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The Moore-Penrose Inverse of

Intuitionistic Fuzzy Matrices

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Abstract

In this paper, we define the various g-inverses of an intuitionistic fuzzy matrices, left (right) cancelable intuitionistic fuzzy matrices and derive the equivalent condition for the existence of the generalized inverses . We also study the relation between the minus-ordering and the various g–inverses of an intuitionistic fuzzy matrix.

Keywords: Intuitionistic fuzzy matrix (IFM), generalized inverse (g-inverse), minus ordering

1. Introduction

In 1965, Zadeh [7] introduced the concept of fuzzy sets which formed the fundamental of fuzzy mathematics. The fuzzy matrices introduced first time by Thomason [5], and he discussed about the convergence of powers of fuzzy matrix. Cen [2] introduced T-ordering in fuzzy matrices and discussed the relationship between the T- ordering and the generalized inverses. Meenakshi. AR and Inbam. C [3] studied the minus ordering for fuzzy matrices and proved that the minus ordering is a partial ordering in the set of all regular fuzzy matrices. Atanassov [1] introduced and studied the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets. Using the idea of intuitionistic fuzzy sets Im and Lee [6] defined the concept of intuitionistic fuzzy matrix as a natural generalization of fuzzy matrices and they introduced the determinant of square intuitionistic fuzzy matrix. Susanta K. Khan and Anita Pal [4] introduced the concept of generalized inverses for intuitionistic fuzzy matrices, minus partial ordering and studied several properties of it.

Definition 1.1: An m x n matrix A = (a_{ij}) whose components are in the unit interval [0,1] is called a fuzzy matrix.

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Definition 1.2[6]: An intuitionistic fuzzy matrix (IFM) A is a matrix of pairs $A = (a_{ij}, a_{ij}')$ of non-negative real numbers satisfying $a_{ij} + a_{ij}' \leq 1$ for all i, j.
The matrix operations

$$A+B = [\max\{a_{ij}, b_{ij}\}, \min\{a_{ij}', b_{ij}'\}]$$
$$AB = [\max\{\min\{a_{ik}, b_{kj}\}\}, \min\{\max\{a_{ik}', b_{kj}'\}\}]$$

are defined for given IFMs A, B. The addition is defined for IFMs of same order, the product AB is defined if and only if the number of columns of A is same as the number of rows of B, A and B are said to be conformable for multiplication. $Ғ_{mn}$ denote the set of all intutionistic fuzzy matrices of order m x n. If m = n, then $Ғ_n$ denote the set of all square IFMs of order n.

Definition 1.3[4] : A matrix $A \in Ғ_{mn}$ is said to be regular if there exists $X \in Ғ_{nm}$ such that AXA=A. In this case X is called a generalized inverse (g-inverse) of A and it is denoted by A^-. A{1} denotes the set of all g – inverses of A.

Definition 1.4: For the intuitionistic fuzzy matrices A and B of order m x n, the minus ordering is defined as $A \leq^- B$ if and only if $A^-A = A^-B$ and $AA^- = BA^-$ for some $A^- \in A\{1\}$. $Ғ_{mn}^- = \{A \in Ғ_{mn} \mid A \text{ has a g\_inverse}\}$.

Property 1:[4]. The following are equivalent for A and B in $Ғ_{mn}$

i) $A \leq^- B$

i i) $A = AA^-B = BA^-B$.

Property 2.[4]. In $Ғ_{mn}^-$, the minus ordering $\leq^-$ is a partial ordering.

Preliminaries

In this section, we define the various g-inverses of IFMs and left, right cancelable IFMs. Also, we discuss existence of the generalized inverses and the relation between the minus-ordering and the Moore-Penrose inverse.

Definition 2.1: For an IFM A of order m x n, an IFM G of order n x m is said to be

{1, 2}-inverse or semi inverse of A , if

$$AGA = A \text{ and } GAG = G \quad (1.1)$$

G is said to be {1,3}-inverse or a least square g-inverse of A ,if

$$AGA = A \text{ and } (AG)^T = AG \quad (1.2)$$

G is said to be {1,4}-inverse or a minimum norm g-inverse of A ,if

$AGA = A$ and $(GA)^T = GA$ (1.3)

G is said to be a Moore-Penrose inverse of A , if

$AGA = A$, $GAG = G$, $(AG)^T = AG$ and $(GA)^T = GA$ (1.4)

The Moore-Penrose inverse of A is denoted by A^+.

Definition 2.2: A{λ} is the set of all λ-inverses of A, where λ is a subset of {1,2,3,4}.

Definition2.3: Let $A\in \mathcal{F}_{mn,}$ A is called left(right) cancelable if $A^TAX_1=A^TAX_2(X_1AA^T=X_2AA^T)$ implies $AX_1=AX_2(X_1A=X_2A)$ for any $X_1,X_2\in \mathcal{F}_{nm}$. A is called cancelable in case it is both left and right cancelable.

Theorem 2.1: For each IFM A of order m x n the following statements are equivalent.

i) $A\{1,3\} \neq \phi$

ii) The intuitionistic fuzzy relation equation

$XA^TA = A$ has solutions.

iii) A is left cancelable and $A^TA\ \{1\} \neq \phi$.

Proof:

(i) ⇒ (ii)

For $B \in A\ \{1,3\}$, then

$A = ABA = (AB)^TA = B^TA^TA$

Thus, B^T is a solution of $A = XA^TA$.

(ii) ⇒ (iii)

Let B be a solution of $XA^TA = A$

Then, $BA^TA = A$

For $X_1, X_2 \in \mathcal{F}_{nm}$,

if $A^TAX_1=A^TAX_2$,then

$AX_1= BA^TA X_1 =BA^TA X_2 = AX_2$

Hence, A is left cancelable and

$A^TA = (BA^TA)^T BA^TA= A^TAB^TBA^TA$

$\Rightarrow B^TB \in A^TA\{1\}$

$\Rightarrow A^TA\ \{1\} \neq \phi$

(iii) ⇒ (ii)

Let $X_1 \in AA^T\{1\}$

$\Rightarrow A^TA\ X_1\ A^TA = A^TA$

$\Rightarrow A\ X_1\ A^TA = A$ by definition 2.3

AX_1 is a solution of $XA^TA = A$.

(ii) $\Rightarrow$ *(i)*

Let B^T be a solution of $XA^TA = A$

$\Rightarrow B^TA^TA = A$.

$A = B^TA^TA = (AB)^TA = ABA$

since, $(AB)^T = (B^TA^TAB)^T = (AB)^T (B^TA^T)^T = B^TA^TAB = AB$

Thus, $(AB)^T = AB$ and $A = ABA$

$\Rightarrow B \in A \{1,3\}$ and $A\{1,3\} \neq \phi$.

Example 2.1.

Let $A = \begin{bmatrix} <1,0> & <0,0> \\ <1,0> & <0,0> \end{bmatrix}$, then $H = \begin{bmatrix} <1,0> & <1,0> \\ <1,0> & <0,0> \end{bmatrix}$

satisfies $AHA = A$, $(AH)^T = AH$ and $HA^TA = A$.

Therefore, $A \{1,3\} \neq \phi \Rightarrow$ there exist a solution for $XA^TA = A$.

Theorem 2.2: For each $A \in$ Ғ$_{mn}$, the following statements are equivalent.

i) $A\{1,4\} \neq \phi$

ii) The intuitionistic fuzzy relation equation $AA^TY = A$ has solutions

iii) A is right cancelable and $AA^T\{1\} \neq \phi$

Proof: Proof is similar to the proof of Theorem 2.1 because $A\{1,3\}\neq Ø$ if and only if $A^T\{1,4\}\neq Ø$.

Example 2.2:

If $A= \begin{bmatrix} <1,0> & <0,0> \\ <0,0> & <1,0> \end{bmatrix}$ then $A = X$ itself is a g-inverse satisfying $AXA=A$, $(XA)^T=XA$.

Theorem 2.3: For each $A \in$ Ғ$_{mn,}$ the following statements are equivalent.

i) A^+ exist

ii) The intuitionistic fuzzy relation equations $XA^TA = A$, $AA^TY = A$ have solutions

iii) $A\{1,3\} \neq \phi$ and $A\{1,4\} \neq \phi$

iv) A is cancelable, $A^TA\{1\} \neq \phi$ and $AA^T\{1\} \neq \phi$

Proof:

It is enough if we prove (ii) $\Rightarrow$ (i)

Let B be a solution of $XA^TA = A$ and C be a solution of $AA^TY = A$.

Then, $A = BA^TA$ and $A = AA^TC$

Set, $G = A^TCB^T$

Now, $AGA=AA^TCB^TA=AB^TA=(BA^TA)B^TA=B(A^TAB^T)A=B(BA^TA)^TA$

$=BA^TA=A$

Thus, $G \in A\{1\}$

$GAG = A^TCB^T(AA^TC)B^T$
$=A^TCB^TAB^T=(AA^TC)^TCB^TAB^T$
$=C^TAA^TCB^TAB^T=C^T(AA^TC)B^TAB^T$
$= C^TAB^TAB^T$
$= C^TAB^T = A^TCB^T = G$

Again, $(AG)^T = (AA^TCB^T)^T$
$= (AB^T)^T = AB^T = AA^TCB^T = AG$

Similarly, $(GA)^T = GA$

Therefore, G is a Moore-Penrose inverse of A

Hence, $A^+ = A^TCB^T = C^TAB^T = C^TBA^T$

Theorem 2.4:

Let $A, B \in Ғ_{mn}$, $A \leq^- B$. If B^+ exists, A is cancelable and $A \in B\{2\}$ then A^+ exists and

$A^+ = (B^+)^TA(B^+)^T$.

Proof:

Given $A \leq^- B \Rightarrow A^-A = A^-B$ and $AA^- = BA^-$. Also, $A= AA^-B = BA^-A$.

Now, $AB^+A = (AA^-B)B^+(BA^-A.)$
$= AA^-(BB^+B)A^-A = AA^-BA^-A=(AA^-B)A^-A= AA^-A=A.$

This shows that $B^+ \in A\{1\}$.

Let $X_1, X_2 \in Ғ_{nm}$

For $A^-AX_1 = A^-AX_2$

$AX_1 = AA^-BX_1 = AA^-AX_1 = AA^-AX_2 = AX_2$.

Therefore $A^-AX_1 = A^-AX_2 \Rightarrow AX_1 = AX_2$.

Take $X_1 = A^-A$, $X_2 = B^+B$

We have $A^-A = A^-B$

$A^-AA^-A = A^-BB^+B$
$= A^-AB^+B$

$\Rightarrow AA^-A = AB^+B$

$\Rightarrow A=AB^+B$

Similarly $A=BB^+A$.

Consider, $AB^+ =BB^+AB^+ =B(B^+AB^+) = BB^+$
$= (BB^+)^T$
$= (BB^+AB^+)^T$
$= ((BB^+A)B^+)^T$
$= (AB^+)^T$

Similarly we can prove $B^+A= (B^+A)^T$

Therefore $B^{+} \in A\{1,2,3\}$. Hence by Theorem 2.3 A^{+} exists and $A^{+}=(B^{+})^{T}A(B^{+})^{T}$.

3. Characterization of minus ordering

Theorem 3.1: Let A, B $\in \mathcal{F}_{mn}$ and A^{+} exists, then the following are equivalent.

i) $A \leq^{-} B$

ii) $A^{+}A = A^{+}B$ and $AA^{+} = BA^{+}$

iii) $AA^{+}B = A = BA^{+}A$

Proof: This proof is evident from property 1 by replacing A^{-} by A^{+}.

Corollary 3.2: If $A \leq^{-} B$ and A^{+} exists, then $B \in A^{+}\{1\}$

Proof:

A^{+} exist $\Rightarrow$ $A^{+} = A^{+}AA^{+}$

$A^{+} = A^{+}BA^{+}$

$\Rightarrow B \in A^{+}\{1\}$.

Theorem 3.3:

Let A, B $\in \mathcal{F}_{mn}$. If A^{+} and B^{+} both exist and $A \in B\{2\}$, then the following conditions are equivalent.

i) $A \leq^{-} B$

ii) $A^{+}A = B^{+}A$ and $AA^{+} = AB^{+}$

iii) $B^{+}AA^{+} = A^{+} = A^{+}AB^{+}$

iv) $A^{T}AB^{+} = A^{T} = A^{T}BA^{+}$

Proof:

(i) $\Rightarrow$ (ii)

Let $A \leq^{-} B$, then $AA^{-} = BA^{-}$ and $A^{-}A = A^{-}B$

By Theorem 2.4

$A = AB^{+}A$

$A^{+}A = A^{+}AB^{+}A$

$(A^{+}A)^{T} = (A^{+}AB^{+}A)^{T}$

$A^{+}A = (B^{+}A)^{T}(A^{+}A)^{T}$

$= B^{+}AA^{+}A = B^{+}A$

Therefore $A^{+}A = B^{+}A$.

Similarly, $AA^{+} = AB^{+}$.

(ii) $\Rightarrow$ (iii)

$A^{+}A = B^{+}A$ and $AA^{+} = AB^{+}$

Now, $A^{+} = A^{+}AA^{+} = B^{+}AA^{+}$

Similarly, $A^+ = A^+AA^+ = A^+AB^+$.

(iii) $\Rightarrow$ (iv)

$A = AA^+A$

$A^T = A^T(AA^+)^T = A^TAA^+ = A^TA(A^+AB^+) = A^T(AA^+A)B^+$

$= A^TAB^+$.

Similarly $A^TBA^+ = A^T$

Therefore, $A^TBA^+ = A^T = A^TAB^+$.

(iv) $\Rightarrow$ (i)

Take

$A^T = A^TAB^+$

$A = AB^+A$

$AB^+ = (AB^+A)B^+ = A(B^+AB^+) = AA^+$

Similarly, $A^+A = A^+B$ .Therefore by Theorem 3.1, (iv) $\Rightarrow$(i) holds.

Corollary 3.4: If $A \leq^{-} B$ and A^+, B^+ both exist, then $B^+ \in A\{1\}$

Proof:

$A = AA^+A = AB^+A$ by Theorem 3.3

$\Rightarrow B^+ \in A\{1\}$.

Theorem 3.5:

If $A \in Ғ_{mn}$ and A^+ exist, then we have

i) $(AA^T)^+$, $(A^TA)^+$ are also exist and $(AA^T)^+ = (A^+)^T A^+$, $(A^TA)^+ = A^+(A^+)^T$

ii)$(AA^+)^+$, $(A^+A)^+$ are also exist and $(AA^+)^+ = AA^+$, $(A^+A)^+ = A^+A$

Proof:

i) $AA^T = AA^+AA^T = (A^+)^T A^TAA^T = (A^+)^T (AA^+A)^T AA^T = (A^+)^T A^+AA^TAA^T$

$\Rightarrow (A^+)^T A^+$ is a solution of $AA^T = XAA^TAA^T$

Therefore, $(A^+)^T A^+ \in AA^T \{1,3\}$

Since, AA^T is symmetric

$(A^+)^T A^+ \in AA^T \{1,4\}$

By Theorem 2.3 and 2.4

$(AA^T)^+$ exist and

$(AA^T)^+ = ((A^+)^T A^+)^T AA^T((A^+)^T A^+)^T$

$= (A^+)^T A^+A A^T(A^+)^T A^+$

$= (A^+)^T A^+A (A^+A)^T A^+$

$= (A^+)^T A^+A A^+A A^+ = (A^+)^T A^+A A^+ = (A^+)^T A^+$

Similarly, $(A^TA)^+$ exist and $(A^TA)^+ = A^+(A^+)^T$

ii) $AA^+ = AA^+A A^+AA^+$

AA^+ is a solution of $AA^+ = XAA^+AA^+$

and $(AA^{+}AA^{+})^{T} = AA^{+}AA^{+}$
Therefore, $AA^{+} \in AA^{+}\ \{1,3\} \cap AA^{+}\ \{1,4\}$
Hence, $(AA^{+})^{+}$ exist
and $(AA^{+})^{+} = (AA^{+})^{T}\ AA^{+}(AA^{+})^{T} = AA^{+}AA^{+}AA^{+} = AA^{+}$
Similarly, $(A^{+}A)^{+} = A^{+}A$.

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