Int. Journal of Math. Analysis, Vol. 4, 2010, no. 36, 1753 - 1764

Fixed Points for Contractive Type Multimaps

Abdul Latif

Department of Mathematics, Faculty of Science
King Abdulaziz University, P.O. Box 80203
Jeddah 21589, Saudi Arabia
latifmath@yahoo.com

Afrah A. N. Abdou

Department of Mathematics, Science Faculty For Girls
King Abdulaziz University, P.O. Box 14884
Jeddah 21434, Saudi Arabia
aabdou@kau.edu.sa

Abstract

Without using the concept of the Hausdorff metric, we prove some results on the existence of fixed points and strict fixed points for multivalued maps satisfying some generalized Latif-Albar type conditions. Consequently, several known fixed point results either generalized or improved including the corresponding recent results of Feng and Liu [J. Math. Anal. Appl. 317 (2006) 103-112], Latif and Albar [Demonstratio Mathematica 41 (2008) 145-150], and Chifu and Petrusel [Fixed Point Theory and Applications, vol. 2007, Article ID 34248, 8 pages].

Mathematics Subject Classification: 47H09, 54H25

Keywords: Fixed point, multivalued contraction, w-distance, metric space

1 Introduction

Let (X, d) be a metric space. Let 2^X be denote a collection of nonempty subsets of X, $Cl(X)$ a collection of nonempty closed subsets of X, $B(X)$ a collection of nonempty bounded subsets of X and $CB(X)$ a collection of nonempty closed bounded subsets of X. Let H be the Hausdorff metric with respect to d, that is,

$$H(A,B) = \max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\},$$

for every $A, B \in CB(X)$, where $d(x,B) = \inf_{y\in B} d(x,y)$.

An element $x \in X$ is called a *fixed point* of a multivalued map $T : X \to 2^X$ if $x \in T(x)$. And $x \in X$ is called a *strict fixed point* of the map T if $T(x) = \{x\}$. We denote $Fix(T) = \{x \in X : x \in T(x)\}$ and $SFix(T) = \{x \in X : T(x) = \{x\}\}$. A map $f : X \to \mathbb{R}$ is called *lower semicontinuous* if for any sequence $\{x_n\} \subset X$ with $x_n \to x \in X$ imply that $f(x) \leq \liminf_{n\to\infty} f(x_n)$.

Using the concept of Hausdorff metric, Nadler [9] established the following multivalued version of the Banach contraction principle.

Theorem 1.1 *Let (X,d) be a complete metric space and let $T : X \to CB(X)$ be a map such that for a fixed constant $h \in (0,1)$ and for each $x, y \in X$,*

$$H(T(x), T(y)) \leq h\ d(x,y).$$

Then $Fix(T) \neq \emptyset$.

Many authors have been using the Hausdorff metric to obtain fixed point results for multivalued maps. But, in fact for most cases the existence part of the results can be proved without using the concept of Hausdorff metric. Recently, Feng and Liu [4] extended Nadler's fixed point theorem without using the concept of the Hausdorff metric. They proved the following result.

Theorem 1.2 *Let (X,d) be a complete metric space and let $T : X \to Cl(X)$ be a map such that for any fixed constants $h, b \in (0,1)$, $h < b$, and for each $x \in X$ there is $y \in T(x)$ satisfying the following conditions:*

$$bd(x,y) \leq d(x,T(x))$$

and

$$d(y,T(y)) \leq hd(x,y),$$

Then $Fix(T) \neq \emptyset$ provided a real-valued function g on X, $g(x) = d(x,T(x))$ is lower semi-continuous.

On the other hand, Kada et al. [5] introduced the concept of w-distance on a metric space as follows:

A function $\omega : X \times X \to [0, \infty)$ is called *w-distance* on X if it satisfies the following for each $x,\ y,\ z \in X$:

(w_1) $\omega(x, z) \leq \omega(x, y) + \omega(y, z)$;

(w_2) a map $\omega(x, .) : X \to [0, \infty)$ is lower semicontinuous;

(w_3) for any $\epsilon > 0$, there exists $\delta > 0$ such that $\omega(z, x) \leq \delta$ and $\omega(z, y) \leq \delta$ imply $d(x, y) \leq \epsilon$.

Note that, in general for $x, y \in X$, $\omega(x, y) \neq \omega(y, x)$ and not either of the implications $\omega(x, y) = 0 \Leftrightarrow x = y$ necessarily hold. Clearly, the metric d is a w-distance on X. Let $(Y, \|.\|)$ be a normed space. Then the functions $\omega_1, \omega_2 : Y \times Y \to [0, \infty)$ defined by $\omega_1(x, y) = \|y\|$ and $\omega_2(x, y) = \|x\| + \|y\|$ for all $x, y \in Y$ are w-distances [5]. Many other examples and properties of the w-distance can be found in [11, 12, 13].

In the sequel we also use the following notations:
Let $x \in X$ and $A \in 2^X$. We denote, $\omega(x, A) = \inf_{y \in A} \omega(x, y)$, and $\delta_\omega(x, A) = \sup_{y \in A} \omega(x, y)$, and for $r > 0$, $\tilde{B}(x, r) = \{y \in X : \omega(x, y) \leq r\}$.

Recently, using the concept of w-distance, Suzuki and Takahashi [11] improved the Nadler's fixed point result as follows:

Theorem 1.3 *Let (X, d) be a complete metric space and let ω be a w-distance on X. Let $T : X \to Cl(X)$ be a map such that any constant $h \in (0, 1)$ and for each $x, y \in X,\ u \in T(x)$ there is $v \in T(y)$ satisfying the following condition*

$$\omega(u, v) \leq h\, \omega(x, y).$$

Then $Fix(T) \neq \emptyset$.

Most recently, Latif and Albar [7] proved the following result which is an improved version of Theorem 1.2.

Theorem 1.4 *Let (X, d) be a complete metric space and let ω be a w-distance on X. Let $T : X \to Cl(X)$ be a map such for any fixed constants $h, b \in (0, 1)$, $h < b$, and for any $x \in X$ there is $y \in T(x)$ satisfying*

$$b\omega(x, y) \leq \omega(x, T(x))$$

and

$$\omega(y, T(y)) \leq h\omega(x, y).$$

Then $Fix(T) \neq \emptyset$ *provided a real-valued function* g *on* X, $g(x) = \omega(x, T(x))$ *is lower semi-continuous.*

Many other results with respect to w-distance can be found in the literature, for example, see [1, 6, 8, 10, 12, 13, 14].

The following lemmas concerning w-distance are crucial for the proofs of our results.

Lemma 1.5 ([5]) *Let* $\{x_n\}$ *and* $\{y_n\}$ *be sequences in* X *and let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be sequences in* $[0, \infty)$ *converging to* 0. *Then, for the w-distance* ω *on* X *the following hold for every* $x, y, z \in X$:

(a) *if* $\omega(x_n, y) \leq \alpha_n$ *and* $\omega(x_n, z) \leq \beta_n$ *for any* $n \in \mathbb{N}$, *then* $y = z$; *in particular, if* $\omega(x, y) = 0$ *and* $\omega(x, z) = 0$, *then* $y = z$;

(b) *if* $\omega(x_n, y_n) \leq \alpha_n$ *and* $\omega(x_n, z) \leq \beta_n$ *for any* $n \in \mathbb{N}$, *then* $\{y_n\}$ *converges to* z;

(c) *if* $\omega(x_n, x_m) \leq \alpha_n$ *for any* $n, m \in \mathbb{N}$ *with* $m > n$, *then* $\{x_n\}$ *is a Cauchy sequence;*

(d) *if* $\omega(y, x_n) \leq \alpha_n$ *for any* $n \in \mathbb{N}$, *then* $\{x_n\}$ *is a Cauchy sequence.*

Lemma 1.6 ([8]) *Let* K *be a closed subset of* X *and* ω *be a w-distance on* X. *Suppose that there exists* $u \in X$ *such that* $\omega(u, u) = 0$. *Then* $\omega(u, K) = 0 \Leftrightarrow u \in K$.

In the present paper, we study the existence of fixed points and strict fixed points for multivalued maps satisfying some generalized Latif-Albar type conditions. Our results either generalize or improve a number of fixed point results including the corresponding results of Feng and Liu [4], Latif and Albar [7], Chifu and Petrusel [2].

2 Main Results

First we prove a local version of the main result of Latif and Albar [7, Theorem 3.3].

Theorem 2.1 *Let* (X, d) *be a complete metric space and let* ω *be a w-distance on* X *such that* $x_0 \in \tilde{B}(x_0, r)$ *for* $x_0 \in X$. *Let* $T : X \to Cl(X)$ *be a multivalued* *map . Suppose that*

(I) *there exist constants* $h, b \in (0,1)$ *with* $h < b$ *such that for each* $x \in \tilde{B}(x_0, r)$ *there exists* $y \in T(x)$ *satisfying the following two conditions*

$$b\omega(x, y) \leq \omega(x, T(x))$$

and

$$\omega(y, T(y)) \leq h\omega(x, y)$$

(II) $$\omega(x_0, T(x_0)) \leq (b - h)r,$$

(III) *a map* $g : X \to [0, \infty)$ *with* $g(x) = \omega(x, T(x))$ *is lower semicontinuous.*

Then there exists $v_0 \in \tilde{B}(x_0, r)$ *such that* $g(v_0) = 0$*. Moreover, if* $\omega(v_0, v_0) = 0$ *then* $Fix(T) \cap \tilde{B}(x_0, r) \neq \emptyset$*.*

Proof. Since $x_0 \in \tilde{B}(x_0, r)$, it follows from (I) and (II) that there exists $x_1 \in T(x_0)$ such that

$$\omega(x_0, x_1) \leq \frac{1}{b}\omega(x_0, T(x_0)) \leq (1 - \frac{h}{b})r,$$

and

$$\omega(x_1, T(x_1)) \leq h\omega(x_0, x_1) \leq \frac{h}{b}\omega(x_0, T(x_0)).$$

Put $\frac{h}{b} = \lambda$. Since $\lambda < 1$, we get $x_1 \in \tilde{B}(x_0, r)$. Thus there exists $x_2 \in T(x_1)$ such that

$$\omega(x_1, x_2) \leq \frac{1}{b}\omega(x_1, T(x_1)) \leq \lambda\omega(x_0, x_1) \leq \lambda(1 - \lambda)r,$$

and

$$\omega(x_2, T(x_2)) \leq h\omega(x_1, x_2) \leq \lambda\omega(x_1, T(x_1)) \leq \lambda^2\omega(x_0, T(x_0)).$$

Note that

$$\omega(x_0, x_2) \leq \omega(x_0, x_1) + \omega(x_1, x_2) \leq (1 - \lambda^2)\, r,$$

and thus $x_2 \in \tilde{B}(x_0, r)$. Continuing this process, we get a sequence $\{x_n\}$ in $\tilde{B}(x_0, r)$ such that $x_n \in T(x_{n-1})$ for all $n \geq 1$ and satisfying the following inequalities.

(i) $\omega(x_n, T(x_n)) \leq \omega(x_{n-1}, T(x_{n-1}))$

(ii) $\omega(x_n, T(x_n)) \leq \lambda^n\omega(x_0, T(x_0))$,

(iii) $\omega(x_n, x_{n+1}) \leq \lambda^n\omega(x_0, x_1)$.

Since $\lambda < 1$, it follows from (i) and (ii) that the sequence $\{\omega(x_n, T(x_n))\}$

is decreasing to 0. Now we show that $\{x_n\}$ is a Cauchy sequence. For any $n, m \in \mathbb{N}$ with $m > n$

$$\begin{aligned} \omega(x_n, x_m) &\leq \omega(x_n, x_{n+1}) + \omega(x_{n+1}, x_{n+2}) + \dots\dots\dots + \omega(x_{m-1}, x_m) \\ &\leq \lambda^n \omega(x_0, x_1) + \lambda^{n+1}\omega(x_0, x_1) + \dots\dots + \lambda^{m-1}\omega(x_0, x_1) \\ &\leq \frac{\lambda^n}{1-\lambda}\omega(x_0, x_1). \end{aligned}$$

and thus by Lemma 1.5 , $\{x_n\}$ is a Cauchy sequence in the closed set $\tilde{B}(x_0, r) \subset X$. Since $\tilde{B}(x_0, r)$ is complete, there exists $v_0 \in \tilde{B}(x_0, r)$ such that the sequence $\{x_n\}$ converges to v_0. Since the function g is lower semicontinuous we have

$$0 \leq g(v_0) \leq \liminf_{n\to\infty} g(x_n) = \liminf_{n\to\infty} \omega(x_n, T(x_n)) = 0.$$

Thus, $g(v_0) = \omega(v_0, T(v_0)) = 0$. Since $\omega(v_0, v_0) = 0$ and $T(v_0)$ is closed, it follows from Lemma 1.6 that $v_0 \in T(v_0)$ and hence $Fix(T) \cap \tilde{B}\ (x_0, r) \neq \emptyset$.

Now, we prove a local version of the result in [7, Theorem 3.5].

Theorem 2.2 *Suppose that all the hypothesis of Theorem 2.1 except (III) hold. Assume that* $\inf\{\omega(x, v) + \omega(x, T(x)) : x \in X\} > 0$, *for every* $v \in X$ *with* $v \notin T(v)$. *Then* $Fix(T) \cap \tilde{B}(x_0, r) \neq \emptyset$.

Proof. Following the proof of Theorem 2.1, there exists a Cauchy sequence $\{x_n\} \subset \tilde{B}(x_0, r)$ with $x_n \in T(x_{n-1})$ and $v_0 \in \tilde{B}(x_0, r)$ such that $\lim_{n\to\infty} x_n = v_0$. Moreover, For any $n, m \in \mathbb{N}$ with $m > n$

$$\omega(x_n, x_m) \leq \frac{\lambda^n}{1-\lambda}\omega(x_0, x_1).$$

and

$$\omega(x_n, T(x_n)) \leq \omega(x_n, x_{n+1}) \leq \lambda^n \omega(x_0, x_1).$$

Since $\omega(x_n, .)$ is lower semicontinuous, we get

$$\omega(x_n, , v_0) \leq \liminf_{m\to\infty} \omega(x_n, x_m) < \frac{\lambda^n}{1-\lambda}\omega(x_0, x_1),$$

Now, suppose that $v_0 \notin T(v_0)$, then we have

$$\begin{aligned} 0 &< \inf\{\omega(x, v_0) + \omega(x, T(x)) : x \in X\} \\ &\leq \inf\{\omega(x_n, v_0) + \omega(x_n, T(x_n)) : n \in \mathbb{N}\} \\ &\leq \inf\{\frac{\lambda^n}{1-\lambda}\omega(x_0, x_1) + \lambda^n \omega(x_0, x_1) : n \in \mathbb{N}\} \\ &= \inf\{\lambda^n : n \in \mathbb{N}\}\frac{2-\lambda}{1-\lambda}\omega(x_0, x_1)\} = 0, \end{aligned}$$

which is impossible, and thus $v_0 \in T(v_0)$. Since $v_0 \in \tilde{B}(x_0, r)$, we get $Fix(T) \cap \tilde{B}(x_0, r) \neq \emptyset$.

Remark A. (a) In particular if we take $r = \infty$, then Theorem 3.3 and Theorem 3.5 of [7] follow from our Theorem 2.1 and Theorem 2.2 respectively.
(b) Our above results generalize Theorem 1.2 and Theorem 1.3. And also generalize the main fixed point result of Chifu and Petrusel [2, Theorem 2.1].

Now, we prove the following fixed point result.

Theorem 2.3 *Let (X, d) be a complete metric space and let ω be a w-distance on X such that $x_0 \in \tilde{B}(x_0, r)$ for $x_0 \in X$. Let $T : X \to Cl(X)$ be a multivalued map . Suppose that*
(I) *there exist constants b, q, $h \in (0, 1)$ with $h < b$, $\frac{h}{b} + q < 1$ such that for each $x \in \tilde{B}(x_0, r)$ there exists $y \in T(x)$ satisfying the following two conditions*

$$b\omega(x, y) \leq \omega(x, T(x))$$

and

$$\omega(y, T(y)) \leq h\omega(x, y) + q\omega(x, T(x)).$$

(II) $$\omega(x_0, T(x_0)) \leq (b - (h + bq))r,$$

(III) *a map $g : X \to [0, \infty)$ with $g(x) = \omega(x, T(x))$ is lower semicontinuous.*

Then there exists $v_0 \in \tilde{B}(x_0, r)$ such that $g(v_0) = 0$. Moreover, if $\omega(v_0, v_0) = 0$ then $Fix(T) \cap \tilde{B}(x_0, r) \neq \emptyset$.

Proof. Since $x_0 \in \tilde{B}(x_0, r)$, it follows from (I) and (II) that there exists $x_1 \in T(x_0)$ such that

$$\omega(x_0, x_1) \leq \frac{1}{b}\omega(x_0, T(x_0)) \leq (1 - (\frac{h}{b} + q))r,$$

and

$$\omega(x_1, T(x_1)) \leq h\omega(x_0, x_1) + q\omega(x_0, T(x_0)) \leq (\frac{h}{b} + q)\omega(x_0, T(x_0))$$

Note that $\omega(x_1, T(x_1)) \leq \omega(x_0, T(x_0))$ and $x_1 \in \tilde{B}(x_0, r)$. Thus, there exists $x_2 \in T(x_1)$ such that

$$\begin{aligned} \omega(x_1, x_2) &\leq \frac{1}{b}\omega(x_1, T(x_1)) \\ &\leq \frac{1}{b}(\frac{h}{b} + q)\omega(x_0, T(x_0)) \\ &\leq (\frac{h}{b} + q)(1 - (\frac{h}{b} + q))r. \end{aligned}$$

and

$$\begin{aligned}\omega(x_2, T(x_2)) &\le h\omega(x_1, x_2) + q\omega(x_1, T(x_1)) \\ &\le \left(\frac{h}{b} + q\right)\omega(x_1, T(x_1)) \\ &\le \left(\frac{h}{b} + q\right)^2 \omega(x_0, T(x_0)).\end{aligned}$$

Note that $\omega(x_2, T(x_2)) \le \omega(x_1, T(x_1))$. Moreover,

$$\begin{aligned}\omega(x_0, x_2) &\le \omega(x_0, x_1) + \omega(x_1, x_2) \\ &\le (1 - (\frac{h}{b} + q))r + (\frac{h}{b} + q)(1 - (\frac{h}{b} + q))r \\ &= [1 - \left(\frac{h}{b} + q\right)^2]r.\end{aligned}$$

Thus $x_2 \in \tilde{B}(x_0, r)$. Continuing this process, we get a sequence $\{x_n\} \subset \tilde{B}(x_0, r)$ such that $x_{n+1} \in T(x_n)$ and satisfying the following properties.

(i) $\omega(x_n, T(x_n)) \le \omega(x_{n-1}, T(x_{n-1}))$

(ii) $\omega(x_n, T(x_n)) \le \lambda^n \omega(x_0, T(x_0))$

(iii) $\omega(x_n, x_{n+1}) \le \frac{\lambda^n}{b}\omega(x_0, x_1)$,

where $\lambda = \frac{h}{b} + q$. The rest of the proof runs similar as the proof of Theorem 2.1 and hence we get $Fix(T) \cap \tilde{B}(x_0, r) \neq \emptyset$.

Using the same method as the proof of Theorem 2.2, we can get the following fixed point result.

Theorem 2.4 *Suppose that all the hypothesis of Theorem 2.3 except (III) hold. Assume that* $\inf\{\omega(x, v) + \omega(x, T(x)) : x \in X\} > 0$, *for every* $v \in X$ *with* $v \notin T(v)$. *Then* $Fix(T) \cap \tilde{B}(x_0, r) \neq \emptyset$.

Remark B. Theorem 2.3 and Theorem 2.4 generalize the fixed point result in [2, Theorem 2.5].

Finally, we prove a result on the existence of strict fixed points.

Theorem 2.5 *Let* (X, d) *be a complete metric space with a w-distance* ω, *and let* $T : X \to B(X)$ *be a multivalued map. Suppose that*

(I) *there exist constants* b, $h \in (0, 1)$ *with* $h < b$ *such that for each* $x \in X$

there exists $y \in T(x)$ *satisfying the following two conditions*

$$b\delta_\omega(x, T(x)) \le \omega(x, y)$$

and

$$\delta_\omega(y, T(y)) \le h \max\{\delta_\omega(x, T(x)), \frac{1}{2}\omega(x, T(y))\}.$$

(II) *a map* $f : X \to [0, \infty)$ *with* $f(x) = \delta_\omega(x, T(x))$ *is lower semicontinuous.*

Then $SFix(T) \neq \emptyset$.

Proof. Let $x_0 \in X$. Suppose that $\delta_\omega(x_0, T(x_0)) > 0$. Then for any $x_0 \in X$, there exists $x_1 \in T(x_0)$ such that the following hold.

$$b\delta_\omega(x_0, T(x_0)) \le \omega(x_0, x_1) \tag{1}$$

and

$$\delta_\omega(x_1, T(x_1)) \le h \max\{\delta_\omega(x_0, T(x_0)), \frac{1}{2}\omega(x_0, T(x_1))\}. \tag{2}$$

Now, here we have two cases:
Case I if $\frac{1}{2}\omega(x_0, T(x_1)) \le \delta_\omega(x_0, T(x_0))$, then we get

$$\delta_\omega(x_1, T(x_1)) \le h\delta_\omega(x_0, T(x_0)) \le \frac{h}{b}\omega(x_0, x_1),$$

Case II if $\delta_\omega(x_0, T(x_0)) \le \frac{1}{2}\omega(x_0, T(x_1))$, then we get

$$\delta_\omega(x_1, T(x_1)) \le \frac{h}{2}\omega(x_0, T(x_1)).$$

Now, note that

$$\begin{aligned}
\omega(x_0, T(x_1)) &\le \omega(x_0, T(x_0)) + \omega(T(x_0), T(x_1)) \\
&\le \delta_\omega(x_0, T(x_0)) + \omega(x_1, T(x_1)) \\
&\le \frac{1}{b}\omega(x_0, x_1) + \delta_\omega(x_1, T(x_1)) \\
&\le \frac{1}{b}\omega(x_0, x_1) + \frac{h}{2}\omega(x_0, T(x_1)) \\
&\le \frac{2}{b(2-h)}\omega(x_0, x_1).
\end{aligned}$$

and thus

$$\delta_\omega(x_1, T(x_1)) \le \frac{h}{b(2-h)}\omega(x_0, x_1).$$

Hence for the both cases (2) becomes

$$\delta_\omega(x_1, T(x_1)) \leq \max\{1, \frac{1}{2-h}\}\frac{h}{b}\omega(x_0, x_1) \;=\; \frac{h}{b}\omega(x_0, x_1).$$

Continuing this process we get a sequence $\{x_n\}$ in X such that $x_{n+1} \in T(x_n)$ satisfying the following properties

$$b\delta_\omega(x_n, T(x_n)) \leq \omega(x_n, x_{n+1})$$

and

$$\delta_\omega(x_n, T(x_n)) \leq h \max\{\delta_\omega(x_{n-1}, T(x_{n-1})), \frac{1}{2}\omega(x_{n-1}, T(x_n))\} \leq \frac{h}{b}\omega(x_{n-1}, x_n).$$

Now,

$$\begin{aligned} \omega(x_n, x_{n+1}) &\leq \delta_\omega(x_n, T(x_n)) \\ &\leq \frac{h}{b}\omega(x_{n-1}, x_n) \\ &\leq \left(\frac{h}{b}\right)^2 \omega(x_{n-2}, x_{n-1}) \\ &\vdots \\ &\leq \left(\frac{h}{b}\right)^n \omega(x_0, x_1), \end{aligned}$$

since $\frac{h}{b} < 1$, we get $\{x_n\}$ is a Cauchy sequence in a complete metric space X. Let $x_n \to z \in X$. Note that

$$f(x_n) = \delta_\omega(x_n, T(x_n)) \leq \frac{1}{b}\omega(x_n, x_{n+1}) \leq \frac{1}{b}\left(\frac{h}{b}\right)^n \omega(x_0, x_1).$$

Thus, $f(x_n) \to 0$ as $n \to \infty$. Since f is lower semicontinuous, we have

$$0 \leq f(z) \leq \liminf_{n\to\infty} f(x_n) = 0.$$

Thus $f(z) = \delta_\omega(z, T(z)) = 0$ and hence $T(z) = \{z\}$.

Remark C. Theorem 2.5 generalizes strict fixed point results of [2] and [3].

References

[1] Q. H. Ansari, Vectorial form of Ekeland-type variational priciple with applications to vector equilibrium problems and fixed point theory, J. Math. Anal. Appl. 334 (2007) 561-575.

[2] C. Chifu and G. Petrusel, Existence and data dependence of fixed points and strict fixed points for contractive-type multivalued operators, Fixed Point Theory and Applications, vol. 2007, Article ID 34248, 8 pages, 2007. doi:10.1155/2007/34248

[3] L. B. Ciric, Generalized contractions and fixed point theorems, Publications de l'Institut Mathematique. Nouvelle Serie, 12 (1971), 19-26.

[4] Y. Feng and S. Liu, Fixed point theorems for multivalued contractive mappings and multivalued Caristi Type mappings, J. Math. Anal. Appl. 317 (2006) 103-112.

[5] O. Kada, T. Susuki and W. Takahashi, Nonconvex minimization theorems and fixed point theorems in complete metric spaces, Math. Japon. 44 (1996) 381-391.

[6] A. Latif and A A. N. Abdou, Fixed points of generalized contractive maps, Fixed Point Theory and Applications, vol. 2009, Article ID 487161, 9 pages, 2009. doi:10.1155/2009/487161.

[7] A. Latif and W. A. Albar, Fixed point results in complete metric spaces, Demonstratio Mathematica 41 (2008) 145-150.

[8] L.J. Lin and W.S. Du, Some equivalent formulations of the generalized Ekland's variational principle and their applications, Nonlinear Anal. 67 (2007) 187-199.

[9] S. B. Nadler, Multivalued contraction mappings, Pacific J. Math. 30 (1969) 475-488.

[10] S. Park, On generalizations of the Ekland-type variational principles, Nonlinear Anal. 39 (2000) 881-889.

[11] T. Suzuki and W. Takahashi, Fixed point Theorems and characterizations of metric completeness, Topol. Methods Nonlinear Anal. 8 (1996) 371-382.

[12] T. Suzuki, Several fixed point theorems in complete metric spaces, Yokohama Math. J. 44 (1997) 61-72.

[13] W. Takahashi, Nonlinear Functional Analysis: Fixed point theory and its applications, Yokohama Publishers, 2000.

[14] J. S. Ume, B. S. Lee and S. J. Cho, Some results on fixed point theorems for multivalued mappings in complete metric spaces, IJMMS 30 (2002) 319-325.

Received: March, 2010