 \end{cases}$$

$$Sx = \begin{cases} x^2, & 0 \le x < \frac{1}{2} \\ 0, & \frac{1}{2} \le x \le 1 \end{cases}, \qquad Tx = \begin{cases} x^2, & 0 \le x < \frac{1}{2} \\ 0, & \frac{1}{2} \le x \le 1 \end{cases}$$

Consider the function p and ψ defined as follows

$$\psi(x) = \begin{cases} x^2, & 0 \le x < \frac{1}{2} \\ \frac{x}{2}, & \frac{1}{2} \le x \le 1 \end{cases} \quad \text{and} \quad p(x, y) = \begin{cases} y, & 0 \le y < \frac{1}{2} \\ 1, & \frac{1}{2} \le y \le 1 \end{cases}$$

$\phi(t) = 1$, for t > 0. Then f, g, S and T satisfies all the conditions of our *theorem* 3.1. Also, we have

$$\int_0^{d\left(S\left(\frac{1}{4}\right), T\left(\frac{1}{3}\right)\right)} \phi(t)dt = \frac{7}{144} >$$

$$\int_0^{\Psi\left[d\left(S\left(\frac{1}{4}\right), f\left(\frac{1}{4}\right)\right), d\left(f\left(\frac{1}{4}\right), g\left(\frac{1}{3}\right)\right), d\left(T\left(\frac{1}{3}\right), g\left(\frac{1}{3}\right)\right), \left\{d\left(T\left(\frac{1}{3}\right), f\left(\frac{1}{4}\right)\right) + d\left(S\left(\frac{1}{4}\right), g\left(\frac{1}{3}\right)\right)\right\}/2\right]} \phi(t)dt = \frac{9}{256}$$

which implies that $\int_0^{d(S(x), T(y))} \phi(t)dt < \int_0^{\Psi(N(x, y))} \phi(t)dt$, does not hold for all x, y $\in$ X, where

N(x, y) = [d(S (x), f(x)), d(f (x), g(y)), d(T (y), g(y)),{d(T(y), f(x)) + d(S (x), g(y))}/2]
However, we have

$$\int_0^{d(S(x), T(y))} \phi(t)dt < \int_0^{\Psi[M(x, y)]} \phi(t)dt$$ $\forall$ x, y $\in$ X and 0 is the unique common fixed point of f, g, S and T.

We prove the second theorem for E-distance as follows:

Theorem 3.2: Let (X, $\Im$) be a Hausdroff uniformly space and p be an E-distance on X such that X is p-bounded and S, f, g and T be four self maps in X such that
(1)• the pairs (S, f) and (T, g) be pointwise converse p-commuting
(2)• the contractive condition

$$\int_0^{p(S(x), T(y)} \phi(t)dt < \int_0^{\Psi[p(S(x), f(x)), p(f(x), g(y)), p(T(y), g(y)), \{p(T(y), f(x) + p (S(x), g(y))\}/2]]} \phi(t)dt$$

holds for all x, y $\in$ X, where $\phi \in$ F*. Then there exist a unique common fixed point of S, f, g and T.

Proof: Since an E-distance function p is an A-distance function, so that S, f, g and T have a unique common fixed point. Suppose z and w are two common fixed points of S, f, g and T such that

S(z) = f(z) = g(z) = T(z) = z and S(w) = f(w) = g(w) = T(w) = w.

If p(z, w) $\neq$ 0, then

$$\int_0^{p(z, w)} \phi(t)dt = \int_0^{p(S(z), T(w))} \phi(t)dt$$

$$< \int_0^{\Psi[p(S(z), f(z)), p(f(z), g(w)), p(T(w), g(w)),\{p(T(w), f(z)) + p(S (z), g(w))\}/2]} \phi(t)dt$$

$$= \int_0^{\Psi[p(z, z), p(z, w), p(w, w), \{p(w, z) + p(z, w)\}/2]} \phi(t)dt$$

$$= \int_0^{\Psi[0, p(z, w), 0 , p(w, z)]} \phi(t)dt < \int_0^{p(S(z), T(w))} \phi(t)dt,$$

which is a contradiction. Thus p(z, w) = 0, similarly we can show that p(w, z) = 0. Consequently by (p_2), we have

$$\int_0^{p(z, z)} \phi(t)dt \le \int_0^{p(z, w)} \phi(t)dt + \int_0^{p(w, z)} \phi(t)dt$$ and therefore p(z, z) = 0.

Now we have p(z, z) = 0 and p(z, w) = 0, which implies z = w. Hence the theorem.

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