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An Iterative Method for Equilibrium Problems of Nonexpansive Semigroups in Hilbert Spaces

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Abstract

In this paper, we introduce an iterative scheme for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive semigroups in Hilbert Spaces.We prove that the approximate solution converges strongly to a solution of variational
inequalities under some appropriate conditions imposed on the parameter.The results of this paper extend and improve the results of Wangkeeree [12] and many others.

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1 Introduction

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. And let C be a nonempty closed convex subset of H. A mapping T of C into itself is said to be *nonexpansive* if $\|Tx-Ty\| \le \|x-y\|$ for each $x, y \in C$. We denote by $F(T)$ the set of fixed points of T. Let $\mathcal{S} = \{T(s) : 0 \le s < \infty\}$ be a family mappings of C into itself is called a *nonexpansive semigroup* on C if it satisfies the following conditions:

(i) $T(0)x = x$ for all $x \in C$;

(ii) $T(s+t) = T(s)T(t)$ for all $s, t \ge 0$;

(iii) $\|T(s)x - T(s)y\| \le \|x-y\|$ for all $x, y \in C$ and $s \ge 0$;

(iv) for all $x \in C, s \mapsto T(s)x$ is continuous.

We denote by $F(\mathcal{S})$ the set of all common fixed points of $\mathcal{S}$, that is, $F(\mathcal{S}) = \{x \in C : T(s)x = x, 0 \leq s < \infty\}$. It is known that $F(\mathcal{S})$ is closed and convex.

In [11], Shioji and Takahashi introduced in a Hilbert space the implicit iteration

$$x \in C, \;\; x_n = \alpha_n x + (1-\alpha_n)\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds, \quad \forall n \in \mathbb{N} \tag{1.1}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$, $\{s_n\}$ is a sequence of positive real numbers which diverges to ∞. Under certain restrictions on the sequence $\{\alpha_n\}$, they proved strong convergence of the sequence $\{x_n\}$ to a member of $F(\mathcal{S})$.

In [10] , Shimizu and Takahashi studied the strong convergence of the sequence $\{x_n\}$ defined by

$$x \in C, \;\; x_{n+1} = \alpha_n x + (1-\alpha_n)\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds, \quad \forall n \in \mathbb{N} \tag{1.2}$$

in a real Hilbert space where $\{T(s) : s \geq 0\}$ is a strongly continuous semigroup of nonexpansive mappings on a closed convex subset C of a Banach space and $\lim_{n \longrightarrow \infty} s_n = \infty$.

On the other hand,Plubtieng and Punpaeng [9] introduced the following iterative method for nonexpansive semigroup $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ in a real Hilbert space. Let $f : C \longrightarrow C$ be a contraction and the sequence $\{x_n\}$ be defined by $x_1 \in C$,

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1-\alpha_n-\beta_n)\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds, \; n \in \mathbb{N} \tag{1.3}$$

where $\{\alpha_n\}, \{\beta_n\}$ are the sequences in $(0,1)$ and $\{s_n\}$ is a positive real divergent sequence. They proved the strong convergence theorem under the conditions $\alpha_n + \beta_n < 1, \lim_{n \longrightarrow \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n \longrightarrow \infty} \beta_n = 0$.

Very recently, Wangkeeree [12] studied the strong convergence of the sequence $\{x_n\}$ defined by $x_1 \in C$,

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds, \; n \in \mathbb{N} \tag{1.4}$$

They prove that under certain appropriate conditions imposed on $\{\alpha_n\}, \{\beta_n\}$, the sequence x_n converges strongly to a point $z \in F$ which is the unique solution of the variational inequality

$$\langle (A - \gamma f)z, x - z\rangle \geq 0, x \in F(S). \tag{1.5}$$

Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The eguilibrium problem for $F : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$F(x, y) \geq 0 \ \ forall \ \ y \in C. \tag{1.6}$$

The set of solutions of (1.3) is denote by $EP(F)$. Given a mapping $T : C \to H$,let $F(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in C$. Then $z \in EP(F)$if and only if $\langle Tz, y - z\rangle \geq 0$ for all $y \in C$,i.e.,z is a solution of the variational inequality. Numerous problem in physics,optimization, and economics reduce to find a solution of (1.3).

In 2005, Combettes and Hirstoaga [3] introduced an iterative scheme of finding the best approximation to fine initial data when $EP(F)$ is nonempty and prove a strong convergence theorem.

By using the viscosity approximation method, Moudafi [6] introduced the following iterative iterative process for nonexpansive mappings (see [13] for further developments in both Hilbert and Banach spaces). Let f be a contraction on H. Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (1 - \sigma_n)Tx_n + \sigma_n f(x_n), \ n \geq 0, \tag{1.7}$$

where $\{\sigma_n\}$ is a sequence in $(0, 1)$. It is proved [6, 13] that under certain appropriate conditions imposed on $\{\sigma_n\}$, the sequence $\{x_n\}$ generated by (1.7) strongly converges to the unique solution x^* in C of the variational inequality

$$\langle (I - f)x^*, x - x^*\rangle \geq 0, x \in C. \tag{1.8}$$

In 2006, Marino and Xu [5] considered the following general iterative method in a real Hilbert space:

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n \gamma f(x_n), n \geq 0. \tag{1.9}$$

where A is a strongly positive bounded linear operator on H. They proved that if the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.9) converges strongly to the unique solution of the variational inequality

$$\langle (A - \gamma f)x^*, x - x^*\rangle \geq 0, x \in C \tag{1.10}$$

which is the optimality condition for the minimization problem

$$\min_{x \in C} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for $\gamma f(i.e., h'(x) = \gamma f(x)$ for $x \in H)$.

On the other hand, Plubtieng and Punpang [8] introduced and considered the following iterative scheme as below for finding a common element of the set $EP(F)$ and the set of fixed points of a nonexpansive mapping in a Hilbert space. Let $S : C \to H$ be a nonexpansive mapping , defined sequences $\{x_n\}$ and $\{u_n\}$ by

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in H, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Su_n, & \forall n \in \mathbb{N}, \end{cases}$$

They proved that under certain appropriate conditions the sequence $\{x_n\}$ converges strongly to the unique solution of the variational inequality

$$\langle (A - \gamma f)z, x - z\rangle \geq 0, x \in F(S) \cap EP(F), \tag{1.11}$$

which is the optimality condition for the minimization problem

$$\min_{x \in F(S)\cap EP(F)} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for γf. Inspired and motivated by the ongoing research in this field, we suggest and analyze an iterative scheme for a nonexpansive semigroup and under some appropriate conditions imposed on the parameters, we prove another strong convergence theorem for a nonexpansive semigroup in a real Hilbert space and show that the approximate solution converges to a unique solution of some variational inequality which is the optimality condition for the minimization problem. The results of this paper extend and improve the results of Wangkeeree [12] and many others.

2 Preliminaries

Let H be a real Hilbert space and let C be a closed convex subset of H.Then, for any $x \in H$, there exists a unique nearest point $u \in C$ such that

$$\|x - u\| \leq \|x - y\|, \forall y \in C.$$

We denote u by $P_C(x)$, where P_C is called the metric projection of H onto C. It is well known that P_C is nonexpansive. Furthermore, for $x \in H$ and $u \in C$,

$$u = P_C(x) \Leftrightarrow \langle x - u, u - y\rangle \geq 0, \forall y \in C$$

A space X is said to satisfy Opial's condition [7] if for each sequence $\{x_n\}$ in X which converges weakly to a point $x \in X$, we have

$$\liminf_{n\longrightarrow\infty} \|x_n - x\| < \liminf_{n\longrightarrow\infty} \|x_n - y\|, \ \ \forall y \in X, y \neq x.$$

Lemma 2.1. *Let H be a real Hilbert space, C be a closed convex subset of H. Given $x \in H$ and $y \in C$. Then $x = P_C x$ if and only if there holds the inequality*

$$\langle x - y, y - z \rangle \geq 0, \forall z \in C.$$

Lemma 2.2. ([2]) *Let E be a uniformly convex Banach space, K be a nonempty closed convex subset of E and $T : K \longrightarrow E$ a nonexpansive mapping. Then $I - T$ is demi-closed at zero.*

Lemma 2.3. ([13]). *Assume $\{a_n\}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \alpha_n) a_n + \delta_n, \ n \geq 0$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

$$(1) \lim_{n \longrightarrow \infty} \alpha_n = 0 \qquad \textit{and} \qquad \sum_{n=1}^{\infty} \alpha_n = \infty$$

$$(2) \limsup_{n \longrightarrow \infty} \frac{\delta_n}{\alpha_n} \leq 0 \qquad \textit{or} \qquad \sum_{n=1}^{\infty} |\delta_n| < \infty.$$

Then $\lim_{n \longrightarrow \infty} a_n = 0$.

Lemma 2.4. ([5]) *Assume A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.*

Lemma 2.5. ([4]) *Let H be a real Hilbert space, C be a closed convex subset of H, and $T : C \rightarrow C$ be a nonexpansive mapping with $F(T) \neq \phi$. If $\{x_n\}$ is a sequence in C weakly converging to x and if $(I - T)x_n$ converge strongly to y , then $(I - T)x = y$*

Lemma 2.6. ([5]) *Let H be a Hilbert space, C a nonempty closed convex subset of H, and $f : H \longrightarrow H$ a contraction with coefficient $0 < \alpha < 1$, and A a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$. Then, for $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$,*

$$\langle x - y, (A - \gamma f)x - (A - \gamma f)y \rangle \geq (\bar{\gamma} - \gamma\alpha)\|x - y\|^2, \ x, y \in H.$$

That is, $A - \gamma f$ is strongly monotone with coefficient $\bar{\gamma} - \gamma\alpha$.

Lemma 2.7. ([10]) *Let C be a nonempty bounded closed convex subset of H and let $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C. For $x \in C$ and $t > 0$. Then, for any $0 \leq h < \infty$,*

$$\lim_{t\longrightarrow\infty} \sup_{x\in C} \|\frac{1}{t}\int_0^t T(s)xds - T(h)(\frac{1}{t}\int_0^t T(s)xds)\| = 0.$$

In this paper, For solving the equilibrium problem, let us assume that the bifunction F satisfies the following conditions (see[1]):

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for any $x, y \in C$;

(A3) F is upper-hemicontinuous, i.e.,for each $x, y, z \in C$,

$$\lim_{t\longrightarrow 0^+} \sup F(tz + (1-t)x, y) \leq F(x, y);$$

(A4) $F(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$.

Lemma 2.8. ([1]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C \longrightarrow \mathbb{R}$ satisfies (A1)-(A4) and let $r > 0$ and $x \in H$. Then there exists unique $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle(y - z, z - x)\rangle \geq 0, \ for\ all\ z \in C.$$

Lemma 2.9. ([3]) *Assume that F satisfies the same assumptions as Lemma 2.8. For $r > 0$ and $x \in C$, define a mapping $T_r = H \to C$ as follow:*

$$T_r(x) = \{y \in C : F(y, z) + \frac{1}{r}\langle(y-z, z-x)\rangle \geq 0,\ for\ all\ z \in C\}\ for\ all\ y \in H$$

Then, the following hold:

(1) T_r is single-valued;

(2) T_r is firmly nonexpansive, i.e., for any $x, y \in H$

$$\|T_r x - T_r y\|^2 \leq \langle(T_r x - T_r y, x - y)\rangle;$$

(3) $F(T_r) = EP(F)$;

(4) $EP(F)$ is closed and convex.

3 Main results

Throughout the rest of this paper, We always assume that C is a nonempty closed convex subset of a real Hilbert space H, f is a contraction of C into itself with coefficient $\alpha \in (0,1)$, and A be a strongly bounded linear operator on H with coefficient $\bar{\gamma}$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$.Let $\{T_{r_n}\}$ be a sequence of a mappings defined as lemma 2.8. Consider a mapping S_n on C defined by:

$$S_n x = \alpha_n \gamma f(x) + \beta_n x + ((1-\beta_n)I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)T_{r_n} x ds, x \in C \ , n \in \mathbb{N} \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\}$ are the sequences in $(0,1)$ and $\{t_n\}$ is a positive real divergent sequence.Indeed, by lemma 2.8, we have

$$\begin{aligned} \|S_n x - S_n y\| &= \|\alpha_n \gamma f(x) + \beta_n x + ((1-\beta_n)I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)T_{r_n} x ds \\ &\quad -\alpha_n \gamma f(y) - \beta_n y - ((1-\beta_n)I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)T_{r_n} y ds\| \\ &\leq \alpha_n \gamma \|f(x) - f(y)\| + \beta_n \|x - y\| \\ &\quad +(1-\beta_n - \alpha_n \bar{\gamma})\|\frac{1}{t_n}(\int_0^{t_n} T(s)T_{r_n} x ds - \int_0^{t_n} T(s)T_{r_n} y ds\| \\ &\leq \alpha_n \gamma \alpha \|x - y\| + (1 - \alpha_n \bar{\gamma})\|x - y\| \\ &\leq (1 - \alpha_n(\bar{\gamma} - \gamma\alpha))\|x - y\|. \end{aligned}$$

Since $0 < 1 - \alpha_n(\bar{\gamma} - \gamma\alpha) < 1$, it follows that S_n is a contraction. Therefor by the Banach contraction principle, S_n has a fixed point $x_n \in C$ such that

$$x_n = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)T_{r_n}(x_n) ds \tag{3.2}$$

Note that x_n indeed depends on f as well, but we will suppress this dependence of x_n on f for simplicity of notation and also always use γ to mean a number in $(0, \frac{\bar{\gamma}}{\alpha})$.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H.Let F be a bifunction from $C \times C \longrightarrow \mathbb{R}$ satisfies (A1)-(A4) and $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \cap EP(F) \neq \emptyset$. Let $f : C \longrightarrow C$ be a contraction with coefficient $\alpha \in (0,1)$, and A be a strongly bounded linear operator on H with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$.Suppose $\{\alpha_n\}, \{\beta_n\}$ are the sequences in $[0,1]$ and $\{r_n\} \subset (0,\infty)$is a real sequence. Suppose the following conditions are satisfied:*

$$\lim_{n\longrightarrow\infty}\alpha_n = 0, \qquad \sum_{n=1}^{\infty}\alpha_n = \infty \qquad and \quad \sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty,$$

$$\lim_{n\longrightarrow\infty}\beta_n = 0 \qquad and \quad \sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n| < \infty,$$

$$\liminf_{n\longrightarrow\infty} r_n > 0 \qquad and \quad \sum_{n=1}^{\infty}|r_{n+1}-r_n| < \infty.$$

Let the sequence $\{x_n\}$ and $\{u_n\}$ be generated by $x_1 \in C$ and

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0, & \forall y \in H, \\ x_{n+1} = \alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds, & \end{cases}$$

for all $n \in \mathbb{N}$, where $u_n = T_{r_n}x_n$ and $\{t_n\}$ is a positive real divergent sequence. Then the sequence $\{x_n\}$ and $\{u_n\}$ converge strongly to a point z where $z = P_{F(S)\cap EF(F)}(I - A + \gamma f)(z)$ is the unique solution of the variational inequality

$$\langle (A-\gamma f)z, z-x\rangle \leq 0, \ \ \forall x \in F(S)\cap EF(F).$$

Proof. From $\lim_{n\longrightarrow\infty}\alpha_n = 0$ and $\lim_{n\longrightarrow\infty}\beta_n = 0$, we assume, without loss of generality, that $\alpha_n \leq (1-\beta_n)\|A\|^{-1}$ for all $n \in \mathbb{N}$. Since A is a strongly positive bounded linear operator on H and lemma 2.4, we have

$$\|A\| = \sup\{|\ \langle Ax, x\rangle\ | : x \in H, \|x\| = 1\}$$

Note that

$$\begin{aligned} \langle((1-\beta_n)I - \alpha_n A)x, x\rangle &= 1-\beta_n - \alpha_n\langle Ax, x\rangle \\ &\geq 1-\beta_n-\alpha_n\|A\| \\ &\geq 0, \ \forall x \in H. \end{aligned}$$

This implies that $(1-\beta_n)I - \alpha_n A$ is positive and hence

$$\begin{aligned} \|(1-\beta_n)I - \alpha_n A\| &= sup\{|\ \langle((1-\beta_n)I-\alpha_n A)x, x\rangle\ | \ \ : x \in H, \|x\| = 1\} \\ &= sup\{1-\beta_n-\alpha_n\langle Ax, x\rangle : x \in H, \|x\| = 1\} \\ &\leq 1-\beta_n-\alpha_n\bar{\gamma}. \end{aligned}$$

Now, let $Q = P_{F(S)\cap EF(F)}$. Note that f is a contraction with coefficient $\alpha \in (0,1)$.For all $x, y \in H$, we have

$$\begin{aligned}\|Q(I-A+\gamma f)(x) - Q(I-A+\gamma f)(y)\| &\leq \|(I-A+\gamma f)(x) - (I-A+\gamma f)(y)\| \\ &\leq \|I-A\|\|x-y\| + \gamma\|f(x)-f(y)\| \\ &\leq (1-\bar{\gamma})\|x-y\| + \gamma\alpha\|x-y\| \\ &= (1-(\bar{\gamma}-\gamma\alpha))\|x-y\|.\end{aligned}$$

Therefore, $Q(I-A+\gamma f)$ is a contraction on H,
which implies that, there exists a unique element $z \in H$
such that $z = Q(I-A+\gamma f)(z) = P_{F(S)\cap EF(F)}(I-A+\gamma f)(z)$.That is

$$\langle (A-\gamma f)z, z-x\rangle \leq 0, \;\; x \in F(\mathcal{S}) \cap EP(F). \tag{3.3}$$

Let $p \in F(S) \cap EF(F)$. Then from $u_n = T_{r_n}x_n$, it follows that

$$\|u_n - p\| = \|T_{r_n}x_n - T_{r_n}p\| \leq \|x_n - p\|$$

Now, we calculate

$$\begin{aligned}\|x_{n+1}-p\| &= \|\alpha_n(\gamma f(x_n) - Ap) + \beta_n(x_n - p) \\ &\quad +((1-\beta_n)I - \alpha_n A)(\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds - p)\| \\ &\leq (1-\beta_n-\alpha_n\bar{\gamma})\|\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds - p\| + \beta_n\|x_n-p\| \\ &\quad +\alpha_n\|\gamma f(x_n) - Ap\| \\ &\leq (1-\beta_n-\alpha_n\bar{\gamma})\|u_n-p\| + \beta_n\|x_n-p\| + \alpha_n\|\gamma f(x_n)-Ap\| \\ &\leq (1-\alpha_n\bar{\gamma})\|x_n-p\| + \alpha_n\gamma\|f(x_n)-f(p)\| + \alpha_n\|\gamma f(p)-Ap\| \\ &\leq (1-\alpha_n\bar{\gamma})\|x_n-p\| + \alpha_n\gamma\alpha\|x_n-p\| + \alpha_n\|\gamma f(p)-Ap\| \\ &= (1-(\bar{\gamma}-\gamma\alpha)\alpha_n)\|x_n-p\| + (\bar{\gamma}-\gamma\alpha)\alpha_n\frac{\|\gamma f(p)-Ap\|}{\bar{\gamma}-\gamma\alpha}.\end{aligned}$$

It follows from induction that

$$\|x_n - p\| \leq \max\left\{\|x_1-p\|, \frac{\|\gamma f(p)-Ap\|}{\bar{\gamma}-\gamma\alpha}\right\}, n \geq 1. \tag{3.4}$$

Hence $\{x_n\}$ is bounded. We also obtain that $\{u_n\}$ and $\{f(x_n)\}$ are all bounded.

Put $z_1 = P_{F(\mathcal{S})\bigcap EP(F)}x_1$ and set $D = \{z \in C : \|z - z_1\| \leq \|x_1 - z_1\| + \frac{\|\gamma f(z_1)-Az_1\|}{\bar{\gamma}-\gamma\alpha}\}$. Then D is a nonempty closed bounded convex subset of C which is $T(s)$-invariant for each $s \in [0,\infty)$ and $\{x_n\} \subset D$. Without loss of generality, we may assume that $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ is a nonexpansive semigroup on D. By Lemma 2.7, we get

$$\lim_{n \longrightarrow \infty} \|\frac{1}{t_n} \int_0^{t_n} T(s)u_n ds - T(h)(\frac{1}{t_n} \int_0^{t_n} T(s)u_n ds)\| = 0.$$

for every $h \in [0, \infty)$. Setting $z_n := \frac{1}{t_n} \int_0^{t_n} T(s)u_n ds$, we have

$$\lim_{n \longrightarrow \infty} \|z_n - T(h)z_n\| = 0, \ \ \forall h \in [0, \infty). \tag{3.5}$$

Next, we show that $\|x_{n+1} - x_n\| \to 0, \quad as \quad n \to \infty$
From definition of $\{x_n\}$, we obtain

$$\begin{aligned}
\|x_{n+2} - x_{n+1}\| &= \|\alpha_{n+1}\gamma f(x_{n+1}) + \beta_{n+1}x_{n+1} + ((1 - \beta_{n+1})I - \alpha_{n+1}A)z_{n+1} \\
&\quad -\alpha_n\gamma f(x_n) - \beta_n x_n - ((1 - \beta_n)I - \alpha_n A)z_n\| \\
&= \|((1 - \beta_{n+1})I - \alpha_{n+1}A)(z_{n+1} - z_n) \\
&\quad +\{(\beta_n - \beta_{n+1})z_n + (\alpha_n - \alpha_{n+1})Az_n\} \\
&\quad +\alpha_{n+1}\gamma(f(x_{n+1}) - f(x_n)) + (\alpha_{n+1} - \alpha_n)\gamma f(x_n) \\
&\quad +\beta_{n+1}(x_{n+1} - x_n) + (\beta_{n+1} - \beta_n)x_n\| \\
&\leq \|(1 - \beta_{n+1})I - \alpha_{n+1}A\|\|z_{n+1} - z_n\| \\
&\quad +|\beta_{n+1} - \beta_n|\|z_n\| + |\alpha_{n+1} - \alpha_n|\|Az_n\| \\
&\quad +\alpha_{n+1}\gamma\|f(x_{n+1}) - f(x_n)\| + |\alpha_{n+1} - \alpha_n|\gamma\|f(x_n)\| \\
&\quad +\beta_{n+1}\|x_{n+1} - x_n\| + |\beta_{n+1} - \beta_n|\|x_n\| \\
&\leq \|(1 - \beta_{n+1})I - \alpha_{n+1}A\|\|u_{n+1} - u_n\| \\
&\quad +|\beta_{n+1} - \beta_n|\|z_n\| + |\alpha_{n+1} - \alpha_n|\|Az_n\| \\
&\quad +\alpha_{n+1}\gamma\alpha\|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n|\gamma\|f(x_n)\| \\
&\quad +\beta_{n+1}\|x_{n+1} - x_n\| + |\beta_{n+1} - \beta_n|\|x_n\| \\
&\leq \|(1 - \beta_{n+1})I - \alpha_{n+1}A\|\|u_{n+1} - u_n\| \\
&\quad +|\beta_{n+1} - \beta_n|\|z_n\| + |\alpha_{n+1} - \alpha_n|\|Az_n\| \\
&\quad +|\beta_{n+1} + \alpha_{n+1}\gamma|\|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n|\gamma\|f(x_n)\| \\
&\quad +|\beta_{n+1} - \beta_n|\|x_n\|
\end{aligned}$$

$$\begin{aligned}
&\leq \|(1-\beta_{n+1})I-\alpha_{n+1}A\|\|u_{n+1}-u_n\| \\
&\quad +|\beta_{n+1}+\alpha_{n+1}\gamma|\|x_{n+1}-x_n\|+|\beta_{n+1}-\beta_n|M \\
&\quad +|\alpha_{n+1}-\alpha_n|M+\delta_n \\
&\leq (1-\beta_{n+1}-\alpha_{n+1}\bar{\gamma}\|u_{n+1}-u_n\| \\
&\quad +|\beta_{n+1}+\alpha_{n+1}\gamma|\|x_{n+1}-x_n\|+|\beta_{n+1}-\beta_n|M \\
&\quad +|\alpha_{n+1}-\alpha_n|M+\delta_n, \qquad (3.6)
\end{aligned}$$

where $M := sup\{max\{\|z_n\|+\|x_n\|, \|Az_n\|+\gamma\|f(x_n)\|\} : n \geq 0\} < \infty$ and $\delta_n = |\alpha_{n+1}-\alpha_n|M + |\beta_{n+1}-\beta_n|M$
Since $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$, and $\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n| < \infty$, we have

$$\sum_{n=1}^{\infty}\delta_n = \sum_{n=1}^{\infty}(|\alpha_{n+1}-\alpha_n|M+|\beta_{n+1}-\beta_n|M) < \infty$$

On the other hand, from $u_n = T_{r_n}x_n$ and $u_{n+1} = T_{r_{n+1}}x_{n+1}$, we have

$$F(u_n, x)+\frac{1}{r_n}\langle x-u_n, u_n-x_n\rangle \geq 0, \forall x \in C, \qquad (3.7)$$

and

$$F(u_{n+1}, x)+\frac{1}{r_{n+1}}\langle x-u_{n+1}, u_{n+1}-x_{n+1}\rangle \geq 0, \forall x \in C, \qquad (3.8)$$

Putting $x = u_{n+1}$ in(3.7) and $x = u_n$ in(3.8), we have

$$F(u_n, u_{n+1})+\frac{1}{r_n}\langle u_{n+1}-u_n, u_n-x_n\rangle \geq 0, \qquad (3.9)$$

and

$$F(u_{n+1}, u_n)+\frac{1}{r_{n+1}}\langle u_n-u_{n+1}, u_{n+1}-x_{n+1}\rangle \geq 0. \qquad (3.10)$$

From (A2),the monotonicity of F, we have $F(u_n, u_{n+1})+F(u_{n+1}, u_n) \leq 0$. So, from (3.9) and (3.10), we can conclude that $\langle u_{n+1}-u_n, \frac{u_n-x_n}{r_n}-\frac{u_{n+1}-x_{n+1}}{r_{n+1}} \geq 0\rangle$ and hence $\langle u_{n+1}-u_n, u_n-u_{n+1}+u_{n+1}-x_n-\frac{r_n}{r_{n+1}}(u_{n+1}-x_{n+1}) \geq 0\rangle$. Since $\lim\ inf_{n\to\infty}\ r_n > 0$, we may assume that there exists a real number b such that $r_n > b > 0$ for all $n \geq 1$. Then we have

$$\begin{aligned}
\|u_{n+1}-u_n\|^2 &\leq \langle u_{n+1}-u_n, x_{n+1}-x_n+(1-\frac{r_n}{r_{n+1}})(u_{n+1}-x_{n+1})\rangle \\
&\leq \|u_{n+1}-u_n\|\{\|x_{n+1}-x_n\|+|1-\frac{r_n}{r_{n+1}}|\|u_{n+1}-x_{n+1}\|\}
\end{aligned}$$

and hence

$$\|u_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + \frac{K}{b}|r_{n+1} - r_n|, \tag{3.11}$$

where $K = sup\{\|u_n - x_n\| : n \in \mathbb{N}\}$. Using (3.11) in (3.6), we get

$$\begin{aligned}
\|x_{n+2} - x_{n+1}\| &\leq (1 - \beta_{n+1} - \alpha_{n+1}\bar{\gamma})[\|x_{n+1} - x_n\| + \frac{K}{b}|r_{n+1} - r_n|] \\
&\quad +|\beta_{n+1} + \alpha_{n+1}\gamma|\|x_{n+1} - x_n\| + |\beta_{n+1} - \beta_n|M \\
&\quad +|\alpha_{n+1} - \alpha_n|M + \delta_n \\
&\leq (1 - (\bar{\gamma} - \gamma)\alpha_{n+1})\|x_{n+1} - x_n\| \\
&\quad +(1 - \beta_{n+1} - \gamma_{n+1}\bar{\gamma})\frac{K}{b}|r_{n+1} - r_n| \\
&\quad +(|\beta_{n+1} - \beta_n| + |\alpha_{n+1} - \alpha_n|)M + \delta_n
\end{aligned}$$

therefore, it follow from Lemma (2.3) that,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0 \tag{3.12}$$

From (3.11), and $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$, we have $\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0$.

Next, we will show that $\lim_{n\to\infty} \|x_n - z_n\| = 0$. From definition of $\{x_n\}$, we get

$$\begin{aligned}
\|x_n - z_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\| \\
&\leq \|x_n - x_{n+1}\| + \|\alpha_n\gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \gamma_n A)z_n - z_n\| \\
&\leq \|x_n - x_{n+1}\| + \alpha_n\|\gamma f(x_n) - Az_n\| + \beta_n\|x_n - z_n\|.
\end{aligned}$$

That is; $\|x_n - z_n\| \leq \frac{1}{1-\beta_n}\|x_n - x_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|\gamma f(x_n) - Az_n\|$. It follows that

$$\lim_{n\to\infty} \|x_n - z_n\| = 0 \tag{3.13}$$

Next, we will show that $\lim_{n\to\infty} \|x_n - u_n\| = 0$
For $p \in F(S)\bigcap EP(F)$, note that $u_n = T_{r_n}x_n$ is firmly nonexpansive; then we have

$$\begin{aligned}
\|u_n - p\|^2 &= \|T_{r_n}x_n - T_{r_n}p\|^2 \\
&\leq \langle T_{r_n}x_n - T_{r_n}p, x_n - p\rangle \\
&= \langle u_n - p, x_n - p\rangle \\
&= \frac{1}{2}(\|u_n - p\|^2 + \|x_n - p\|^2 + \|x_n - u_n\|^2)
\end{aligned}$$

and hence $\|u_n - p\|^2 \leq \|x_n - p\|^2 + \|x_n - u_n\|^2$.
Therefore, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n(\gamma f(x_n) - Ap) + \beta_n(x_n - z_n) + (I - \alpha_n A)(z_n - p)\|^2 \\
&\leq \|(I - \alpha_n A)(z_n - p) + \beta_n(x_n - z_n)\|^2 \\
&\quad +2\alpha_n\langle \gamma f(x_n) - Ap, x_{n+1} - p\rangle \\
&\leq [\|(I - \alpha_n A)(z_n - p)\| + \beta_n\|(x_n - z_n)\|]^2 \\
&\quad +2\alpha_n\|\gamma f(x_n) - Ap\|\|x_{n+1} - p\| \\
&\leq [(1 - \alpha_n\bar{\gamma})\|u_n - p\| + \beta_n\|(x_n - z_n)\|]^2 \\
&\quad +2\alpha_n\|\gamma f(x_n) - Ap\|\|x_{n+1} - p\| \\
&\leq (1 - \alpha_n\bar{\gamma})^2\{\|x_n - p\|^2 - \|x_n - u_n\|^2\} + \beta_n^2\|(x_n - z_n)\|^2 \\
&\quad +2(1 - \alpha_n\bar{\gamma})\beta_n\|u_n - p\|\|(x_n - z_n)\| \\
&\quad +2\alpha_n\|\gamma f(x_n) - Ap\|\|x_{n+1} - p\| \\
&= (1 - 2\alpha_n\bar{\gamma} + (\alpha_n\bar{\gamma})^2)\|x_n - p\|^2 \\
&\quad -(1 - \alpha_n\bar{\gamma})^2\|x_n - u_n\|^2 + \beta_n^2\|(x_n - z_n)\|^2 \\
&\quad +2(1 - \alpha_n\bar{\gamma})\beta_n\|u_n - p\|\|(x_n - z_n)\| \\
&\quad +2\alpha_n\|\gamma f(x_n) - Ap\|\|x_{n+1} - p\| \\
&\leq \|x_n - p\|^2 + \alpha_n\bar{\gamma}^2\|x_n - p\|^2 - (1 - \alpha_n\bar{\gamma})^2\|x_n - u_n\|^2 \\
&\quad +\beta_n^2\|(x_n - z_n)\|^2 + 2(1 - \alpha_n\bar{\gamma})\beta_n\|u_n - p\|\|(x_n - z_n)\| \\
&\quad +2\alpha_n\|\gamma f(x_n) - Ap\|\|x_{n+1} - p\|.
\end{aligned}$$

Then we have

$$\begin{aligned}
(1 - \alpha_n\bar{\gamma})^2\|x_n - u_n\|^2 &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \beta_n^2\|(x_n - z_n)\|^2 \\
&\quad +\alpha_n\bar{\gamma}^2\|x_n - p\|^2 + 2\alpha_n\|\gamma f(x_n) - Ap\|\|x_{n+1} - p\| \\
&\quad +2(1 - \alpha_n\bar{\gamma})\beta_n\|u_n - p\|\|(x_n - z_n) \\
&\leq (\|x_n - p\| + \|x_{n+1} - p\|)\|x_{n+1} - x_n\| + \alpha_n\bar{\gamma}^2\|x_n - p\|^2 \\
&\quad +\beta_n^2\|(x_n - z_n)\|^2 + 2(1 - \alpha_n\bar{\gamma})\beta_n\|u_n - p\|\|(x_n - z_n)\| \\
&\quad +2\alpha_n\|\gamma f(x_n) - Ap\|\|x_{n+1} - p\|. \qquad (3.14)
\end{aligned}$$

So from (3.12)-(3.14), we have $\lim_{n\to\infty}\|x_n - u_n\| = 0$.
From $\|z_n - u_n\| \leq \|z_n - x_n\| + \|x_n - u_n\|$,
we also have $\|z_n - u_n\| \to 0$.

Next, we show that $\lim sup_{n\to\infty}\langle (A - \gamma f)z, z - z_n\rangle \leq 0$,
where $z = P_{F(S)\bigcap EP(F)}(I - A + \gamma f)(z)$ is a unique solution of a variational inequality.

Since $\{z_n\} \subseteq D$ is bounded, there is a subsequence $\{z_{n_j}\}$ of $\{z_n\}$ such that

$$\lim_{j\longrightarrow\infty} \langle (A-\gamma f)z, z-z_{n_j}\rangle = \limsup_{n\longrightarrow\infty} \langle (A-\gamma f)z, z-z_n\rangle. \qquad (3.15)$$

Since $\{z_{n_j}\}$ is bounded, there exists a subsequence $\{z_{n_{j_i}}\}$ of $\{z_{n_j}\}$ which converges weakly to $\omega \in C$. Without loss of generality, we can assume that $z_{n_j} \rightharpoonup \omega$.

First, we show that $\omega \in EP(F)$. From $\{u_n\}$ is bounded, there exists a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ which converges weakly to ω and we have

$$F(u_n, x) + \frac{1}{r_n}\langle x-u_n, u_n-x_n\rangle \geq 0, \forall x \in C.$$

From (A2), we have

$$\frac{1}{r_n}\langle x-u_n, u_n-x_n\rangle \geq -F(u_n, x) \geq F(x, u_n),$$

and hence $\langle x-u_{n_i}, \frac{u_{n_i}-x_{n_i}}{r_{n_i}}\rangle \geq F(x, u_{n_i})$

Since $\frac{u_{n_i}-x_{n_i}}{r_{n_i}} \to 0$ and $u_{n_i} \rightharpoonup \omega$.From (A4), we have $F(x,\omega) \leq 0$ for all $x \in C$. For t with $0 < t \leq 0$ and $x \in C$, let $x_t = tx + (1-t)\omega$. Since $x \in C$ and $\omega \in C$, we have $x_t \in C$ and hence $F(x_t, \omega) \leq 0$. So, from (A4), we have

$$0 = F(x_t, x_t) \leq tF(x_t, x) + (1-t)F(x_t, \omega) \leq tF(x_t, x),$$

and hence $F(x_t, x) \geq 0$, then we get $F(\omega, x) \geq 0$ for all $x \in C$. Hence $\omega \in EP(F)$.

Now, we will show that $\omega \in F(\mathcal{S})$. Assume that $\omega \neq T(h)\omega$ for some $h \in [0,\infty)$. It follows from (3.5) and the Opial's condition that

$$\begin{aligned}
\liminf_{j\longrightarrow\infty} \|z_{n_j} - \omega\| &< \liminf_{j\longrightarrow\infty} \|z_{n_j} - T(h)\omega\| \\
&\leq \liminf_{j\longrightarrow\infty} (\|z_{n_j} - T(h)z_{n_j}\| + \|T(h)z_{n_j} - T(h)\omega\|) \\
&\leq \liminf_{j\longrightarrow\infty} \|z_{n_j} - \omega\|
\end{aligned}$$

This is a contradiction. Hence, $\omega = T(h)\omega$ for each $h \geq 0$, that is $\omega \in F(\mathcal{S})$. Therefore, $\omega \in F(S) \bigcap EP(F)$. Using (3.3) and (3.15) , we obtain

$$\limsup_{n\longrightarrow\infty} \langle (A-\gamma f)z, z-z_n\rangle = \langle (A-\gamma f)z, z-\omega\rangle \leq 0$$

as required.

Finally, we prove that $\{x_n\}$ and $\{u_n\}$ converge strongly to z.

From the definition of $\{x_n\}$, we calculate

$$\begin{aligned}
\|x_{n+1}-z\|^2 &= \|\alpha_n\gamma f(x_n)+\beta_n x_n+((1-\beta_n)I-\alpha_n A)z_n-z\|^2\\
&= \|\alpha_n(\gamma f(x_n)-Az)+((1-\beta_n)I-\alpha_n A)(z_n-z)\\
&\quad +\beta_n(x_n-z)\|^2\\
&= \|((1-\beta_n)I-\alpha_n A)(z_n-z)+\beta_n(x_n-z)\|^2\\
&\quad +\alpha_n^2\|\gamma f(x_n)-Az\|^2+2\beta_n\alpha_n\langle x_n-z,\gamma f(x_n)-Az\rangle\\
&\quad +2\alpha_n\langle((1-\beta_n)I-\alpha_n A)(z_n-z,\gamma f(x_n)-Az\rangle\\
&\le ((1-\beta_n-\alpha_n\bar{\gamma})\|z_n-z\|+\beta_n\|x_n-z\|)^2\\
&\quad +\alpha_n^2\|\gamma f(x_n)-Az\|^2+2\beta_n\gamma\alpha_n\langle x_n-z,f(x_n)-f(z)\rangle\\
&\quad +2\beta_n\alpha_n\langle x_n-z,\gamma f(z)-Az\rangle\\
&\quad +2(1-\beta_n)\gamma\alpha_n\langle z_n-z,f(x_n)-f(z)\rangle\\
&\quad +2(1-\beta_n)\alpha_n\langle z_n-z,\gamma f(z)-Az\rangle\\
&\quad -2\alpha_n^2\langle A(z_n-z),\gamma f(x_n)-Az\rangle
\end{aligned}$$

Which implies that

$$\begin{aligned}
\|x_{n+1}-z\|^2 &\le ((1-\alpha_n\bar{\gamma})^2+2\alpha\beta_n\gamma\alpha_n+2\alpha(1-\beta_n)\gamma\alpha_n)\|x_n-z\|^2\\
&\quad +\alpha_n^2\|\gamma f(x_n)-Az\|^2+2\beta_n\alpha_n\langle x_n-z,\gamma f(z)-Az\rangle\\
&\quad +2(1-\beta_n)\alpha_n\langle z_n-z,\gamma f(z)-Az\rangle\\
&\quad -2\alpha_n^2\langle A(z_n-z),\gamma f(x_n)-Az\rangle\\
&\le (1-2(\bar{\gamma}-\alpha\gamma)\alpha_n)\|x_n-z\|^2+\bar{\gamma}^2\alpha_n^2\|x_n-z\|^2\\
&\quad +\alpha_n^2\|\gamma f(x_n)-Az\|^2+2\beta_n\alpha_n\langle x_n-z,\gamma f(z)-Az\rangle\\
&\quad +2(1-\beta_n)\alpha_n\langle z_n-z,\gamma f(z)-Az\rangle\\
&\quad +2\alpha_n^2\|A(z_n-z)\|\|\gamma f(x_n)-Az)\|\\
&= (1-2(\bar{\gamma}-\alpha\gamma)\alpha_n)\|x_n-z\|^2+\alpha_n\{\alpha_n(\bar{\gamma}^2\|x_n-z\|^2\\
&\quad +\|\gamma f(x_n)-Az\|^2+2\|A(z_n-z)\|\|\gamma f(x_n)-Az\|)\\
&\quad +2\beta_n\langle x_n-z,\gamma f(z)-Az\rangle\\
&\quad +2(1-\beta_n)\langle z_n-z_o,\gamma f(z)-Az\rangle\}
\end{aligned}$$

Since $\{x_n\}$, $\{f(x_n)\}$ and $\{z_n\}$ are bounded, we can take a constant $L>0$ such that

$$L\ge\bar{\gamma}^2\|x_n-z\|^2+\|\gamma f(x_n)-Az\|^2+2\|A(z_n-z)\|\|\gamma f(x_n)-Az\|,\forall n\ge 0$$

It then follows that

$$\|x_{n+1}-z\|^2\le(1-2(\bar{\gamma}-\alpha\gamma)\alpha_n)\|x_n-z\|^2+\alpha_n\xi_n \tag{3.16}$$

where

$$\xi_n = 2\beta_n\langle x_n - z, \gamma f(z) - Az\rangle + 2(1-\beta_n)\langle z_n - z, \gamma f(z) - Az\rangle + L\alpha_n$$

We get $\lim_{n\longrightarrow\infty}\sup\xi_n \leq 0$.
Now applying lemma 2.3 to (3.16) concluded that

$$x_n \rightarrow z \text{ as } n \rightarrow \infty. \tag{3.17}$$

Consequently, we can obtian that $\{u_n\}$ also converges to z.
This completes the proof. □

Corollary 3.2. *[9, Theorem 3.3] Let C be a nonempty closed convex subset of a real Hilbert space H and $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n\}$ be a sequence in $(0,1)$ which satisfies $\alpha_n + \beta_n < 1, \lim_{n\longrightarrow\infty}\alpha_n = 0, \lim_{n\longrightarrow\infty}\beta_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$. Let $f : C \longrightarrow C$ be a contraction mapping with coefficient $\alpha \in (0,1)$ and $\{s_n\}$ a positive real divergent sequence. Then the sequence $\{x_n\}$ defined by*

$$x_1 \in C, \quad \text{and} \quad x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1-\alpha_n-\beta_n)\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds \quad \forall n \geq 1$$

converges strongly to z, which is the unique solution in $F(\mathcal{S})$ of the variational inequality

$$\langle (I-f)z, x-z\rangle \geq 0, \; x \in F(\mathcal{S})$$

or equivalently $z = P_{F(\mathcal{S})}(z))$, where P is a metric projection mapping from H onto $F(\mathcal{S})$.

Proof. Put $A = I, \gamma \equiv 1, F(x,y) = 0$ for all $x, y \in C$ and $r_n = 1$. We get $u_n = x_n$ in theorem 3.1 . Then from theorem 3.1 the sequence $\{x_n\}$ generated in corollary 3.2 converges strongly to $z \in F(\mathcal{S})$. □

Corollary 3.3. *[12, Theorem 3.1] Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \longrightarrow C$ be a contraction with coefficient $\alpha \in (0,1)$, and A be a strongly bounded linear operator on H with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let the sequence $\{x_n\}$ be generated by*

$$x_1 \in C, \quad \text{and} \quad x_{n+1} = \alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds, \quad n \geq 1,$$

where $\{\alpha_n\}, \{\beta_n\}$ *are the sequences in* $[0, 1]$ *satisfying* $\alpha_n+\beta_n < 1, \lim_{n\longrightarrow\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$*,* $\lim_{n\longrightarrow\infty} \beta_n = 0$ *and* $\{s_n\}$ *is a positive real divergent sequence. Then the sequence* $\{x_n\}$ *converges strongly to a point* $z \in F$ *which is the unique solution of the variational inequality*

$$\langle (A - \gamma f)z, z - x\rangle \leq 0, \;\; x \in F(\mathcal{S}).$$

Equivalently, we have $z = P_{F(\mathcal{S})}(I - A + \gamma f)(z)$*.*

Proof. Put $F(x, y) = 0$ for all $x, y \in C$ and $r_n = 1$. We get $u_n = x_n$ in theorem 3.1 . Then from theorem 3.1 the sequence $\{x_n\}$ generated in corollary 3.3 converges strongly to $z \in F(\mathcal{S})$. □

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