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Some Coefficient Inequalities for Certain Subclasses of Analytic Functions with respect to k-Symmetric Points

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Abstract

In the present paper, we introduce two new subclasses $B_\lambda^{(k)}(\alpha)$ and $L_\lambda^{(k)}(\alpha)$ of analytic functions with respect to k-symmetric points. Some coefficient inequalities for functions belonging to these classes and their subclasses with positive coefficients are provided.

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1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk $\Delta = \{z \in C : |z| < 1\}$.

Let $\mathcal{M}(\alpha)$ be the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy the inequality

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} < \alpha \quad (z \in \Delta),$$

for some α $(\alpha > 1)$ and let $\mathcal{N}(\alpha)$ be the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy the inequality

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \alpha \quad (z \in \Delta),$$

for some α $(\alpha > 1)$. The classes $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$ were introduced and investigated recently by Owa and Nishiwaki [1] (see also Srivastava and Attiya [2]).

Motivated by $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$, the following two subclasses of analytic functions with respect to k-symmetric points were introduced and some interesting results were obtained by Zhi-Gang Wang et al. [3].

A function $f(z) \in \mathcal{A}$ is in the class $\mathcal{M}^{(k)}(\alpha)$ if

$$\Re\left\{\frac{zf'(z)}{f_k(z)}\right\} < \alpha \quad (z \in \Delta),$$

where $\alpha > 1$, $k \geq 1$ is a fixed positive integer and $f_k(z)$ is defined by the following equality

$$f_k(z) = \frac{1}{k}\sum_{\nu=0}^{k-1} \varepsilon^{-\nu} f(\varepsilon^{\nu} z), \quad (\varepsilon^k = 1; z \in \Delta) \tag{1}$$

And a function $f(z) \in \mathcal{A}$ is in the class $\mathcal{N}^{(k)}(\alpha)$ if and only if $zf'(z) \in \mathcal{M}^{(k)}(\alpha)$.

We now provide the following two classes $\mathcal{M}_1^{(k)}(\alpha)$ and $\mathcal{N}_1^{(k)}(\alpha)$, which are subclasses with positive coefficients of the classes $\mathcal{M}^{(k)}(\alpha)$ and $\mathcal{N}^{(k)}(\alpha)$ respectively.

$$\mathcal{M}_1^{(k)}(\alpha) = \left\{f(z) \in \mathcal{M}^{(k)}(\alpha) : f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \text{ with } a_n \geq 0 (n \geq 2)\right\}$$

and

$$\mathcal{N}_1^{(k)}(\alpha) = \left\{f(z) \in \mathcal{N}^{(k)}(\alpha) : f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \text{ with } a_n \geq 0 (n \geq 2)\right\}.$$

The subclasses $\mathcal{M}_1^{(k)}(\alpha)$ and $\mathcal{N}_1^{(k)}(\alpha)$ were introduced and studied by Zhi-Gang Wang et al. [3].

Definition 1.1 *A function $f(z) \in \mathcal{A}$ is in the class $B_\lambda^{(k)}(\alpha)$ if*

$$\Re \left\{ \frac{z^{(1-\lambda)} f'(z)}{[f_k(z)]^{(1-\lambda)}} \right\} < \alpha \quad (\lambda \geq 0, z \in \Delta)$$

where $\alpha > 1$, $k \geq 1$ is a fixed positive integer and $f_k(z)$ is given by (1).

Definition 1.2 *A function $f(z) \in \mathcal{A}$ is in the class $L_\lambda^{(k)}(\alpha)$ if*

$$\Re \left\{ \frac{zf'(z) + \lambda z^2 f''(z)}{f_k(z)} \right\} < \alpha \quad (\lambda \geq 0, z \in \Delta)$$

where $\alpha > 1$, $k \geq 1$ is a fixed positive integer and $f_k(z)$ is given by (1).

In the present paper, we shall provide some coefficient inequalities for functions belonging to the classes $B_\lambda^{(k)}(\alpha)$ and $L_\lambda^{(k)}(\alpha)$ and their subclasses with positive coefficients.

2 Main Results

Theorem 2.1 *Let $\alpha > 1$. If $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=1}^{\infty} \left[(nk+1) + |(nk+1) - 2\alpha(1-\lambda)|\right] |a_{nk+1}| + \sum_{\substack{n=2 \\ n \neq \ell k+1}}^{\infty} 2n|a_n| \leq 2(\alpha - 1) \quad (2)$$

then $f(z) \in B_\lambda^{(k)}(\alpha)$.

Proof Suppose that $f(z) \in \mathcal{A}$ with $\alpha > 1$, it suffices to show that,

$$\left| \frac{z^{(1-\lambda)} f'(z)}{[f_k(z)]^{(1-\lambda)}} \right| < \left| \frac{z^{(1-\lambda)} f'(z)}{[f_k(z)]^{(1-\lambda)}} - 2\alpha \right|, \quad (z \in \Delta).$$

Let M be denoted by

$$\begin{aligned} M &= \left|z^{(1-\lambda)} f'(z)\right| - \left|z^{(1-\lambda)} f'(z) - 2\alpha [f_k(z)]^{(1-\lambda)}\right| \\ &= \left| z^{(1-\lambda)} + \sum_{n=2}^{\infty} n a_n z^{n-\lambda} \right| \\ &\quad - \left| z^{(1-\lambda)} + \sum_{n=2}^{\infty} n a_n z^{n-\lambda} - 2\alpha z^{(1-\lambda)} - 2\alpha(1-\lambda) \sum_{n=2}^{\infty} a_n b_n z^{(n-\lambda)} \right| \end{aligned}$$

where $b_n = \dfrac{1}{k} \displaystyle\sum_{\nu=0}^{k-1} \varepsilon^{(n-1)\nu}$, $(\varepsilon^k = 1)$.

Thus, for $|z| = r < 1$, we have,

$$\begin{aligned} M &\leq r^{1-\lambda} + \sum_{n=2}^{\infty} n|a_n|r^{n-\lambda} \\ &\quad - \left[(2\alpha-1)r^{1-\lambda} - \sum_{n=2}^{\infty} |n - 2\alpha(1-\lambda)b_n|\,|a_n|r^{n-\lambda}\right] \\ &< \left\{\sum_{n=2}^{\infty}[n + |n-2\alpha(1-\lambda)b_n|]|a_n| - 2(\alpha-1)\right\} r \end{aligned} \tag{3}$$

From the definition of b_n, we know

$$b_n = \begin{cases} 1, & n = \ell k + 1 \\ 0, & n \neq \ell k + 1 \end{cases} \tag{4}$$

Substituting (4) into inequality (3), we get,

$$\begin{aligned} M < \Bigg\{ &\sum_{n=1}^{\infty} [(nk+1) + |(nk+1) - 2\alpha(1-\lambda)|]\,|a_{nk+1}| \\ &+ \sum_{\substack{n=2 \\ n\neq \ell k+1}}^{\infty} 2n|a_n| - 2(\alpha-1)\Bigg\} r \end{aligned}$$

From (2), we know that $M < 0$. Thus we have,

$$\Re\left\{\frac{z^{(1-\lambda)}f'(z)}{[f_k(z)]^{(1-\lambda)}}\right\} < \alpha, \quad (z \in \Delta),$$

that is $f(z) \in B_\lambda^{(k)}(\alpha)$.

This completes the proof of Theorem 2.1.

Theorem 2.2 *Let $\alpha > 1$. If $f(z) \in \mathcal{A}$ satisfies*

$$\begin{aligned} &\sum_{n=1}^{\infty} [((nk+1) + \lambda(nk+1)(nk)) + |(nk+1) + \lambda(nk+1)(nk) - 2\alpha|]\,|a_{nk+1}| \\ &\quad + \sum_{\substack{n=2 \\ n\neq \ell k+1}}^{\infty} 2[n + \lambda n(n-1)]|a_n| \leq 2(\alpha-1) \end{aligned} \tag{5}$$

then $f(z) \in L_\lambda^{(k)}(\alpha)$.

The proof of Theorem 2.2 is similar to Theorem 2.1, so the details are omitted.

Corollary 2.3 *By substituting $\lambda = 0$ in Theorem 2.1 and Theorem 2.2, we have, for $\alpha > 1$, $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=1}^{\infty} [(nk+1) + |(nk+1) - 2\alpha|] \, |a_{nk+1}| + \sum_{\substack{n=2 \\ n \neq \ell k+1}}^{\infty} 2n|a_n| \leq 2(\alpha - 1) \tag{6}$$

then $f(z) \in \mathcal{M}^{(k)}(\alpha)$ which was studied by Zhi-Gang Wang et al. [3].

We now provide the necessary and sufficient coefficient conditions for the following two classes $B_{\lambda,1}^{(k)}(\alpha)$ and $L_{\lambda,1}^{(k)}(\alpha)$, which are subclasses with positive coefficients of the classes $B_{\lambda}^{(k)}(\alpha)$ and $L_{\lambda}^{(k)}(\alpha)$ respectively.

$$B_{\lambda,1}^{(k)}(\alpha) = \left\{ f(z) \in B_{\lambda}^{(k)}(\alpha) : f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \text{ with } a_n \geq 0 \ (n \geq 2) \right\}$$

and

$$L_{\lambda,1}^{(k)}(\alpha) = \left\{ f(z) \in L_{\lambda}^{(k)}(\alpha) : f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \text{ with } a_n \geq 0 \ (n \geq 2) \right\}.$$

Theorem 2.4 *Let $k \geq 2$, $1 < \alpha \leq k+1$ and $f(z) \in \mathcal{A}$, then $f(z) \in B_{\lambda,1}^{(k)}(\alpha)$ if and only if*

$$\sum_{n=2}^{\infty} n a_n - \alpha(1-\lambda) \sum_{\ell=1}^{\infty} a_{\ell k+1} \leq \alpha - 1$$

Proof In view of Theorem 2.1, we need only to prove the necessity. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in B_{\lambda,1}^{(k)}(\alpha)$, then $a_n \geq 0$ for $n \geq 2$ and

$$\Re \left\{ \frac{z^{(1-\lambda)} f'(z)}{[f_k(z)]^{(1-\lambda)}} \right\} < \alpha$$

this is equivalent to

$$\left| \frac{z^{(1-\lambda)} f'(z)}{[f_k(z)]^{(1-\lambda)}} \right| < \left| \frac{z^{(1-\lambda)} f'(z)}{[f_k(z)]^{(1-\lambda)}} - 2\alpha \right|$$

or equivalently,

$$|z^{(1-\lambda)} f'(z)| < |z^{(1-\lambda)} f'(z) - 2\alpha [f_k(z)]^{(1-\lambda)}|.$$

Hence,

$$\left|1+\sum_{n=2}^{\infty} na_n z^{n-1}\right| < \left|1+\sum_{n=2}^{\infty} na_n z^{n-1} - 2\alpha - 2\alpha(1-\lambda)\sum_{\ell=1}^{\infty} a_{\ell k+1} z^{\ell k}\right|$$

Setting $z \to 1^-$, noting that $a_n \geq 0$ for $n \geq 2$ and $\alpha > 1$, we have,

$$1+\sum_{n=2}^{\infty} na_n \leq 2\alpha - 1 + 2\alpha(1-\lambda)\sum_{\ell=1}^{\infty} a_{\ell k+1} - \sum_{n=2}^{\infty} na_n$$

that is,

$$\sum_{n=2}^{\infty} na_n - \alpha(1-\lambda)\sum_{\ell=1}^{\infty} a_{\ell k+1} \leq \alpha - 1.$$

Hence the proof of Theorem 2.4 is complete.

Theorem 2.5 *Let $k \geq 2$, $1 < \alpha \leq k+1$ and $f(z) \in \mathcal{A}$, then $f(z) \in L^{(k)}_{\lambda,1}(\alpha)$ if and only if*

$$\sum_{n=2}^{\infty}[n+\lambda n(n-1)]a_n - \alpha\sum_{\ell=1}^{\infty} a_{\ell k+1} \leq \alpha - 1$$

The proof of Theorem 2.5 is similar to Theorem 2.4, so the details are omitted.

Corollary 2.6 *By substituting $\lambda = 0$ in Theorem 2.4 and Theorem 2.5, we have for $k \geq 2$, $1 < \alpha \leq k+1$ and $f(z) \in \mathcal{A}$, then $f(z) \in \mathcal{M}^{(k)}_1(\alpha)$ if and only if*

$$\sum_{n=2}^{\infty} na_n - \alpha\sum_{\ell=1}^{\infty} a_{\ell k+1} \leq \alpha - 1$$

which was studied by Zhi-Gang Wang et al. [3].

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