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Carleson Measures and Composition Operators on Bergman-Orlicz Spaces of the Unit Ball[1]

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Abstract. Let $\mathbb{B}$ be the open unit ball in the n dimension Euclidean space $\mathbb{C}^n$ and dv be the normalized volume measure on $\mathbb{B}$. For $\alpha > -1$, define the weighted Lebesgue measure dv_α by $dv_\alpha(z) = c_\alpha(1-|z|^2)^\alpha dv(z)$, where $c_\alpha = \Gamma(n+\alpha+1)/n!\Gamma(\alpha+1)$. Let $\Phi : [0,\infty) \to [0,\infty)$ be a strictly increasing, convex function satisfying the global Δ_2-condition with $\Phi(0) = 0$, and Φ^{-1} differentiable. For such a function Φ, the Bergman-Orlicz space is define by

$$A^\Phi_\alpha = \{f \in H(\mathbb{B}) : \|f\|_\Phi = \int_{\mathbb{B}} \Phi(|f(z)|)dv_\alpha(z) < \infty\}.$$

This paper characterizes Carleson measures on Bergman-Orlicz spaces, bounded and compact composition operators on these spaces.

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1. Introduction

Let $\mathbb{B}$ be the open unit ball in the n dimension Euclidean space $\mathbb{C}^n$ and $H(\mathbb{B})$ be the space of all holomorphic functions on $\mathbb{B}$. Let dv denote the normalized volume measure on $\mathbb{B}$. For $\alpha > -1$, define the weighted Lebesgue measure dv_α by $dv_\alpha(z) = c_\alpha(1-|z|^2)^\alpha dv(z)$, where $c_\alpha = \Gamma(n+\alpha+1)/n!\Gamma(\alpha+1)$. For $\alpha > -1$ and $p > 0$ the weighted Bergman space $A^p_\alpha(\mathbb{B})$ consists of holomorphic functions f in $L^p(\mathbb{B}, dv_\alpha)$, that is

$$A^p_\alpha(\mathbb{B}) = L^p(\mathbb{B}, dv_\alpha) \cap H(\mathbb{B}).$$

Let φ be a holomorphic self-map of $\mathbb{B}$. For $f \in H(\mathbb{B})$, the composition operator is defined by

$$C_\varphi f(z) = f(\varphi(z)), \quad z \in \mathbb{B}.$$

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The idea of studying the general properties of composition operators was due to Nordgren [1]. As a sequence of Littlewood's subordinate theorem, each φ induces a bounded composition operator on the Hardy spaces $H^p(\mathbb{D})$ and the weighted Bergman spaces $A^p_\alpha(\mathbb{D})$. However, there are different properties of composition operators on $H^p(\mathbb{B})$ and $A^p_\alpha(\mathbb{B})$, for example, there exists unbounded composition operators on $H^p(\mathbb{B})$ and $A^p_\alpha(\mathbb{B})$ (see [2], [3] and [4]). Cima and Wogen [5], Luecking [6] and Zhu [7, 8] characterized all positive Borel measures μ on $\mathbb{B}$ such that

$$\int_{\mathbb{B}} |f(z)|^p d\mu(z) \leq C_1 \int_{\mathbb{B}} |f(z)| dv_\alpha(z) \tag{1}$$

for some constant $C_1 > 0$ and all $f \in A^p_\alpha(\mathbb{B})$. A natural question comes up: What can we say about the same question if we replace condition (1) in the definition of Bergman space by more general condition

$$\int_{\mathbb{B}} \Phi(\gamma |f(z)|) dv_\alpha(z) < \infty \tag{2}$$

for some $\gamma > 0$ depending on f, where $\Phi : [0, \infty) \to [0, \infty)$ is a continuous non-decreasing function such that $\Phi(0) = 0$ and $\lim_{t\to\infty} \Phi(t) = \infty$.

The set A^Φ_α of all holomorphic functions on $\mathbb{B}$ satisfying condition (2) is an $F-$ space which is called Bergman-Orlicz space, analogous to Hardy-Orlicz space [9, 10, 11]. If Φ is a convex function, then A^Φ_α is a Banach space under the Luxemburg norm

$$\|f\|_\Phi = \inf\Big\{k > 0 : \int_{\mathbb{B}} \Phi\Big(\frac{|f|}{k}\Big) dv_\alpha(z) \leq 1\Big\}.$$

A function Φ satisfies the global "Δ_2-condition" if there is constant $K > 1$ such that $\Phi(2t) \leq K\Phi(t)$ for all $t \geq 0$. If Φ is convex, then A^Φ_α coincides with the set of all holomorphic functions f on $\mathbb{B}$ that satisfy

$$\|f\|_\Phi = \int_{\mathbb{B}} \Phi(|f|) dv_\alpha(z) < \infty.$$

In this paper we assume that $\Phi : [0, \infty) \to [0, \infty)$ is a strictly increasing, convex function satisfying the global "Δ_2-condition" and $\Phi(0) = 0$. Furthermore, we also assume that Φ^{-1} is differentiable. In fact, if Φ^{-1} is not differentiable, then the function

$$\Psi(t) = \int_{t/2}^{t} \frac{\Phi(x)}{x} dx$$

is differentiable on $[0, \infty)$. Since $\Phi(x)/x$ is increasing, we have

$$\Phi(t) \geq \Psi(t) \geq \int_{t/2}^{t} \frac{\Phi(x)}{x} dx \geq \Phi(\frac{t}{2}),$$

hence $A^\Phi_\alpha = A^\Psi_\alpha$.

In section 2 of this paper, we offer some facts concerning pseudo-hyperbolic metric on $\mathbb{B}$ and holomorphic automorphism of $\mathbb{B}$. In section 3, we first obtain a technical lemma constructing a holomorphic function on $\mathbb{B}$, which will be used in characterizing the positive Borel measure μ satisfying

$$\int_{\mathbb{B}} \Phi(|f(z)|)d\mu(z) \le C_1 \int_{\mathbb{B}} dv_\alpha(z)$$

for all $f \in A^\Phi_\alpha$, then we character boundedness of composition operator on A^Φ_α. In section 4, we apply the function constructed in Lemma 3.1 to investigate compactness of composition operator on A^Φ_α.

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other.

2. Pseudo-hyperbolic metric

We recall that for $a \in \mathbb{B}$, an automorphism of the unit ball $\mathbb{B}$ is given by

$$\varphi_a(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a\rangle}, \quad z \in \mathbb{B},$$

where $s_a = (1-|a|^2)^{1/2}$, P_a is the orthogonal projection from $\mathbb{C}^n$ onto the one dimensional subspace $[a]$ generated by a, and Q_a is the orthogonal projection from $\mathbb{C}^n$ onto $\mathbb{C}^n \ominus [a]$. The mapping φ_a maps the ball $B(0,r)$ onto the pseudo-hyperbolic metric ball $\Delta(a,r) = \{z \in \mathbb{B} : |\varphi_a(z)| < r\}$. Moreover, the following property holds:

$$\sup\{|\varphi_a'(z)|^2 : z \in \Delta(a,r)\} = \frac{(1+r|a|)^{2n+2}}{(1-|a|^2)^{n+1}}.$$

Also, there exists a positive integer N such that for each $r \in (0,1)$, we can find a sequence $\{a_k\}$ in $\mathbb{B}$ with the following properties:
(a) $\mathbb{B} = \cup_{k=1}^\infty \Delta(a_k, r)$,
(b) if $k \neq m$, then $|\varphi_{a_k}(a_m)| > r/4$,
(c) each point $z \in \mathbb{B}$ belongs to at most N of the sets $\Delta(a_k, 4r)$.

A sequence $\{a_k\}$ in $\mathbb{B}$ with property (b) above is called separated and r called the separation constant for $\{a_k\}$. For the pseudo-hyperbolic metric ball $\Delta(z,r) = \{w \in \mathbb{B} : |\varphi_z(w)| < r\}$, we have the volume formula

$$v(\Delta(z,r)) = \frac{r^{2n}(1-|z|^2)^{n+1}}{(1-r^2|z|^2)^{n+1}}.$$

Moreover,

$$C_1(1-|z|^2)^{n+1+\alpha} \le v_\alpha(\Delta(z,r)) \le C_2(1-|z|^2)^{n+1+\alpha},$$

and for $a \in \Delta(z,r)$,

$$C^{-1} \le \frac{1-|a|^2}{1-|z|^2} \le C$$

and

$$C^{-1} \le \frac{1-|z|^2}{|1-\langle z,a\rangle|} \le C.$$

For some related information of pseudo-hyperbolic metric on $\mathbb{B}$ can be found in [8] and [12].

3. Carleson Measure

A finite positive Borel measure μ on $\mathbb{B}$ is called a *Carleson measure* if there exits a constant $C > 0$ such that $\mu(\Delta(a, r)) \leq C v_\alpha(\Delta(a, r))$ for all $0 < r < 1$ and $a \in \mathbb{B}$. It is well-known that the measure μ on $\mathbb{B}$ is a Carleson measure if and only if the Bergman space $A^p_\alpha(\mathbb{B})$ is boundedly contained in $L^p(\mathbb{B}, d\mu)$ (see [8]). In this section we obtain the similar result for Bergman-Orlicz space case. For this goal we need two the following lemmas.

Lemma 3.1 *For each $a \in \mathbb{B}$, there is a holomorphic function f on $\mathbb{B}$ such that*

$$\Phi(|f(z)|) = \Big(\frac{1 - |a|^2}{|1 - \langle z, a\rangle|^2}\Big)^{n+1+\alpha}.$$

Proof. For $z = (z_1, z_2, ..., z_n) \in \mathbb{B}$, taking $u(z) = \Phi^{-1}\Big(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\Big)^{n+1+\alpha}$, then $u(z)$ is a real and continuously differentiable function. We set $f(z) = u(z) \exp iv(z)$. In order to turn f into a holomorphic function on $\mathbb{B}$, its real and imaginary parts must satisfy the Cauchy-Riemann equations for each variable $z_j = x_j + y_j$; that is,

$$\begin{cases} u_{x_1}\cos(v) - u\sin(v)v_{x_1} = u_{y_1}\sin(v) + u\cos(v)v_{y_1}, \\ u_{y_1}\cos(v) - u\sin(v)v_{y_1} = u_{x_1}\sin(v) + u\cos(v)v_{x_1}, \\ \qquad\qquad \cdots\cdots\cdots\cdots\cdot\cdot \\ u_{x_n}\cos(v) - u\sin(v)v_{x_n} = u_{y_n}\sin(v) + u\cos(v)v_{y_n}, \\ u_{y_n}\cos(v) - u\sin(v)v_{y_n} = u_{x_n}\sin(v) + u\cos(v)v_{x_n}; \end{cases}$$

which can be satisfied if the relations $uv_{x_j} = -u_{y_j}$ and $uv_{y_j} = u_{x_j}$ hold for $j = 1, 2, ..., n$. Thus we can find a such function v by solving the following equations:

$$\begin{cases} uv_{x_1} = -u_{y_1}, \\ uv_{y_1} = u_{x_1}, \\ \quad \cdots\cdots \\ uv_{x_n} = -u_{y_n}, \\ uv_{y_n} = u_{x_n}. \end{cases}$$

Of course, it is easy to find a v that satisfies from above equations.

Remark 3.2 By a calculation it shows that the function f constructed of Lemma 3.1 belongs to A^Φ_α and $\|f\|_\Phi = 1$.

The following lemma is a version of [7] . Here we give the proof for completeness.

Lemma 3.3 *For any $r \in (0, 1)$ and $a \in \mathbb{B}$, there is a positive constant C depending on r such that*

$$\Phi(|f(a)|) \leq \frac{C}{v_\alpha(\Delta(a, r))} \int_{\Delta(a,r)} \Phi(|f(z)|) dv_\alpha(z)$$

for any holomorphic function on $\mathbb{B}$.

Proof. Since $|f\circ\varphi_a|$ is a subharmonic function on $\mathbb{B}$, we have

$$|f\circ\varphi_a(0)| \le \frac{1}{v_{\alpha(\Delta(0,r))}}\int_{\Delta(0,r)}|f\circ\varphi_a(w)|dv_\alpha(w). \tag{1}$$

Because Φ is increasing and convex, by (1) we obtain

$$\Phi(|f\circ\varphi_a(0)|) \le \frac{1}{v_\alpha(\Delta(0,r))}\int_{\Delta(0,r)}\Phi(|f\circ\varphi_a(w)|)dv_\alpha(w).$$

Making a change of variables $z=\varphi_a(w)$, then

$$\begin{aligned}\Phi(|f(a)|) &\le \frac{1}{v_\alpha(\Delta(0,r))}\int_{\Delta(a,r)}\Phi(|f(z)|)|\varphi_a'(z)|^2dv_\alpha(z)\\ &= \frac{1}{v_\alpha(\Delta(0,r))}\int_{\Delta(a,r)}\Phi(|f(z)|)\Big(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\Big)^{n+\alpha+1}dv_\alpha(z)\\ &\le \frac{1}{v_\alpha(\Delta(0,r))(1-|a|^2)^{n+1+\alpha}}\int_{\Delta(a,r)}\Phi(|f(z)|)dv_\alpha(z)\\ &\le \frac{C}{v_\alpha(\Delta(a,r))}\int_{\Delta(a,r)}\Phi(|f(z)|)dv_\alpha(z).\end{aligned}$$

Remark 3.4 By Lemma 3.3, we get the following growth estimation

$$|f(z)| \le \Phi^{-1}\Big(\frac{C\|f\|_\Phi}{(1-|z|^2)^{n+1+\alpha}}\Big), \quad z\in\mathbb{B}.$$

Therefore, Lemma 3.3 shows that the counting functional $\delta_z : f\mapsto f(z)$ is continuous on A^Φ_α.

Theorem 3.5 *Suppose that μ is a finite positive Borel measure on $\mathbb{B}$, then μ is a Carleson measure if and only if there exists a constant $C>0$ such that*

$$\int_{\mathbb{B}}\Phi(|f(z)|)d\mu(z) \le C\int_{\mathbb{B}}\Phi(|f(z)|)dv_\alpha(z)$$

for all $f\in A^\Phi_\alpha$.

Proof. First assume that μ is a Carleson measure. Then for $r\in(0,1)$, there exists a constant $C>0$ such that $\mu(\Delta(a,r))\le Cv_\alpha(\Delta(a,r))$ for any $a\in\mathbb{B}$. For such $r\in(0,1)$, choose a r-separated sequence $\{a_k\}$ and a positive integer N such that each point in $\mathbb{B}$ belongs to at most N sets in $\{\Delta(a_k,4r)\}$; thus, since $\mathbb{B}=\cup_k\Delta(a_k,r)$, we have

$$\begin{aligned}\int_{\mathbb{B}}\Phi(|f(z)|)d\mu(z) &\le \sum_{k=1}^\infty\int_{\Delta(a_k,r)}\Phi(|f(z)|)d\mu(z)\\ &\le \sum_{k=1}^\infty\mu(\Delta(a_k,r))\sup\{\Phi(|f(z)|) : z\in\Delta(a_k,r)\}\\ &\le C_1\sum_{k=1}^\infty\frac{\mu(\Delta(a_k,r))}{v_\alpha(\Delta(a_k,r))}\int_{\Delta(a_k,4r)}\Phi(|f(z)|)dv_\alpha(z)\end{aligned}$$

$$\leq CC_1 \sum_{k=1}^{\infty} \int_{\Delta(a_k,4r)} \Phi(|f(z)|) dv_\alpha(z)$$
$$\leq CC_1 N \int_{\mathbb{B}} \Phi(|f(z)|) dv_\alpha(z),$$

where we have used Lemma 3.1 and the facts of section 2.

Next, assume that there exists a constant $C > 0$ such that

$$\int_{\mathbb{B}} \Phi(|f(z)|) d\mu(z) \leq C \int_{\mathbb{B}} \Phi(|f(z)|) dv_\alpha(z)$$

for all $f \in A_\alpha^\Phi$. Fix $r \in (0,1)$ and $a \in \mathbb{B}_n$. By Lemma 3.1 we get a holomorphic function f on $\mathbb{B}$ such that

$$\Phi(|f(z)|) = \Big(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2} \Big)^{n+1+\alpha}$$

Applying the hypothesis to the function f, the fact that there is a constant $C > 0$ depending on r such that $v_\alpha(\Delta(a,r)) \geq C(1-|a|^2)^{n+1+\alpha}$, and the identity

$$\int_{\mathbb{B}} \Big(\frac{1-|a|^2}{|1-<z,a>|^2} \Big)^{n+1+\alpha} dv_\alpha(z) = 1$$

we get

$$\begin{aligned} C &\geq \int_{\mathbb{B}} \Big(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2} \Big)^{n+1+\alpha} d\mu(z) \geq \int_{\Delta(a,r)} \Big(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2} \Big)^{n+1+\alpha} d\mu(z) \\ &\geq \frac{C}{(1-|a|^2)^{n+1+\alpha}} \mu(\Delta(a,r)) \geq \frac{C\mu(\Delta(a,r))}{v_\alpha(\Delta(a,r))}. \end{aligned}$$

The proof of Theorem 3.5 is complete.

Since

$$\|C_\varphi f\|_\Phi = \int_{\mathbb{B}} \Phi(|f \circ \varphi(z)|) dv_\alpha(z) = \int_{\mathbb{B}} \Phi(|f(z)|) dv_\alpha \circ \varphi^{-1}(z)$$

for all $f \in A_\alpha^\Phi$, by Theorem 3.5 we have the following corollary.

Corollary 3.6 *Suppose that φ is a holomorphic self-map of $\mathbb{B}$, then C_φ is bounded on A_α^Φ if and only if the measure $v_\alpha \circ \varphi^{-1}$ is a Carleson measure.*

4. Compactness

In this section, we investigate compact composition operator on Bergman-Orlicz space. We begin with the following lemma, which characterizes compact composition operator in terms of sequential convergence.

Lemma 4.1 *Suppose that φ is a holomorphic self-map of $\mathbb{B}$, then C_φ is compact on A_α^Φ if and only if for each bounded sequence $\{f_k\}$ in A_α^Φ which converges to 0 uniformly on compact subsets of $\mathbb{B}$, one has $\|C_\varphi f_k\|_\Phi \to 0$.*

Proof. By Lemma 3.3 and standard technique of Montel Theorem the lemma follows easily from the definition of compact operator.

Remark 4.2 It is known, for who are interested in composition operators on analytic function spaces, that often using sequential convergence analogous to the Lemma 4.1 characterizes the compactness of these operators. Here we offer a general compactness criterion. We say that a Banach space of holomorphic functions on open subset Ω of $\mathbb{C}^n$ has the *Fatou property* if X is continuously embedded into $H(\Omega)$, the space $H(\Omega)$ with it's natural topology of compact convergence, and if X has the following property: for each bounded sequence $\{f_k\}$ in X which converges uniformly on any compact subset of Ω to function f, one has $f \in X$. By this criterion, it follows that C_φ is compact on X if and only if for each bounded sequence $\{f_k\}$ in X which converges to 0 uniformly on compact subsets of Ω, one has $\|C_\varphi f_k\|_X \to 0$.

Theorem 4.3 *Suppose that φ is a holomorphic self-map of $\mathbb{B}$, then the bounded operator C_φ is compact on A^Φ_α if and only if*

$$\lim_{\delta\to 0} \frac{v_\alpha \circ \varphi^{-1}(Q(\xi,\delta))}{\delta^{n+1+\alpha}} = 0.$$

Proof. First assume that C_φ is compact on A^Φ_α. Let $a = (1-\delta)\xi$, $\xi \in \partial\mathbb{B}$, $\delta \in (0,1)$ and $f_a(z) = f(z)$, where $f(z)$ is the function of Lemma 3.1. Since Φ^{-1} is continuous at $t = 0$ with $\Phi^{-1}(0) = 0$, $f_a \to 0$ uniformly on compact subsets of $\mathbb{B}$ as $|a| \to 1$. Moreover,

$$\|f_a\|_\Phi = \int_{\mathbb{B}} \Phi(|f_a(z)|)dv_\alpha(z) = \int_{\mathbb{B}} \Phi(|f(z)|)dv_\alpha(z) = 1.$$

On the other hand, if $|1 - \langle z,\zeta\rangle|/(1-|a|) < \gamma$ for some fixed $0 < \gamma < 1/4$, where $\zeta = a/|a|$, i.e. $z \in Q(\gamma\delta,\zeta)$, then

$$\begin{aligned}\frac{1-|a|^2}{|1-\langle z,a\rangle|^2} &= \frac{1-|a|^2}{(1-|a|)^2}\frac{(1-|a|)^2}{|1-\langle z,a\rangle|^2} = \frac{1-|a|^2}{(1-|a|)^2}\left|1+\frac{|a|(1-\langle z,a\rangle)}{1-|a|}\right|^{-2} \\ &\geq \frac{1}{(1+\gamma)^2}\frac{1-|a|^2}{(1-|a|)^2} > \frac{1}{4}\frac{1-|a|^2}{(1-|a|)^2} \geq \frac{1}{4\delta}.\end{aligned}$$

Hence, for $z \in Q(\gamma\delta,\zeta)$ we have

$$\Phi^{-1}\Big(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\Big)^{n+1+\alpha} \geq \Phi^{-1}\Big(\frac{1}{4\delta}\Big)^{n+1+\alpha}$$

and

$$\Phi(|f_a(z)|) = \Big(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\Big)^{n+1+\alpha} \geq \Big(\frac{1}{4\delta}\Big)^{n+1+\alpha}.$$

So, for all $\xi \in \partial\mathbb{B}$ and $\delta \in (0,1)$,

$$\Big(\frac{1}{4\delta}\Big)^{n+1+\alpha} v_\alpha \circ \varphi^{-1}(Q(\gamma\delta,\xi)) \leq \int_{Q(\gamma\delta,\xi)} \Phi(|f_a(z)|)dv_\alpha \circ \varphi^{-1}(z) \leq \int_{\mathbb{B}} \Phi(|f_a(z)|)dv_\alpha \circ \varphi^{-1}(z)$$

$$= \int_{\mathbb{B}} \Phi(|f_a \circ \varphi(z)|) dv_\alpha(z) = \|C_\varphi f_a\|_\Phi.$$

Because of the compactness of C_φ, it must have $\|C_\varphi f_a\|_\Phi \to 0$, which means that

$$\lim_{\delta\to 0} \frac{v_\varphi \circ \varphi^{-1}(Q(\gamma\delta, \xi))}{\delta^{n+1+\alpha}} = 0.$$

Conversely, let $\{f_k\}$ be a bounded sequence by a constant $M > 0$ in A^Φ_α and converges to zero uniformly on compact subsets of $\mathbb{B}$. By Lemma 4.1, it is enough to show that $\|f_k \circ \varphi\|_\Phi \to 0$ as $k \to \infty$. For $\varepsilon > 0$, there is a constant $r \in (0, 1)$ whenever $r < |a| < 1$, we have

$$\frac{v_\varphi \circ \varphi^{-1}(Q(1 - |a|, \xi))}{(1 - |a|^2)^{n+1+\alpha}} < \varepsilon.$$

For above $r \in (0, 1)$, we can choose a sequence $\{a_k\}$ of section 2 such that whenever $k > K_0$, it follows that

$$\frac{v_\varphi \circ \varphi^{-1}(Q(1 - |a_k|, \xi))}{(1 - |a_k|^2)^{n+1+\alpha}} < \varepsilon.$$

Hence, for such k we have

$$\begin{aligned} \|C_\varphi f_k\|_\Phi &= \int_{\mathbb{B}} \Phi(|f_k(z)|) dv_\alpha \circ \varphi^{-1}(z) \le \sum_{k=1}^{\infty} \int_{\Delta(a_k,r)} \Phi(|f_k(z)|) dv_\alpha \circ \varphi^{-1}(z) \\ &= \sum_{k=1}^{K_0} \int_{\Delta(a_k,r)} \Phi(|f_k(z)|) dv_\alpha \circ \varphi^{-1}(z) + \sum_{k=K_0+1}^{\infty} \int_{\Delta(a_k,r)} \Phi(|f_k(z)|) dv_\alpha \circ \varphi^{-1}(z) \end{aligned}$$

For the second term in above equality, we have

$$\begin{aligned} \sum_{k=K_0+1}^{\infty} \int_{\Delta(a_k,r)} \Phi(|f_k(z)|) dv_\alpha \circ \varphi^{-1}(z) &\le C \sum_{k=K_0+1}^{\infty} \frac{v_\alpha \varphi^{-1}(\Delta(a_k, r))}{v_\alpha(\Delta(a_k, r))} \int_{\Delta(a_k,4r)} \Phi(|f_k(z)|) dv_\alpha \circ \varphi^{-1}(z) \\ &\le \varepsilon C \sum_{k=K_0+1}^{\infty} \int_{\Delta(a_k,4r)} \Phi(|f_k(z)|) dv_\alpha \circ \varphi^{-1}(z) \\ &\le \varepsilon C \int_{\mathbb{B}} \Phi(|f_k(z)|) dv_\alpha \circ \varphi^{-1}(z) \\ &\le \varepsilon CM. \end{aligned}$$

Hence the second term in above equality is bounded by a constant multiple of ε. Since $f_k \to 0$ uniformly on compact subsets of $\mathbb{B}$, the first term in above equality is bounded by a constant multiple of ε.

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