Int. Journal of Math. Analysis, Vol. 4, 2010, no. 34, 1651 - 1659

On Approximate Solutions of a Volterra Type Integrodifferential Equation on Time Scales

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Abstract

The main objective of the present paper is to study the approximate solutions of a certain Volterra type dynamic integrodifferential equation on time scales which unifies the study of the corresponding continuous and discrete versions of the same. The tool employed in the analysis is based on the application of a time scale analogue of a certain integral inequality with explicit estimates.

Mathematics Subject Classification: 39A10, 39A12

Keywords: Integral equations, time scales, qualitative properties, explicit estimates, existence and uniqueness

1. Introduction

In [3] Stefan Hilger initiated the study of dynamic equations on time scales which effectively manage both continuous and discrete time combination. During the past few years many authors have studied some basic qualitative aspects of various dynamic equations on time scales by using different techniques, see [4, 7, 9, 10] and the references cited therein. In the present paper we shall study the approximate solutions of the following dynamic integrodifferential equation on time scales

$$x^{\Delta}(t) = f\left(t, x(t), \int_{t_0}^{t} g(t, \tau, x(\tau))\,\Delta\tau\right), \tag{1.1}$$

with the given initial condition

$$x(t_0) = x_0, \tag{1.2}$$

where x is the unknown function to be found, $\tau \leq t$, $g : I_{\mathbb{T}}^2 \times R^n \to R^n$, $f : I_{\mathbb{T}} \times R^n \times R^n \to R^n$, x^{Δ} is the generalized delta derivative of x, t is from a time scale $\mathbb{T}$, which is a known nonempty closed subset of R, the set of real numbers and $I_{\mathbb{T}} = I \cap \mathbb{T}$, $I = [t_0, \infty)$ be the given subset of R, R^n the real n-dimensional Euclidean space with appropriate norm defined by $|.|$. The problem of existence of a unique solution to (1.1)-(1.2) is dealt with by using a fixed point technique (see also [6, 8, 9, 10]). In dealing with equations like (1.1)-(1.2), the basic questions to be answered are: (i) if a solution do exist, then what are their nature ?, (ii) how can we find them or closely approximate them?. The study of such questions is interesting and needs a fresh outlook for handling the equations of the form (1.1)-(1.2).Here we offer the conditions for error evaluation of approximate solutions of $(1.1) - (1.2)$ by establishing some new bounds and convergence properties of approximate problems. Our proofs rely on the application of a time scale analogue of a certain integral inequality established by the present author in [7].

2. Preliminaries

In what follows, R denotes the set of real numbers, Z the set of integers and $\mathbb{T}$ denotes the arbitrary time scale and $t_0 \in \mathbb{T}$. Define the jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

If $\sigma(t) = t$ and $\sigma(t) < t$, then the point $t \in \mathbb{T}$ is left-dense and left-scattered. If $\rho(t) = t$ and $\rho(t) < t$, then the point $t \in \mathbb{T}$ is right-dense and right-scattered. If $\mathbb{T}$ has a right-scattered minimum m, define $\mathbb{T}^k := \mathbb{T} - m$, otherwise $\mathbb{T}^k := \mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum M, define $\mathbb{T}^k := \mathbb{T} - M$, otherwise $\mathbb{T}^k := \mathbb{T}$. The graininess function $\mu : \mathbb{T} \to R_+ = [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$. We say that $p : \mathbb{T} \to \mathbb{R}$ is regressive provided $1 + \mu(t)\, p(t) \neq 0$ for all $t \in \mathbb{T}$. For $f : \mathbb{T} \to R$ and $t \in \mathbb{T}^k$, the delta derivative of f at t denoted by $f^{\Delta}(t)$ is the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t such that

$$\left| f(\sigma(t), \tau) - f(s, \tau) - f^{\Delta}(t, \tau)(\sigma(t) - s) \right| \leq \in |\sigma(t) - s|,$$

for all $s \in U$. For $\mathbb{T} = R$, $f^{\Delta}(t) = f'(t)$ the usual derivative; for $\mathbb{T} = Z$ the delta derivative is the forward difference operator, $f^{\Delta}(t) = f(t+1) - f(t)$. A function $f : \mathbb{T} \to R$ is right dense continuous or rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist(finite) at left-dense points in $\mathbb{T}$ and its left-sided limit exist(finite) at left-dense points in $\mathbb{T}$. If $\mathbb{T} = R$ then f is rd-continuous if and only if f is continuous. It is known [1, Theorem 1.74] that if f is right-dense continuous, then there is a function F such that $f^{\Delta}(t) = f(t)$ and

$$\int_a^b f(t)\Delta t = F(b) - F(a),$$

where $a, b \in \mathbb{T}$. Note that when $\mathbb{T} = R$, $\sigma(t) = t$, $\mu(t) = 0$, $f^{\Delta} = f'$, $\int_a^b f(t)\Delta t = \int_a^b f(t)dt$, while $\mathbb{T} = Z$, $\sigma(t) = t+1$, $\mu(t) = 1$, $f^{\Delta} = \Delta f$, $\int_a^b f(t)\Delta t = \sum_{t=a}^{p-1} f(t)$. We denote by $\Re$ the set of all regressive and rd-continuous functions and $\Re^{+} = \{p \in \mathbb{R} : 1 + \mu(t)p(t) > 0 \ for\ all\ t \in \mathbb{T}\}$. For $p \in \Re$, we define (see [1], Theorem 2.35) the exponential function $e_p(., t_0)$ on time scale $\mathbb{T}$ as the unique solution to the scalar initial value problem

$$x^{\Delta} = p(t)x, \quad x(t_0) = 1.$$

If $p \in \Re^{+}$ then $e_p(t, t_0) > 0$ for all $t \in \mathbb{T}$ (see [1], Theorem 2.41). As usual, the set of rd-continuous functions is denoted by C_{rd}. For more detailed information concerning the time scales and dynamic equations on time scales, see the recent monographs [1, 2].

Let $x : I_{\mathbb{T}} \to R$ be a continuous function on $I_{\mathbb{T}}$, x^{Δ} exists, rd-continuous on $I_{\mathbb{T}}$ and satisfies the inequality

$$\left| x^{\Delta}(t) - f\left(t, x(t), \int_{t_0}^{t} g(t, \tau, x(\tau))\Delta\tau\right) \right| \le \epsilon,$$

for a given constant $\epsilon \ge 0$, where $x(t_0) = x_0$. Then we call $x(t)$ the ϵ-approximate solution with respect to the equation (1.1).

We need the following variant of the inequality recently established by the present author in [7] (see also [5]).

Lemma 2.1. *Assume that $u, p, c \in C_{rd}$, $u \ge 0$, $p \ge 0$, $c \ge 0$. Let $k(t,s)$ is rd-continuous on $[t_0, \sigma(t)]$. Suppose that for each $\epsilon > 0$, there exists a neighborhood N of t, independent of $\tau \in [t_0, \sigma(t)]$ such that*

$$\left| k(\sigma(t), \tau) - k(s, \tau) - k^{\Delta}(t, \tau)(\sigma(t) - s) \right| \le \varepsilon \left| \sigma(t) - s \right|,$$

for all $s \in N$, where k^{Δ} denotes the derivative of k with respect to the first variable, $k(\sigma(t), t) \ge 0$, $k^{\Delta}(t,s) \ge 0$ for $s, t \in \mathbb{T}$ with $s \le t$. If $c(t)$ is nondecreasing for $t \in \mathbb{T}$ and

$$u(t) \le c(t) + \int_{t_0}^{t} p(s)\left[u(s) + \int_{t_0}^{s} k(s, \tau)u(\tau)\Delta\tau\right]\Delta s, \tag{2.1}$$

for $t \in \mathbb{T}$, then

$$u(t) \le c(t)\left[1 + \int_{t_0}^{t} p(s)e_{p+A}(s, t_0)\Delta s\right], \tag{2.2}$$

for $t \in \mathbb{T}$, *where*

$$A(t) = k(\sigma(t), t) + \int_{t_0}^{t} k^{\Delta}(t, \tau)\,\Delta\tau. \tag{2.3}$$

3. Main Results

Our main result deals with the estimate on the difference between the two approximate solutions of equation (1.1).

Theorem 3.1. *Suppose that the functions* f, g *in equation* (1.1) *are rd-continuous and satisfy the conditions*

$$|f(t, u, v) - f(t, \bar{u}, \bar{v})| \le p(t)\left[|u - \bar{u}| + |v - \bar{v}|\right], \tag{3.1}$$

$$|g(t, \tau, u) - g(t, \tau, v)| \le k(t, \tau)\,|u - v|, \tag{3.2}$$

where p, k *be as in Lemma 2.1. For i=1,2, let* $x_i(t)$ *be respectively* ϵ_i *approximate solutions of equation* (1.1) *on* $I_{\mathbb{T}}$ *with* $x_i(0) = \bar{x}_i$ *such that*

$$|\bar{x}_1 - \bar{x}_2| \le \delta,$$

where $\delta \ge 0$ *is a constant, then*

$$|x_1(t) - x_2(t)| \le n(t)\left[1 + \int_{t_0}^{t} p(s)\,e_{p+A}(s, t_0)\,\Delta s\right], \tag{3.3}$$

for $t \in I_{\mathbb{T}}$, *where*

$$n(t) = (\epsilon_1 + \epsilon_2)(t - t_0) + \delta, \tag{3.4}$$

and $A(t)$ *is given by* (2.3).

Proof. Since $x_i(t), t \in I_{\mathbb{T}}$ are respectively ϵ_i are approximate solutions of equation (1.1) with $x_i(0) = \bar{x}_i$. We have

$$\left| x_i^{\Delta}(t) - f\left(t, x_i(t), \int_{t_0}^{t} g(t, \tau, x_i(\tau))\,\Delta\tau\right)\right| \le \epsilon_i, \tag{3.5}$$

for i=1,2. By taking $t = s$ in (3.5) and delta integrating both sides of (3.5) from t_0 to t we have

$$
\begin{aligned}
\epsilon_i (t-t_0) &\geq \int_{t_0}^{t} \left| x_i^{\Delta}(s) - f\left(s, x_i(s), \int_{t_0}^{s} g(s,\tau,x_i(\tau))\,\Delta\tau\right)\right| \Delta s \\
&\geq \left| \int_{t_0}^{t} \left\{ x_i^{\Delta}(s) - f\left(s, x_i(s), \int_{t_0}^{s} g(s,\tau,x_i(\tau))\,\Delta\tau\right)\right\} \Delta s \right| \\
&= \left| \left\{ x_i(t) - x_i(t_0) - \int_{t_0}^{t} f\left(s, x_i(s), \int_{t_0}^{s} g(s,\tau,x_i(\tau))\,\Delta\tau\right) \Delta s \right\}\right|. \qquad (3.6)
\end{aligned}
$$

From (3.6) and using the basic inequalities

$$
|v-z| \leq |v| + |z|, \ |v| - |z| \leq |v-z|, \qquad (3.7)
$$

we have

$$
\begin{aligned}
(\epsilon_1+\epsilon_2)(t-t_0) &\geq \left| \left\{ x_1(t) - x_1(t_0) - \int_{t_0}^{t} f\left(s, x_1(s), \int_{t_0}^{s} g(s,\tau,x_1(\tau))\,\Delta\tau\right) \Delta s \right\}\right| \\
&\quad + \left| \left\{ x_2(t) - x_2(t_0) - \int_{t_0}^{t} f\left(s, x_2(s), \int_{t_0}^{s} g(s,\tau,x_2(\tau))\,\Delta\tau\right) \Delta s \right\}\right| \\
&\geq \left| \left\{ x_1(t) - x_1(t_0) - \int_{t_0}^{t} f\left(s, x_1(s), \int_{t_0}^{s} g(s,\tau,x_1(\tau))\,\Delta\tau\right) \Delta s \right\} \right. \\
&\quad \left. - \left\{ x_2(t) - x_2(t_0) - \int_{t_0}^{t} f\left(s, x_2(s), \int_{t_0}^{s} g(s,\tau,x_2(\tau))\,\Delta\tau\right) \Delta s \right\}\right| \\
&\geq |x_1(t) - x_2(t)| - |x_1(t_0) - x_2(t_0)| \\
&\quad - \left| \int_{t_0}^{t} f\left(s, x_1(s), \int_{t_0}^{s} g(s,\tau,x_1(\tau))\,\Delta\tau\right) \Delta s \right. \\
&\quad \left. - \int_{t_0}^{t} f\left(s, x_2(s), \int_{t_0}^{s} g(s,\tau,x_2(\tau))\,\Delta\tau\right) \Delta s \right|. \qquad (3.8)
\end{aligned}
$$

Let $u(t) = |x_1(t) - x_2(t)|$, $t \in I_{\mathbb{T}}$. From (3.8) and using the hypotheses, we observe that

$$u(t) \leq (\epsilon_1 + \epsilon_2)(t - t_0) + u(t_0) + \int_{t_0}^{t} \Bigg| f\left(s, x_1(s), \int_{t_0}^{s} g(s, \tau, x_1(\tau))\Delta\tau\right)$$

$$- \int_{t_0}^{t} f\left(s, x_2(s), \int_{t_0}^{s} g(s, \tau, x_2(\tau))\Delta\tau\right)\Bigg| \Delta s$$

$$\leq (\epsilon_1 + \epsilon_2)(t - t_0) + \delta + \int_{t_0}^{t} p(s)\left[|x_1(s) - x_2(s)| + \int_{t_0}^{s} k(s, \tau)|x_1(\tau) - x_2(\tau)|\Delta\tau\right]\Delta s$$

(3.9)

$$= n(t) + \int_{t_0}^{t} p(s)\left[u(s) + \int_{t_0}^{s} k(s, \tau)u(\tau)\Delta\tau\right]\Delta s,$$

where $n(t)$ is given by (3.4). Clearly $n(t)$ is nonnegative and nondecreasing for $t \in \mathbb{T}$. Applying lemma 2.1 to (3.9) we get (3.3). □

Remark 3.2. *We note that in case $x_1(t)$ is a solution of* $(1.1) - (1.2)$, *then we have $\epsilon_1 = 0$ and from* (3.3) *we see that $x_2(t) \to x_1(t)$ as $\epsilon_2 \to 0$ and $\delta \to 0$. If we put (i) $\epsilon_1 = \epsilon_2 = 0, \bar{x}_1 = \bar{x}_2$ in* (3.3), *then the uniqueness of solutions of equation* (1.1) *is established and (ii) $\epsilon_1 = \epsilon_2 = 0$ in* (3.3), *then we get the bound which shows the dependency of solutions of equation* (1.1) *on given initial values.*

Consider equations $(1.1) - (1.2)$ together with the following integrodifferential equation

$$y^{\Delta}(t) = \bar{f}\left(t, y(\tau), \int_{t_0}^{t} g(t, \tau, y(t))\Delta\tau\right), \tag{3.10}$$

with the given initial condition

$$y(t_0) = y_0, \tag{3.11}$$

where $\bar{f} : I_{\mathbb{T}} \times R^n \times R^n \to R^n$ is rd-continuous and g is as in equation (1.1) and rd-continuous. The following theorem concerning the closeness of solutions equations $(1.1) - (1.2)$ and $(3.10) - (3.11)$ holds.

Theorem 3.3. *Suppose that the functions f, g in equation* (1.1) *are rd-continuous and satisfy the conditions* (3.1), (3.2) *and their exist constants $\bar{\epsilon} \geq 0, \bar{\delta} \geq 0$ such that*

$$\left|f(t, u, v) - \bar{f}(t, u, v)\right| \leq \bar{\epsilon}, \tag{3.12}$$

$$|x_0 - y_0| \leq \bar{\delta}, \tag{3.13}$$

where f, x_0 *and* $\bar{f}, y_0$ *are as in equations* (1.1) – (1.2) *and* (3.10) – (3.11). *Let* $x(t)$, $y(t)$ *be respectively, solutions of equations* (1.1) – (1.2) *and* (3.10) – (3.11) *on* $I_{\mathbb{T}}$. *Then*

$$|x(t) - y(t)| \leq m(t) \left[1 + \int_{t_0}^{t} p(s)\, e_{p+A}(s, t_0)\, \Delta s \right], \tag{3.14}$$

for $t \in I_{\mathbb{T}}$, *where*

$$m(t) = \bar{\epsilon}(t - t_0) + \bar{\delta}, \tag{3.15}$$

and $A(t)$ *is given by* (2.3).

Proof. Let $u(t) = |x(t) - y(t)|$, $t \in I_{\mathbb{T}}$. Using the facts that $x(t), y(t)$ are the solutions of equations (1.1) – (1.2) and (3.10) – (3.11) and hypotheses, we observe that

$$\begin{aligned} u(t) \leq{} & |x_0 - y_0| + \int_{t_0}^{t} \left| f\left(s, x(s), \int_{t_0}^{s} g(s, \tau, x(\tau))\, \Delta\tau \right) \right. \\ & \left. - f\left(s, y(s), \int_{t_0}^{s} g(s, \tau, y(\tau))\, \Delta\tau \right) \right| \Delta s \\ & + \int_{t_0}^{t} \left| f\left(s, y(s), \int_{t_0}^{s} g(s, \tau, y(\tau))\, \Delta\tau \right) \right. \\ & \left. - \bar{f}\left(s, y(s), \int_{t_0}^{s} g(s, \tau, y(\tau))\, \Delta\tau \right) \right| \Delta s \\ \leq{} & m(t) + \int_{t_0}^{t} p(s) \left[u(s) + \int_{t_0}^{t} k(t, \tau)\, u(\tau)\, \Delta\tau \right] \Delta s, \end{aligned} \tag{3.16}$$

where $m(t)$ is given by (3.15). Clearly $m(t)$ is nonnegative and nondecreasing for $t \in I_{\mathbb{T}}$. Now applying Lemma 2.1 to (3.16) yields (3.14). □

Remark 3.4. *The result given in Theorem 3.3 relates the solutions of equations* (1.1) – (1.2) *and* (3.10) – (3.11) *in the sense that if* f *is close to* $\bar{f}$ *and* x_0 *is close to* y_0, *then the solutions of equations* (1.1) – (1.2) *and* (3.10) – (3.11) *are also close together.*

A slight variant of Theorem 3.3 is given in the following theorem.

Theorem 3.5. *Suppose that the functions* $f, g, \bar{f}$ *are rd-continuous and*

$$\left| f(t, u, v) - \bar{f}(t, \bar{u}, \bar{v}) \right| \leq r(t) \left[|u - \bar{u}| + |v - \bar{v}| \right], \tag{3.17}$$

where $r \in C_{rd}$ *and the conditions* (3.2) *and* (3.13) *hold. Then*

$$|x(t) - y(t)| \leq \bar{\delta}\left[1 + \int_{t_0}^{t} r(s)\, e_{r+A}(s, t_0)\Delta s\right], \tag{3.18}$$

for $t \in I_{\mathbb{T}}$, *where* $A(t)$ *is given by* (2.3).

Proof. Let $w(t) = |x(t) - y(t)|$ for $t \in I_{\mathbb{T}}$. Using the facts that $x(t)$ and $y(t)$ are respectively, solutions of (1.1) – (1.2) and (3.10) – (3.11), and hypotheses, we observe that

$$w(t) \leq |x_0 - y_0| + \int_{t_0}^{t} \left| f\left(s, x(s), \int_{t_0}^{s} g(s, \tau, x(\tau))\Delta\tau\right) - f\left(s, y(s), \int_{t_0}^{s} g(s, \tau, y(\tau))\Delta\tau\right)\right| \Delta s$$

$$\leq \bar{\delta} + \int_{t_0}^{t} r(s)\left[w(s) + \int_{t_0}^{s} k(s, \tau)\, w(\tau)\Delta\tau\right]\Delta s. \tag{3.19}$$

Now an application of Lemma 2.1 to (3.19) yields (3.18). □

Remark 3.6. *We note that the idea used in this paper can be very easily extended to study the approximate solutions of the Volterra type dynamic equation*

$$y(t) = f(t) + \int_{t_0}^{t} h(t, \tau, y(\tau))\Delta\tau, \tag{3.20}$$

under some suitable conditions on the functions involved in (3.20). *The details of the formulation of the results similar to those of given in Theorems* 3.1, 3.3, 3.5 *for the solutions of equation* (3.20) *is very close to those of given above with suitable modifications and here we omit the details.*

ACKNOWLEDGEMENTS. The research in the present paper is supported by UGC(New Delhi,India) project F.NO 37-538/2009.

References

[1] M. Bohner and A. Peterson, Dynamic equations on Time Scales, *Birkhauser Boston/Berlin*, 2001.

[2] M. Bohner and A. Peterson, Advances in Dynamic equations on Time Scales, *Birkhauser Boston/Berlin*, 2003.

[3] S. Hilger, Analysis on Measure chain-A unified approch to continuous and discrete calculus, *Results Math.* **18**:18-56, 1990.

[4] T. Kulik and C.C. Tisdell, Volterra integral equations on time scales: Basic qualitative and quantitative results with applications to initial value problems on unbounded domains, *Int.J. Difference. Eqn.* **3**(2008),No 1, 103-133.

[5] D.B. Pachpatte, Explicit estimates on integral inequalities with time scale, *J.Inequal. Pure and Appl.Math*, **6(1**(2005), Art 143.

[6] D.B. Pachpatte, Properties of Solutions to nonlinear dynamic integral equations on time scales, *Electron. J. Diff Eqns. Vol. 2008(2008)*, No **136**, pp1-8.

[7] D.B. Pachpatte, On a nonlinear dynamic integrodifferential equations on time scales, *Journal of Applied Analysis* Vol 16, No 2,(2010) *To appear.*

[8] D.B. Pachpatte, On a nonstandard Volterra type dynamic integral equation on time scales, *E. J. Qualitative Theory of Diff. Equ.*, No. **72**. (2009), pp. 1-14.

[9] C.C. Tisdell and Atiya Zaidi, Basic qualitative and quantitative results for solutions to nonlinear dynamic equations on time scales with an application to economic modelling, *Nonlinear Anal.* **68**(2008), 3504-3524.

[10] C.C. Tisdell and Atiya Zaidi, Successive approximations to solutions of dynamic equations on time scales, *Comm. Appl. Nonlinear Anal.* **16** (2009), No. 1, 61–87.

Received: March, 2010