Some Coefficient Inequalities for Certain Subclasses of Analytic Functions with respect to \(k\)-Symmetric Points

V. G. Shanthi\(^1\), B. Srutha Keerthi\(^2\) and B. Adolf Stephen\(^3\)

\(^1\) Department of Mathematics
S.D.N.B. Vaishnav College for Women
Chromepet, Chennai - 600 044, India
vg.shanthi1@gmail.com

\(^2\) Department of Applied Mathematics
Sri Venkateswara College of Engineering
Sriperumbudur, Chennai - 602 105, India
sruthilaya06@yahoo.co.in

\(^3\) Department of Mathematics
Madras Christian College, Tambaram
Chennai - 600 059, India
adolfmcc2003@yahoo.co.in

Abstract

In the present paper, we introduce two new subclasses \(B_{\lambda}^{(k)}(\alpha)\) and \(L_{\lambda}^{(k)}(\alpha)\) of analytic functions with respect to \(k\)-symmetric points. Some coefficient inequalities for functions belonging to these classes and their subclasses with positive coefficients are provided.

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1 Introduction

Let \( \mathcal{A} \) denote the class of functions of the form

\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \]
which are analytic in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$.

Let $\mathcal{M}(\alpha)$ be the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy the inequality

$$\mathcal{R} \left\{ \frac{zf'(z)}{f(z)} \right\} < \alpha \quad (z \in \Delta),$$

for some $\alpha \ (\alpha > 1)$ and let $\mathcal{N}(\alpha)$ be the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy the inequality

$$\mathcal{R} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} < \alpha \quad (z \in \Delta),$$

for some $\alpha \ (\alpha > 1)$. The classes $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$ were introduced and investigated recently by Owa and Nishiwaki [1] (see also Srivastava and Attiya [2]).

Motivated by $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$, the following two subclasses of analytic functions with respect to $k$-symmetric points were introduced and some interesting results were obtained by Zhi-Gang Wang et al. [3].

A function $f(z) \in \mathcal{A}$ is in the class $\mathcal{M}^{(k)}(\alpha)$ if

$$\mathcal{R} \left\{ \frac{zf'(z)}{f_k(z)} \right\} < \alpha \quad (z \in \Delta),$$

where $\alpha > 1, \ k \geq 1$ is a fixed positive integer and $f_k(z)$ is defined by the following equality

$$f_k(z) = \frac{1}{k} \sum_{\nu=0}^{k-1} \varepsilon^{-\nu} f(\varepsilon^\nu z), \quad (\varepsilon^k = 1; z \in \Delta) \quad (1)$$

And a function $f(z) \in \mathcal{A}$ is in the class $\mathcal{N}^{(k)}(\alpha)$ if and only if $zf'(z) \in \mathcal{M}^{(k)}(\alpha)$.

We now provide the following two classes $\mathcal{M}_1^{(k)}(\alpha)$ and $\mathcal{N}_1^{(k)}(\alpha)$, which are subclasses with positive coefficients of the classes $\mathcal{M}^{(k)}(\alpha)$ and $\mathcal{N}^{(k)}(\alpha)$ respectively.

$$\mathcal{M}_1^{(k)}(\alpha) = \left\{ f(z) \in \mathcal{M}^{(k)}(\alpha) : f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \text{ with } a_n \geq 0 (n \geq 2) \right\}$$

and

$$\mathcal{N}_1^{(k)}(\alpha) = \left\{ f(z) \in \mathcal{N}^{(k)}(\alpha) : f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \text{ with } a_n \geq 0 (n \geq 2) \right\} .$$

The subclasses $\mathcal{M}_1^{(k)}(\alpha)$ and $\mathcal{N}_1^{(k)}(\alpha)$ were introduced and studied by Zhi-Gang Wang et al. [3].
Definition 1.1 A function $f(z) \in A$ is in the class $B_{\lambda}^{(k)}(\alpha)$ if
\[
\Re \left\{ \frac{z^{(1-\lambda)} f'(z)}{[f_k(z)]^{(1-\lambda)}} \right\} < \alpha \quad (\lambda \geq 0, z \in \Delta)
\]
where $\alpha > 1$, $k \geq 1$ is a fixed positive integer and $f_k(z)$ is given by (1).

Definition 1.2 A function $f(z) \in A$ is in the class $L_{\lambda}^{(k)}(\alpha)$ if
\[
\Re \left\{ \frac{zf'(z) + \lambda z^2 f''(z)}{f_k(z)} \right\} < \alpha \quad (\lambda \geq 0, z \in \Delta)
\]
where $\alpha > 1$, $k \geq 1$ is a fixed positive integer and $f_k(z)$ is given by (1).

In the present paper, we shall provide some coefficient inequalities for functions belonging to the classes $B_{\lambda}^{(k)}(\alpha)$ and $L_{\lambda}^{(k)}(\alpha)$ and their subclasses with positive coefficients.

2 Main Results

Theorem 2.1 Let $\alpha > 1$. If $f(z) \in A$ satisfies
\[
\sum_{n=1}^{\infty} [(nk + 1) + |(nk + 1) - 2\alpha(1 - \lambda)|] |a_{nk+1}| + \sum_{n=2 \atop n \not\equiv k+1}^{\infty} 2n|a_n| \leq 2(\alpha - 1) \quad (2)
\]
then $f(z) \in B_{\lambda}^{(k)}(\alpha)$.

Proof Suppose that $f(z) \in A$ with $\alpha > 1$, it suffices to show that,
\[
\left| \frac{z^{(1-\lambda)} f'(z)}{[f_k(z)]^{(1-\lambda)}} \right| < \left| \frac{z^{(1-\lambda)} f'(z)}{[f_k(z)]^{(1-\lambda)}} - 2\alpha \right|, \quad (z \in \Delta).
\]

Let $M$ be denoted by
\[
M = \left| z^{(1-\lambda)} f'(z) \right| - \left| z^{(1-\lambda)} f'(z) - 2\alpha f_k(z)^{(1-\lambda)} \right| = z^{(1-\lambda)} + \sum_{n=2}^{\infty} n a_n z^{n-\lambda} - z^{(1-\lambda)} + \sum_{n=2}^{\infty} n a_n z^{n-\lambda} - 2\alpha z^{(1-\lambda)} - 2\alpha(1 - \lambda) \sum_{n=2}^{\infty} a_n b_n z^{(n-\lambda)}
\]
where $b_n = \frac{1}{k} \sum_{\nu=0}^{k-1} \epsilon^{(n-1)\nu}$, $(\epsilon^k = 1)$.
Thus, for $|z| = r < 1$, we have,

$$M \leq r^{1-\lambda} + \sum_{n=2}^{\infty} n |a_n| r^{n-\lambda}$$

$$- \left[ (2\alpha - 1) r^{1-\lambda} - \sum_{n=2}^{\infty} |n - 2\alpha (1 - \lambda) b_n| |a_n| r^{n-\lambda} \right]$$

$$< \left\{ \sum_{n=2}^{\infty} [n + |n - 2\alpha (1 - \lambda) b_n| |a_n| - 2(\alpha - 1) \right\} r$$

(3)

From the definition of $b_n$, we know

$$b_n = \begin{cases} 1, & n = \ell k + 1 \\ 0, & n \neq \ell k + 1 \end{cases}$$

(4)

Substituting (4) into inequality (3), we get,

$$M < \left\{ \sum_{n=1}^{\infty} [(nk + 1) + |(nk + 1) - 2\alpha (1 - \lambda)| |a_{nk+1}|$$

$$+ \sum_{n=2}^{\infty} \sum_{n \neq \ell k + 1} 2n |a_n| - 2(\alpha - 1) \right\} r$$

From (2), we know that $M < 0$. Thus we have,

$$\Re \left\{ \frac{z^{1-\lambda} f'(z)}{[f_k(z)]^{(1-\lambda)}} \right\} < \alpha, \quad (z \in \Delta),$$

that is $f(z) \in B_{\lambda}^{(k)}(\alpha)$. This completes the proof of Theorem 2.1.

**Theorem 2.2** Let $\alpha > 1$. If $f(z) \in \mathcal{A}$ satisfies

$$\sum_{n=1}^{\infty} [(nk + 1) + \lambda (nk + 1)(nk)) + |(nk + 1) + \lambda (nk + 1)(nk) - 2\alpha|] |a_{nk+1}|$$

$$+ \sum_{n=2}^{\infty} \sum_{n \neq \ell k + 1} 2[n + \lambda n(n - 1)] |a_n| \leq 2(\alpha - 1)$$

(5)

then $f(z) \in L_{\lambda}^{(k)}(\alpha)$. 
The proof of Theorem 2.2 is similar to Theorem 2.1, so the details are omitted.

**Corollary 2.3** By substituting $\lambda = 0$ in Theorem 2.1 and Theorem 2.2, we have, for $\alpha > 1$, $f(z) \in A$ satisfies

$$\sum_{n=1}^{\infty} [(nk + 1) + |(nk + 1) - 2\alpha|] |a_{nk+1}| + \sum_{n=2}^{\infty} 2n|a_n| \leq 2(\alpha - 1) \quad (6)$$

then $f(z) \in M^{(k)}(\alpha)$ which was studied by Zhi-Gang Wang et al. [3].

We now provide the necessary and sufficient coefficient conditions for the following two classes $B_{\lambda,1}^{(k)}(\alpha)$ and $L_{\lambda,1}^{(k)}(\alpha)$, which are subclasses with positive coefficients of the classes $B_{\lambda}^{(k)}(\alpha)$ and $L_{\lambda}^{(k)}(\alpha)$ respectively.

$$B_{\lambda,1}^{(k)}(\alpha) = \left\{ f(z) \in B_{\lambda}^{(k)}(\alpha) : f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \text{ with } a_n \geq 0 \ (n \geq 2) \right\}$$

and

$$L_{\lambda,1}^{(k)}(\alpha) = \left\{ f(z) \in L_{\lambda}^{(k)}(\alpha) : f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \text{ with } a_n \geq 0 \ (n \geq 2) \right\}.$$

**Theorem 2.4** Let $k \geq 2$, $1 < \alpha \leq k + 1$ and $f(z) \in A$, then $f(z) \in B_{\lambda,1}^{(k)}(\alpha)$ if and only if

$$\sum_{n=2}^{\infty} n a_n - \alpha(1 - \lambda) \sum_{\ell=1}^{\infty} a_{\ell k+1} \leq \alpha - 1$$

**Proof** In view of Theorem 2.1, we need only to prove the necessity. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in B_{\lambda,1}^{(k)}(\alpha)$, then $a_n \geq 0$ for $n \geq 2$ and

$$\Re \left\{ \frac{z^{(1-\lambda)} f'(z)}{[f_k(z)]^{(1-\alpha)}} \right\} < \alpha$$

this is equivalent to

$$\left| \frac{z^{(1-\lambda)} f'(z)}{[f_k(z)]^{(1-\alpha)}} \right| < \left| \frac{z^{(1-\lambda)} f'(z)}{[f_k(z)]^{(1-\alpha)}} - 2\alpha \right|$$

or equivalently,

$$|z^{(1-\lambda)} f'(z)| < |z^{(1-\lambda)} f'(z) - 2\alpha[f_k(z)]^{(1-\lambda)}|. $$
Hence,

\[ 1 + \sum_{n=2}^{\infty} n a_n z^{n-1} < 1 + \sum_{n=2}^{\infty} n a_n z^{n-1} - 2\alpha - 2\alpha(1 - \lambda) \sum_{\ell=1}^{\infty} a_{\ell k+1} z^{\ell k} \]

Setting \( z \to 1^- \), noting that \( a_n \geq 0 \) for \( n \geq 2 \) and \( \alpha > 1 \), we have,

\[ 1 + \sum_{n=2}^{\infty} n a_n \leq 2\alpha - 1 + 2\alpha(1 - \lambda) \sum_{\ell=1}^{\infty} a_{\ell k+1} - \sum_{n=2}^{\infty} n a_n \]

that is,

\[ \sum_{n=2}^{\infty} n a_n - \alpha(1 - \lambda) \sum_{\ell=1}^{\infty} a_{\ell k+1} \leq \alpha - 1. \]

Hence the proof of Theorem 2.4 is complete.

**Theorem 2.5** Let \( k \geq 2, 1 < \alpha \leq k + 1 \) and \( f(z) \in \mathcal{A} \), then \( f(z) \in L_{\lambda,1}^{(k)}(\alpha) \) if and only if

\[ \sum_{n=2}^{\infty} [n + \lambda n(n - 1)] a_n - \alpha \sum_{\ell=1}^{\infty} a_{\ell k+1} \leq \alpha - 1 \]

The proof of Theorem 2.5 is similar to Theorem 2.4, so the details are omitted.

**Corollary 2.6** By substituting \( \lambda = 0 \) in Theorem 2.4 and Theorem 2.5, we have for \( k \geq 2, 1 < \alpha \leq k + 1 \) and \( f(z) \in \mathcal{A} \), then \( f(z) \in M_{1}^{(k)}(\alpha) \) if and only if

\[ \sum_{n=2}^{\infty} n a_n - \alpha \sum_{\ell=1}^{\infty} a_{\ell k+1} \leq \alpha - 1 \]

which was studied by Zhi-Gang Wang et al. [3].

**References**


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