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A Note on the Positive Nonoscillatory Solutions of the Difference Equation

$$x_{n+1} = \frac{\alpha}{\sum_{i=0}^{k-1} c_i x_{n-i}} + \left(\frac{x_{n-k}}{\sum_{i=0}^{k-1} c_i x_{n-i}}\right)^p$$

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Abstract

The aim of this note is to show that the following difference equation

$$x_{n+1} = \frac{\alpha}{\sum_{i=0}^{k-1} c_i x_{n-i}} + \left(\frac{x_{n-k}}{\sum_{i=0}^{k-1} c_i x_{n-i}}\right)^p$$

where $\alpha,\ p > 0,\ k \in \mathbb{N},\ c_i \geq 0,\ i = 0, \cdots, k-1,\ \sum_{i=0}^{k-1} c_i = 1$, has positive nonoscillatory solutions which converge to the positive equilibrium $\overline{x} = \dfrac{1+\sqrt{1+4\alpha}}{2}$. In the proof of the result we use a method developed by L. Berg and S. Stević [1-4], [7-15].

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1 Introduction

Recently, there has been a lot of interest in studying the global attractivity, the boundedness character and the periodic nature of nonlinear difference equations. For some recent results see, for example [5-15].

In [5] the authors have studied the behavior of all positive solutions of the difference equation

$$x_{n+1} = \frac{p}{x_n} + \frac{x_{n-2}}{x_n},$$

where p is a positive real parameter and the initial conditions x_{-2}, x_{-1}, x_0 are positive real numbers. For every the value of the positive parameter p, there exists a unique positive equilibrium $\overline{x}$ which satisfies the equation $\overline{x}^2 = \overline{x} + p$.

In this note we will investigate the behavior of the positive solution of the difference equation

$$x_{n+1} = \frac{\alpha}{\sum_{i=0}^{k-1} c_i x_{n-i}} + \left(\frac{x_{n-k}}{\sum_{i=0}^{k-1} c_i x_{n-i}}\right)^p \tag{1}$$

where $\alpha,\ p > 0,\ k \in \mathbb{N},\ c_i \geq 0,\ i = 0, ..., k-1,\ \sum_{i=0}^{k-1} c_i = 1$, and the initial conditions $x_{-k}, x_{-k+1}, \cdots, x_{-1}, x_0$ are arbitrary positive real numbers. Note that the positive equilibrium of Eq (1) also satisfies the equation $\overline{x}^2 = \overline{x} + \alpha$

We say that a solution (x_n) of equation (1) is bounded and persists if there exists positive constants P and Q such that

$$P \leq x_n \leq Q \quad \text{for} \quad n = -k, -k+1, ..., 0, 1, ...$$

A positive semicycle of a solution (x_n) consists of a "string" of terms $\{x_l, x_{l+1}, ..., x_m\}$, all greater than or equal to $\overline{x}$, with $l \geq -k$ and $m \leq +\infty$ and such that

$$\text{either} \ \ l = -k, \ \text{or} \ \ l > -k \ \text{and} \ \ x_{l-1} < \overline{x},$$

and

$$\text{either} \ \ m = \infty, \ \text{or} \ \ m < \infty \ \text{and} \ \ x_{m+1} < \overline{x}.$$

A negative semicycle of a solution (x_n) consists of a "string" of terms $\{x_l, x_{l+1}, ..., x_m\}$, all less than to $\overline{x}$, with $l \geq -k$ and $m \leq \infty$ and such that

$$\text{either} \ \ l = -k, \ \text{or} \ \ l > -k \ \text{and} \ \ x_{l-1} \geq \overline{x},$$

and

$$\text{either} \ \ m = \infty, \ \text{or} \ \ m < \infty \ \text{and} \ \ x_{m+1} \geq \overline{x}.$$

The first semicycle of a solutions starts with the term x_{-k} and is positive if $x_{-k} \geq \overline{x}$.

We now investigate oscillation of positive solutions of the difference equation (1). We shall prove the following theorem, which is similar to the result from paper [5].

Theorem 1.1. *Let $\{x_n\}_{n=-k}^{+\infty}$ be a positive solution of Eq(1) for which there exists $N \geq -k$ such that $x_N < \overline{x}$ and $x_{N+1} \geq \overline{x}$, or $x_N \geq \overline{x}$ and $x_{N+1} < \overline{x}$. Then the solution $\{x_n\}_{n=-k}^{+\infty}$ oscillates about the equilibrium $\overline{x}$ with every semicycle (except possibly the first) having at most k terms.*

Proof. Let $N \geq -k$ such that $x_N < \overline{x} \leq x_{N+1}$. The case where $x_{N+1} < \overline{x} \leq x_N$ is similar and will be omitted. Now suppose that the positive semicycle beginning with the term x_{N+1} has k terms. Then $x_N < \overline{x} \leq x_{N+i},\ i = 1, 2, ..., k$ and so

$$x_{N+k+1} = \frac{\alpha}{\sum_{i=0}^{k-1} c_i x_{N+k-i}} + \left(\frac{x_N}{\sum_{i=0}^{k-1} c_i x_{N+k-i}}\right)^p \leq \frac{\alpha}{\overline{x}} + \left(\frac{x_N}{\min\limits_{i=\overline{0,k-1}} \{x_{N+k-i}\}}\right)^p < \frac{\alpha}{\overline{x}} + 1 = \overline{x}.$$

The proof is complete. □

Before we investigate the local stability of the solution of Eq(1) we quote the following well known result (see [17]).

Lemma 1.1. *Assume that $\sum_{i=0}^{k} |p_i| < 1$. Then the zero equilibrium of the difference equation*

$$y_{n+1} + \sum_{i=0}^{k} p_i y_{n-i} = 0$$

is globally asymptotically stable.

In this section we study the local stability of the solutions of Eq(1).
Eq(1) has two equilibriums

$$\overline{x} = \frac{1 + \sqrt{1 + 4\alpha}}{2},\ \overline{x}_1 = \frac{1 - \sqrt{1 + 4\alpha}}{2}$$

We have the linearized equation for Eq. (1) about the positive equilibrium $\overline{x} = \frac{1+\sqrt{1+4\alpha}}{2}$ is

$$y_{n+1} = -\sum_{i=0}^{k-1} \left[\frac{\alpha c_i}{\overline{x}^2} + \frac{p c_i}{\overline{x}}\right] y_{n-i} + \frac{p}{\overline{x}} y_{n-k}$$

$$y_{n+1} + \sum_{i=0}^{k-1} \left[\frac{\alpha c_i}{\overline{x}^2} + \frac{p c_i}{\overline{x}}\right] y_{n-i} - \frac{p}{\overline{x}} y_{n-k} = 0 \tag{2}$$

The characteristic polynomial associated with Eq(2) is

$$t^{k+1} + \sum_{i=0}^{k-1} \left[\frac{\alpha c_i}{\overline{x}^2} + \frac{p c_i}{\overline{x}}\right] t^{k-i} - \frac{p}{\overline{x}} = 0 \tag{3}$$

Since

$$0<\sum_{i=0}^{k-1}\left[\frac{\alpha c_i}{\overline{x}^2}+\frac{pc_i}{\overline{x}}\right]+\frac{p}{\overline{x}}<1$$
$$\alpha+p\overline{x}+p\overline{x}<\overline{x}^2=\overline{x}+\alpha$$
$$2p\overline{x}<\overline{x}\Leftrightarrow p<\frac{1}{2}$$

by Lemma 1.2 we obtain that the equilibrium $\overline{x}$ is locally asymptotically stable with $0<p<\frac{1}{2}$. In the case $p>\frac{1}{2}$, $\overline{x}$ is unstable.
The linearized equation for Eq(1) about the positive equilibrium $\overline{x}_1=\frac{1-\sqrt{1+4\alpha}}{2}<0$ is

$$y_{n+1}=-\sum_{i=0}^{k-1}\left[\frac{\alpha c_i}{\overline{x}_1^{\,2}}+\frac{pc_i}{\overline{x}_1}\right]y_{n-i}+\frac{p}{\overline{x}_1}y_{n-k}$$
$$y_{n+1}+\sum_{i=0}^{k-1}\left[\frac{\alpha c_i}{\overline{x}_1^{\,2}}+\frac{pc_i}{\overline{x}_1}\right]y_{n-i}-\frac{p}{\overline{x}_1}y_{n-k}=0 \qquad (4)$$

The characteristic polynomial associated with Eq(4) is

$$t^{k+1}+\sum_{i=0}^{k-1}\left[\frac{\alpha c_i}{\overline{x}_1^{\,2}}+\frac{pc_i}{\overline{x}_1}\right]t^{k-i}-\frac{p}{\overline{x}_1}=0 \qquad (5)$$

It is easy to see that

$$\sum_{i=0}^{k-1}\left[\frac{\alpha c_i}{\overline{x}_1^{\,2}}+\frac{pc_i}{\overline{x}_1}\right]-\frac{p}{\overline{x}_1}>1$$

and consequently the equilibrium $\overline{x}_1$ is unstable.

2 On the positive nonoscillatory solutions of the difference equation (1)

Our aim in this note is to solve the following problem. Do there exists nonoscillatory solutions of Eq(1)? We will solve this problem by a method due to L. Berg and S. Stević, see, for example, [2], [7-15].

Note that the linearized equation for Eq(1) about the positive equilibrium $\overline{x}$ can be written in the following equivalent form:

$$\overline{x}^2 y_{n+1} + \sum_{i=0}^{k-1} c_i(\alpha + p\overline{x})y_{n-i} - p\overline{x}y_{n-k} = 0 \tag{6}$$

The characteristic polynomial associated with Eq(6) is

$$g(t) = \overline{x}^2 t^{k+1} + \sum_{i=0}^{k-1} c_i(\alpha + p\overline{x})t^{k-i} - p\overline{x} = 0 \tag{7}$$

Since $g(0) = -p\overline{x} < 0, g(1) = \overline{x}^2 + \sum_{i=0}^{k-1} c_i(\alpha + p\overline{x}) - p\overline{x} = \overline{x}^2 + \alpha > 0$ and

$$g'(t) = \overline{x}^2(k+1)t^k + \sum_{i=0}^{k-1} c_i(k-i)(\alpha + p\overline{x})t^{k-i-1} > 0$$

when $t \in (0, 1]$, it follows that for each $p > 0, \alpha > 0$, there is unique positive root t_0 of the polynomial belonging to the interval $(0, 1)$. As suggested by Stević in [7], this fact motivated us to believe that there are solutions of Eq(1) which have the following asymptotics

$$x_n = \overline{x} + at_0^n + o(t_0^n) \tag{8}$$

where $a \in \mathbb{R}$ and t_0 is the above mentioned root of the polynomial (7). Asymptotics for solutions of difference equations have been investigated by L. Berg and S. Stević, see, for example, [1-4], [7-15] and the reference therein. The problem is solved by constructing two appropriate sequences y_n and z_n with

$$y_n \le x_n \le z_n \tag{9}$$

for sufficiently large n. In [1], [2] some methods can be found for the construction of these bounds, see, also [3, 4].

From [5] and results in Berg's paper [3, 4] we expect that for $k \ge 2$ such solutions have the first three members in their asymptotics in the following form

$$\varphi_n = \overline{x} + at^n + bt^{2n} \tag{10}$$

The following result plays a crucial part in proving the main result. The proof of the result is similar to that of Theorem 1 in [16], we will give a proof for the benefit of the reader.

Theorem 2.1. *Let $f : I^{k+2} \to I$ be a continuous and nondecreasing function in each argument on the interval $I \subset \mathbb{R}$, and let (y_n) and (z_n) be sequences with $y_n < z_n$ for $n \geq n_0$ and such that*

$$y_{n-k} \leq f(n, y_{n-k+1}, ..., y_{n+1}),$$
$$f(n, z_{n-k+1}, ..., z_{n+1}) \leq z_{n-k}, \text{ for } n > n_0 + k - 1 \tag{11}$$

Then there is a solution of the following difference equation

$$x_{n-k} = f(n, x_{n-k+1}, ..., x_{n+1}) \tag{12}$$

with property

$$y_n \leq x_n \leq z_n \text{ for } n \geq n_0. \tag{13}$$

Proof. Let N be an arbitrary integer such that $N > n_0 + k - 1$. The solution (x_n) of (12) with given initial values $x_N, x_{N+1}, \dots, x_{N+k}$ satisfying condition (13) for $n \in \{N, N+1, \dots, N+k\}$ can be continued by (12) to all $n < N$. Inequalities (11) and the monotonic character of f imply that (13) holds for all $n \in \{n_0, \dots, N+k\}$. Let A_N be the set of all $(k+1)$-tuples $x_{n_0}, \dots, x_{n_0+k}$ such that there exist solution (x_n) of (12) with these initial values satisfying (13) for all $n \in \{n_0, \dots, N+k\}$. It is clear that A_N is a closed nonempty set for every $N > n_0 + k - 1$, and that $A_{N+1} \subset A_N$. It follows that the set $A = \cap_{N=n_0+k}^{\infty} A_N$ is a nonempty subset of $\mathbb{R}^{k+1}$ and that if $(x_{n_0}, \dots, x_{n_0+k}) \in A$, then the corresponding solution of (12) satisfy (11) for all $n \geq n_0$, as desired. □

3 The main result

In this section, we prove the main result in this note.

Theorem 3.1. *For each $\alpha, p > 0$ there is a nonoscillatory solution of Eq(1) converging to the positive equilibrium*

$$\overline{x} = \frac{1 + \sqrt{1 + 4\alpha}}{2},$$

with the asymptotic behavior (10).

Proof. First note that Eq(1) can be written in the following equivalent form:

$$x_{n-k} = \left(x_{n+1} - \frac{\alpha}{\sum_{i=0}^{k-1} c_i x_{n-i}}\right)^{\frac{1}{p}} \sum_{i=0}^{k-1} c_i x_{n-i}$$

since

$$x_{n+1}\Big(\sum_{i=0}^{k-1} c_i x_{n-i}\Big) = \alpha + x_{n-k}^{p}\Big(\sum_{i=0}^{k-1} c_i x_{n-i}\Big)^{1-p}$$

We have

$$x_{n+1}\Big(\sum_{i=0}^{k-1} c_i x_{n-i}\Big) > \alpha$$

$$x_{n-k} = \left[x_{n+1}(\sum_{i=0}^{k-1} c_i x_{n-i}) - \alpha\right]^{\frac{1}{p}}(\sum_{i=0}^{k-1} c_i x_{n-i})^{1-\frac{1}{p}}$$

and

$$\begin{aligned} &F(x_{n-k}, x_{n-k+1}, \ldots, x_n, x_{n+1}) = \\ &= \left[x_{n+1}(\sum_{i=0}^{k-1} c_i x_{n-i}) - \alpha\right]^{\frac{1}{p}}(\sum_{i=0}^{k-1} c_i x_{n-i})^{1-\frac{1}{p}} - x_{n-k} = 0 \end{aligned} \tag{14}$$

Let

$$\begin{aligned} f(u_{n+1}, u_n, \ldots, u_{n-k+2}, u_{n-k+1}) &= \left[u_{n+1}(\sum_{i=0}^{k-1} c_i u_{n-i}) - \alpha\right]^{\frac{1}{p}}(\sum_{i=0}^{k-1} c_i u_{n-i})^{1-\frac{1}{p}} \\ &= \left[u_{n+1}(\sum_{i=0}^{k-1} c_i u_{n-i})^{p} - \alpha(\sum_{i=0}^{k-1} c_i u_{n-i})^{p-1}\right]^{\frac{1}{p}} \end{aligned}$$

defines on the set

$$A = \{(u_{n+1}, u_n, \ldots, u_{n-k+2}, u_{n-k+1}) \in \mathbb{R}_{+}^{k+1} : u_{n+1}(\sum_{i=0}^{k-1} c_i u_{n-i}) > \alpha\}$$

We have

$$\frac{\partial f}{\partial u_{n+1}} = \frac{1}{p}\left[u_{n+1}(\textstyle\sum_{i=0}^{k-1} c_i u_{n-i})^{p} - \alpha(\sum_{i=0}^{k-1} c_i u_{n-i})^{p-1}\right]^{\frac{1}{p}-1} .(\textstyle\sum_{i=0}^{k-1} c_i u_{n-i})^{p} > 0$$

$$\begin{aligned} \frac{\partial f}{\partial u_{n-i}} &= \frac{1}{p}\left[u_{n+1}(\textstyle\sum_{i=0}^{k-1} c_i u_{n-i})^{p} - \alpha(\sum_{i=0}^{k-1} c_i u_{n-i})^{p-1}\right]^{\frac{1}{p}-1} \times \\ &\times \left[p c_i u_{n+1}(\textstyle\sum_{i=0}^{k-1} c_i u_{n-i})^{p-1} - \alpha(p-1)c_i(\sum_{i=0}^{k-1} c_i u_{n-i})^{p-2}\right] = \\ &= \frac{1}{p}\left[u_{n+1}(\textstyle\sum_{i=0}^{k-1} c_i u_{n-i})^{p} - \alpha(\sum_{i=0}^{k-1} c_i u_{n-i})^{p-1}\right]^{\frac{1}{p}-1} \times \\ &\times c_i(\textstyle\sum_{i=0}^{k-1} c_i u_{n-i})^{p-2}\left[p u_{n+1}(\sum_{i=0}^{k-1} c_i u_{n-i}) + \alpha(1-p)\right] > 0 \ \forall p \in (0,1], \end{aligned}$$

$i = 0, 1, ..., k-1$
On the other hand,

$$\frac{\partial f}{\partial u_{n-i}} = \frac{1}{p}\left[u_{n+1}(\sum_{i=0}^{k-1} c_i u_{n-i})^p - \alpha(\sum_{i=0}^{k-1} c_i u_{n-i})^{p-1}\right]^{\frac{1}{p}-1} \times$$
$$\times c_i(\sum_{i=0}^{k-1} c_i u_{n-i})^{p-2}\left\{p\left[u_{n+1}(\sum_{i=0}^{k-1} c_i u_{n-i}) - \alpha\right] + \alpha\right\} > 0$$

on the set A, also for $p > 1$.
Let $I = [\overline{x}, \infty)$. Since for $u_{n+1}, u_n, \dots, u_{n-k+2}, u_{n-k+1} \in [\overline{x}, \infty) \Rightarrow u_{n+1}(\sum_{i=0}^{k-1} c_i u_{n-i}) > \overline{x}^2 = \overline{x} + \alpha > \alpha$, we have that $[\overline{x}, \infty)^{k+1} \in A$, so that f increases in each argument on $[\overline{x}, \infty)$ and

$$\min_{(u_{n+1}, u_n, \dots, u_{n-k+2}, u_{n-k+1}) \in [\overline{x}, \infty)^{k+1}} f(u_{n+1}, u_n, \dots, u_{n-k+2}, u_{n-k+1}) = f(\overline{x}, \overline{x}, \dots, \overline{x}) = \overline{x}$$

that is,

$$f : I^{k+1} \to I$$

We expect that solutions of Eq(1) have the asymptotics approximation (10). Thus, we can calculate $F(\varphi_{n-k}, \varphi_{n-k+1}, \dots, \varphi_{n+1}, \varphi_{n+1})$. We have

$$F = \left[(\overline{x} + at^{n+1} + bt^{2n+2})\sum_{i=0}^{k-1} c_i(\overline{x} + at^{n-i} + bt^{2n-2i}) - \alpha\right]^{\frac{1}{p}} \times$$
$$\times \left[c_i(\overline{x} + at^{n-i} + bt^{2n-2i})\right]^{1-\frac{1}{p}} - (\overline{x} + at^{n-k} + bt^{2n-2k})$$
$$F = \left[(\overline{x} + at^{n+1} + bt^{2n+2})\left(\overline{x} + a\sum_{i=0}^{k-1} c_i t^{n-i} + b\sum_{i=0}^{k-1} t^{2n-2i}\right) - \alpha\right]^{\frac{1}{p}} \times$$
$$\times \left[\overline{x} + a\sum_{i=0}^{k-1} c_i t^{n-i} + b\sum_{i=0}^{k-1} t^{2n-2i}\right]^{1-\frac{1}{p}} - (\overline{x} + at^{n-k} + bt^{2n-2k})$$
$$F = \left[\overline{x}^2 + a\overline{x}t^{n+1} + b\overline{x}t^{2n+2} + a\overline{x}\sum_{i=0}^{k-1} c_i t^{n-i} + a^2\sum_{i=0}^{k-1} c_i t^{2n-i+1} + ab\sum_{i=0}^{k-1} c_i t^{3n-i+2} + \right.$$
$$\left. + b\overline{x}\sum_{i=0}^{k-1} c_i t^{2n-2i} + ab\sum_{i=0}^{k-1} c_i t^{3n-2i+1} + b^2\sum_{i=0}^{k-1} c_i t^{4n-2i+2} - \alpha\right]^{\frac{1}{p}} \times$$
$$\times \overline{x}^{\frac{p-1}{p}}\left[1 + \frac{a\sum_{i=0}^{k-1} c_i t^{n-i} + b\sum_{i=0}^{k-1} c_i t^{2n-2i}}{\overline{x}}\right]^{\frac{p-1}{p}} - (\overline{x} + at^{n-k} + bt^{2n-2k})$$

From $\overline{x}^2 = \overline{x} + \alpha$, we have

$$F = \overline{x}\Big[1 + \frac{1}{p\overline{x}}\Big(a\overline{x}t^{n+1} + b\overline{x}t^{2n+2} + a\overline{x}\sum_{i=0}^{k-1} c_i t^{n-i} + a^2 \sum_{i=0}^{k-1} c_i t^{2n-i+1} + ab\sum_{i=0}^{k-1} c_i t^{3n-i+2} +$$

$$+ b\overline{x}\sum_{i=0}^{k-1} c_i t^{2n-2i} + ab\sum_{i=0}^{k-1} c_i t^{3n-2i+1} + b^2 \sum_{i=0}^{k-1} c_i t^{4n-2i+2}\Big) +$$

$$+ \frac{1-p}{2p^2\overline{x}^2}\Big(a\overline{x}t^{n+1} + b\overline{x}t^{2n+2} + a\overline{x}\sum_{i=0}^{k-1} c_i t^{n-i} + a^2 \sum_{i=0}^{k-1} c_i t^{2n-i+1} + ab\sum_{i=0}^{k-1} c_i t^{3n-i+2} +$$

$$+ b\overline{x}\sum_{i=0}^{k-1} c_i t^{2n-2i} + ab\sum_{i=0}^{k-1} c_i t^{3n-2i+1} + b^2 \sum_{i=0}^{k-1} c_i t^{4n-2i+2}\Big)^2 + \cdots\Big] \times$$

$$\times \left[1 + \frac{p-1}{p\overline{x}}\left(a\sum_{i=0}^{k-1} c_i t^{n-i} + b\sum_{i=0}^{k-1} c_i t^{2n-2i}\right) + \frac{1-p}{2p^2\overline{x}^2}\left(a\sum_{i=0}^{k-1} c_i t^{n-i} + b\sum_{i=0}^{k-1} c_i t^{2n-2i}\right)^2 + \cdots\right] -$$

$$- (\overline{x} + at^{n-k} + bt^{2n-2k})$$

$$F = a\left[\frac{\overline{x}^2 t^{k+1} + \sum_{i=0}^{k-1} c_i(\alpha + p\overline{x})t^{k-i} - p\overline{x}}{p\overline{x}t^k}\right]t^n + \left\{b\left[\frac{\overline{x}t^{2k+2} + \sum_{i=0}^{k-1} c_i(\alpha + p\overline{x})t^{2k-2i} - p\overline{x}}{p\overline{x}t^{2k}}\right] +\right.$$

$$\left. + a^2\left[\frac{p + (1-p)\overline{x}}{p^2}\sum_{i=0}^{k-1} c_i t^{-i+1} + \frac{(1-p)\overline{x}^2 + 1 - p}{2p^2\overline{x}}\Big(\sum_{i=0}^{k-1} c_i t^{-i}\Big)^2 + \frac{(1-p)\overline{x}}{2p^2}t^2\right]\right\}t^{2n} + o(t^{2n})$$

We have

$$\frac{\overline{x}^2 t^{k+1} + \sum_{i=0}^{k-1} c_i(\alpha + p\overline{x})t^{k-i} - p\overline{x}}{p\overline{x}t^k} = \frac{g(t)}{p\overline{x}t^k},$$

where $g(t)$ is the characteristic polynomial (7). We know that there exists the unique root $t_0 \in (0,1)$ such that $g(t_0) = 0$. Let

$$\frac{g(t_0^2)}{p\overline{x}t_0^{2k}} = \frac{\overline{x}t_0^{2k+2} + \sum_{i=0}^{k-1} c_i(\alpha + p\overline{x})t_0^{2k-2i} - p\overline{x}}{p\overline{x}t_0^{2k}}.$$

From this, with $t = t_0$ we have

$$F = \left\{b\frac{g(t_0^2)}{p\overline{x}t_0^{2k}} + a^2\left[\frac{p + (1-p)\overline{x}}{p^2}\sum_{i=0}^{k-1} c_i t_0^{-i+1} + \frac{(1-p)\overline{x}^2 + 1 - p}{2p^2\overline{x}}\Big(\sum_{i=0}^{k-1} c_i t_0^{-i}\Big)^2 + \frac{(1-p)\overline{x}}{2p^2}t_0^2\right]\right\}t_0^{2n} +$$

$$+ o(t_0^{2n}),\ \ 0 < t_0^2 < t_0 < 1,\ \ g(t_0^2) < g(t_0) = 0$$

Thus, the coefficient of b is negative: $\dfrac{g(t_0^2)}{p\overline{x}t_0^{2k}} < 0$.

We set

$$A = \frac{p + (1-p)\overline{x}}{p^2}\sum_{i=0}^{k-1} c_i t_0^{-i+1} + \frac{(1-p)\overline{x}^2 + 1 - p}{2p^2\overline{x}}\Big(\sum_{i=0}^{k-1} c_i t_0^{-i}\Big)^2 + \frac{(1-p)\overline{x}}{2p^2}t_0^2$$

Then

$$F = \left[b\frac{g(t_0^2)}{p\overline{x}t_0^{2k}} + a^2A\right]t_0^{2n} + o(t_0^{2n}).$$

Set

$$q = -\frac{a^2.Ap\overline{x}t_0^{2k}}{g(t_0^2)} \quad \text{and} \quad H_{t_0}(b) = b\frac{g(t_0^2)}{p\overline{x}t_0^{2k}} + a^2A.$$

Note that

$$H'_{t_0}(b) = \frac{g(t_0^2)}{p\overline{x}t_0^{2k}} < 0 \text{ and } H_{t_0}(q) = 0.$$

If

$$\hat{\varphi}_n = \overline{x} + at_0^n + bt_0^{2n} = \frac{1+\sqrt{4\alpha+1}}{2} + at_0^n + bt_0^{2n},$$

we obtain

$$F(\hat{\varphi}_{n-k}, \hat{\varphi}_{n-k+1}, \ldots, \hat{\varphi}_n, \hat{\varphi}_{n+1}) \sim \left[b\frac{g(t_0^2)}{p\overline{x}t_0^{2k}} + a^2A\right]t_0^{2n} = H_{t_0}(b)t_0^{2n}.$$

Since $H'_{t_0}(b) = \dfrac{g(t_0^2)}{p\overline{x}t_0^{2k}} < 0$, we obtain that there are $q_1 < q$ and $q_2 > q$ such that $H_{t_0}(q_1) > 0$ and $H_{t_0}(q_2) < 0$.
With the notations

$$y_n = \overline{x} + at_0^n + q_1t_0^{2n}, \;\; z_n = \overline{x} + at_0^n + q_2t_0^{2n}.$$

We get

$$F(y_{n-k}, y_{n-k+1}, \ldots, y_n, y_{n+1}) \sim \left[q_1\frac{g(t_0^2)}{p\overline{x}t_0^{2k}} + a^2A\right]t_0^{2n} > 0$$
$$F(z_{n-k}, z_{n-k+1}, \ldots, z_n, z_{n+1}) \sim \left[q_2\frac{g(t_0^2)}{p\overline{x}t_0^{2k}} + a^2A\right]t_0^{2n} < 0.$$

These relations show that inequalities (11) are satisfied for sufficiently large n, where $f = F + x_{n-k}$ and F is given by (14). Since for all n, $y_n > 0$, we can apply Theorem 2.1 with $I = [\overline{x}, \infty)$ and see that there is an $n_0 > 0$ and a solution of Eq(1) with the asymptotics $x_n = \hat{\varphi}_n + o(t_0^{2n})$, for $n \geq n_0$, where $\hat{\varphi}_n$ is defined by (10) and $b = q$. In particular, the solution converges monotonically to the positive equilibrium

$$\overline{x} = \frac{1+\sqrt{1+4\alpha}}{2}, \text{ for } n \geq n_0.$$

Hence, the solution x_{n+n_0+k} is also such a solution when $n \geq -k$. □

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