The Moore-Penrose Inverse of

Intuitionistic Fuzzy Matrices

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Abstract

In this paper, we define the various g-inverses of an intuitionistic fuzzy matrices, left (right) cancelable intuitionistic fuzzy matrices and derive the equivalent condition for the existence of the generalized inverses. We also study the relation between the minus-ordering and the various g–inverses of an intuitionistic fuzzy matrix.

Keywords: Intuitionistic fuzzy matrix (IFM), generalized inverse (g-inverse), minus ordering

1. Introduction


Definition 1.1: An m x n matrix A = (a_{ij}) whose components are in the unit interval [0,1] is called a fuzzy matrix.
**Definition 1.2[6]:** An intuitionistic fuzzy matrix (IFM) $A$ is a matrix of pairs $A = (a_{ij}, a'_{ij})$ of non-negative real numbers satisfying $a_{ij} + a'_{ij} \leq 1$ for all $i, j$.

The matrix operations

$$A + B = [\max \{a_{ij}, b_{ij}\}, \min \{a'_{ij}, b'_{ij}\}]$$

$$AB = [\max \{\min \{a_{ik}, b_{kj}\}, \min \{a'_{ik}, b'_{kj}\}\}]$$

are defined for given IFMs $A, B$. The addition is defined for IFMs of same order, the product $AB$ is defined if and only if the number of columns of $A$ is same as the number of rows of $B$, $A$ and $B$ are said to be conformable for multiplication. $F_{mn}$ denote the set of all intuitionistic fuzzy matrices of order $m \times n$. If $m = n$, then $F_n$ denote the set of all square IFMs of order $n$.

**Definition 1.3[4]:** A matrix $A \in F_{mn}$ is said to be regular if there exists $X \in F_{nm}$ such that $AXA = A$. In this case $X$ is called a generalized inverse (g-inverse) of $A$ and it is denoted by $A^\sim$. $A\{1\}$ denotes the set of all g – inverses of $A$.

**Definition 1.4:** For the intuitionistic fuzzy matrices $A$ and $B$ of order $m \times n$, the minus ordering is defined as $A \leq B$ if and only if $A - A = A - B$ and $AA = BA$ for some $A^\sim \in A\{1\}$. $F_{mn}^- = \{A \in F_{mn} | A$ has a g-inverse $\}$.

**Property 1:[4].** The following are equivalent for $A$ and $B$ in $F_{mn}$

i) $A \leq B$

ii) $A = A^\sim B = B A^\sim B$.

**Property 2.[4].** In $F_{mn}^-$, the minus ordering $\leq -$ is a partial ordering.

**Preliminaries**

In this section, we define the various g-inverses of IFMs and left, right cancelable IFMs. Also, we discuss existence of the generalized inverses and the relation between the minus-ordering and the Moore-Penrose inverse.

**Definition 2.1:** For an IFM $A$ of order $m \times n$, an IFM $G$ of order $n \times m$ is said to be

- \{1, 2\}-inverse or semi inverse of $A$, if
  $$AGA = A \text{ and } GAG = G$$
  \tag{1.1}

- a left $\{1,3\}$-inverse or a least square g-inverse of $A$, if
  $$AGA = A \text{ and } (AG)^T = AG$$
  \tag{1.2}

- \{1,4\}-inverse or a minimum norm g-inverse of $A$, if
Moore-Penrose inverse of intuitionistic fuzzy matrices

AGA = A and \((GA)^T = GA\) \hspace{1cm} (1.3)

G is said to be a Moore-Penrose inverse of A , if
AGA = A, GAG = G, \((AG)^T = AG\) and \((GA)^T = GA\) \hspace{1cm} (1.4)

The Moore-Penrose inverse of A is denoted by \(A^+\).

**Definition 2.2:** \(A\{\lambda\}\) is the set of all \(\lambda\)-inverses of A, where \(\lambda\) is a subset of \{1,2,3,4\}.

**Definition 2.3:** Let \(A \in \mathbb{F}_{mn}\), A is called left(right) cancelable if \(A^TAX_1 = A^TAX_2\) (\(X_1AAT = X_2AAT\)) implies \(AX_1 = AX_2\) for any \(X_1, X_2 \in \mathbb{F}_{nm}\). A is called cancelable in case it is both left and right cancelable.

**Theorem 2.1:** For each IFM A of order \(m \times n\) the following statements are equivalent.

i) \(A\{1,3\} \neq \phi\)

ii) The intuitionistic fuzzy relation equation \(XA^TA = A\) has solutions.

iii) A is left cancelable and \(A^T\{1\} \neq \phi\).

**Proof:**

(i) \(\Rightarrow\) (ii)
For \(B \in A\{1,3\}\), then
\[A = ABA = (AB)^T A = B^T A^T A\]
Thus, \(B^T\) is a solution of \(A = XA^T A\).

(ii) \(\Rightarrow\) (iii)
Let \(B\) be a solution of \(XA^T A = A\)
Then, \(BA^T A = A\)
For \(X_1, X_2 \in \mathbb{F}_{nm}\),
if \(A^T AX_1 = A^T AX_2\), then
\[AX_1 = BA^T A X_1 = BA^T A X_2 = AX_2\]
Hence, A is left cancelable and
\[A^T A = (BA^T A)^T BA^T A = A^T AB^T BA^T A\]
\(\Rightarrow\) \(B^T B \in A^T \{1\}\)
\(\Rightarrow\) \(A^T \{1\} \neq \phi\)

(iii) \(\Rightarrow\) (ii)
Let \(X_1 \in A A^T \{1\}\)
\(\Rightarrow\) \(A^T A X_1 A^T A = A^T A\)
\(\Rightarrow\) \(A X_1 A^T A = A\) by definition 2.3
AX_1 is a solution of XA^T A = A.

(ii) ⇒ (i)
Let B^T be a solution of XA^T A = A
⇒ B^T A^T A = A.
A = B^T A^T A = (AB)^T A = ABA
since, (AB)^T = (B^T A^T AB)^T = (AB)^T (B^T A^T)^T = B^T A^T AB = AB
Thus, (AB)^T = AB and A = ABA
⇒ B ∈ A \{1,3\} and A(1,3) ≠ φ.

Example 2.1.
Let A = \begin{bmatrix} <1,0> & <0,0> \\ <1,0> & <0,0> \end{bmatrix} , then H = \begin{bmatrix} <1,0> & <1,0> \\ <1,0> & <0,0> \end{bmatrix}
satisfies AHA = A, (AH)^T = AH and HATA = A.
Therefore, A \{1,3\} ≠ φ ⇒ there exist a solution for XA^T A = A.

**Theorem 2.2:** For each A ∈ F_{mn}, the following statements are equivalent.
i) A(1,4) ≠ φ
ii) The intuitionistic fuzzy relation equation AA^T Y = A has solutions
iii) A is right cancelable and AA^T {1} ≠ φ

**Proof:** Proof is similar to the proof of Theorem 2.1 because A{1,3} ≠ Ø if and only if AT{1,4} ≠ Ø.

**Example 2.2:**
If A = \begin{bmatrix} <1,0> & <0,0> \\ <0,0> & <1,0> \end{bmatrix} , then A = X itself is a g-inverse satisfying
AXA = A, (XA)^T = XA.

**Theorem 2.3:** For each A ∈ F_{mn}, the following statements are equivalent.
i) A^+ exist
ii) The intuitionistic fuzzy relation equations XA^T A = A, AA^T Y = A have solutions
iii) A{1,3} ≠ φ and A{1,4} ≠ φ
iv) A is cancelable, A^T A{1} ≠ φ and AA^T {1} ≠ φ

**Proof:**
It is enough if we prove (ii) ⇒ (i)
Let B be a solution of XA^T A = A and C be a solution of AA^T Y = A.
Then, A = BA^T A and A = AA^T C
Set, G = A^T CB^T
Now, AGA = AA^T CB^T A = AB^T A = (BA^T A)B^T A = B(A^T AB^T)A = B(BA^T A)^T A
Thus, $G \in A \{1\}$

\[
G A G = A^T C B^T C B A^T = (A A^T C) B^T C B A^T
\]

Again, $(A G)^T = (A A^T C B^T)^T = A A^T C B^T = A G$

Therefore, $G$ is a Moore-Penrose inverse of $A$

**Theorem 2.4:**

Let $A, B \in F_{mn}, A \leq B$. If $B^+$ exists, $A$ is cancelable and $A \in B\{2\}$ then $A^+$ exists and

\[
A^+ = (B^+)^T A (B^+)^T.
\]

**Proof:**

Given $A \leq B \Rightarrow A^\sim A = A^\sim B$ and $A A^\sim = B A^\sim$. Also, $A = A A^\sim B = B A^\sim A$.

Now, $A B^\sim A = (A A^\sim B) B^\sim (B A^\sim A)$

\[
= A A^\sim (B B^\sim B) A^\sim A = A A^\sim B A^\sim A = (A A^\sim B) A^\sim A = A A^\sim A = A.
\]

This shows that $B^\sim \in A \{1\}$.

Let $X_1, X_2 \in F_{nm}$

For $A^\sim A X_1 = A^\sim X_2$

$A X_1 = A A^\sim B X_1 = A A^\sim A X_1 = A A^\sim A X_2 = A X_2$.

Therefore $A^\sim A X_1 = A^\sim A X_2 \Rightarrow A X_1 = A X_2$.

Take $X_1 = A^\sim A, X_2 = B^\sim B$

We have $A^\sim A = A^\sim B$

\[
A^\sim A A^\sim A = A^\sim B B^\sim B
\]

\[
= A^\sim A B^\sim B
\]

\[
\Rightarrow A A^\sim A = A B^\sim B
\]

\[
\Rightarrow A = A B^\sim B
\]

Similarly $A = B^+ A$

Consider, $A B^+ = B B^+ A B^+ = B (B^+ A B^+)$

\[
= B B^+ = (B B^+)^T = (B B^+ A B^+)^T = ((B A) B^+)^T = (A B^+)^T
\]

Similarly we can prove $B^+ A = (B^+ A)^T$
Therefore $B^+ \in A\{1,2,3\}$. Hence by Theorem 2.3 $A^+$ exists and $A^+ = (B^+)^T A^T (B^+)^T$.

3. Characterization of minus ordering

**Theorem 3.1:** Let $A, B \in F_{mn}$ and $A^+$ exists, then the following are equivalent.

i) $A \leq B$

ii) $A^+ A = A^+ B$ and $AA^+ = BA^+$

iii) $AA^+ B = A = BA^+ A$

**Proof:** This proof is evident from property 1 by replacing $A^-$ by $A^+$.

**Corollary 3.2:** If $A \leq -B$ and $A^+$ exists, then $B \in A^+ \{1\}$

**Proof:**

$A^+$ exist $\Rightarrow A^+ = A^+ A A^+$

$A^+ = A^+ B A^+$

$\Rightarrow B \in A^+ \{1\}$.

**Theorem 3.3:**

Let $A, B \in F_{mn}$. If $A^+$ and $B^+$ both exist and $A \in B\{2\}$, then the following conditions are equivalent.

i) $A \leq -B$

ii) $A^+ A = B^+ A$ and $AA^+ = AB^+$

iii) $B^+ AA^+ = A^+ = A^+ AB^+$

iv) $A^T AB^+ = A^T = A^T BA^+$

**Proof:**

(i) $\Rightarrow$ (ii)

Let $A \leq B$, then $AA^+ = BA^+$ and $A^+ A = A^+ B$

By Theorem 2.4

$A = AB^+ A$

$A^+ A = A^+ A B^+ A$

$(A^+ A)^T = (A^+ AB^+ A)^T$

$A^+ A = (B^+ A)^T (A^+ A)^T$

$= B^+ AA^+ A^+ B^+ A$

Therefore $A^+ A = B^+ A$.

Similarly, $AA^+ = AB^+$.

(ii) $\Rightarrow$ (iii)

$A^+ A = B^+ A$ and $AA^+ = AB^+$

Now, $A^+ = A^+ AA^+ = B^+ AA^+$
Similarly, \( A^+ = A^+A^+ = A^+AB^+ \).

(iii) \( \Rightarrow \) (iv)

\[
A = AA^+A \\
\]

Similarly \( A^TBA^+ = A^T \)

Therefore, \( A^TBA^+ = A^T = A^TAB^+ \).

(iv) \( \Rightarrow \) (i)

Take

\[
A^T = A^TAB^+ \\
A = AB^+A \\
AB^+ = (AB^+A)B^+ = A(B^+B^+) = AA^+ 
\]

Similarly, \( A + A = A + B \). Therefore by Theorem 3.1, (iv) \( \Rightarrow \) (i) holds.

**Corollary 3.4:** If \( A \leq B \) and \( A^+ \), \( B^+ \) both exist, then \( B^+ \in A \{1\} \)

**Proof:**

\( A = AA^+A = AB^+A \) by Theorem 3.3

\( \Rightarrow \) \( B^+ \in A \{1\} \).

**Theorem 3.5:**

If \( A \in F_{mn} \) and \( A^+ \) exist, then we have

i) \((AA^+)\), \((A^TA)\) are also exist and \((AA^+)\)\(^+\) = \((A^+)\)\(^T\)A\(^+\)\(^\), \( (A^TA)\)\(^+\) = \( A^+\)\((A^+)\)\(^T\)

ii) \((AA^+)\)\(^+\), \((A^+A)\)\(^+\) are also exist and \((AA^+)\)\(^+\) = \( AA^+ \), \( (A^+A)\)\(^+\) = \( A^+A \)

**Proof:**

i) \( AA^T = AA^+A^T = (A^+)\)\(^T\)A\(^+\)\(^ T\) = (A^+)\)\(^T\) (AA^+)\)\(^T\) AA^T = (A^+)\)\(^T\) A^+ AA^TAA^T

\( \Rightarrow \) \( (A^+)\)\(^T\) A^+ is a solution of \( AA^T = XAA^TAA^T \)

Therefore, \((A^+)\)\(^T\) A^+ \( \in AA^T \{1,3\} \)

Since, \( AA^T \) is symmetric

\( (A^+)\)\(^T\) A^+ \( \in AA^T \{1,4\} \)

By Theorem 2.3 and 2.4

\( (AA^+)\)\(^+\) exist and

\[
(AA^+)\)\(^+\) = ((A^+)\)\(^T\)A^+\)\(^ T\) AA^T((A^+)\)\(^T\)A^+)\)\(^ T\) \\
= (A^+)\)\(^T\) A^+ A^+ A^T(A^+)\)\(^T\) A^+ \\
= (A^+)\)\(^T\) A^+ A^+ A^T A^+ \\
= (A^+)\)\(^T\) A^+ A^+ A^+ A^+ = (A^+)\)\(^T\) A^+ A^+ A^+ = (A^+)\)\(^T\) A^+ 
\]

Similarly, \( (A^TA)\)\(^+\) exists and \( (A^TA)\)\(^+\) = \( A^+(A^+)\)\(^T\)

ii) \( AA^+ = AA^+A^+AA^+ \)

\( AA^+ \) is a solution of \( AA^+ = XAA^+AA^+ \)
and $(AA^+AA^+)^T = AA^+AA^+
Therefore, $AA^+ \in AA^+ \{1,3\} \cap AA^+ \{1,4\}$
Hence, $(AA^+)^+$ exist
and $(AA^+) = (AA^+)^T AA^+(AA^+)^T = AA^+AA^+AA^+ = AA^+$
Similarly, $(A^+A)^+ = A^+A$.

References


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