Class with Negative Coefficients and Starlike with Respect to Conjugate Points

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Abstract

Let \( S^*_{cT}(A, B) \) denote the class of functions \( f \) which are analytic in an open unit disc \( D = \{ z : |z| < 1 \} \) and satisfying the condition

\[
\frac{2zf'(z)}{f(z) + f(\overline{z})} \preceq 1 + Az + Bz, \quad -1 \leq B < A \leq 1, \quad z \in D.
\]

The aim of paper is to determine coefficient estimates for the class \( S^*_{cT}(A, B) \).

Mathematics Subject Classification: Primary 30C45

Keywords: starlike with respect to conjugate points, coefficient estimates

1 Introduction

Let \( U \) be the class of functions which are analytic in the open unit disc \( D = \{ z : |z| < 1 \} \) given by

\[
w(z) = \sum_{k=1}^{\infty} b_k z^k
\]

and satisfying the conditions

\[
w(0) = 0, \quad |w(z)| < 1, \quad z \in D.
\]

Let \( S \) denote the class of functions \( f \) which are analytic and univalent in \( D \) of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in D.
\]

Also, let \( S^*_s \) be the subclass of \( S \) consisting of functions given by (1) satisfying

\[
\text{Re} \left\{ \frac{zf'(z)}{f(z) - f(-z)} \right\} > 0, \quad z \in D.
\]
These functions are called starlike with respect to symmetric points and were introduced by Sakaguchi in 1959. El-Ashwah and Thomas in [1], introduced two other classes namely the class $S^*_c$ consisting of functions starlike with respect to conjugate points and $S^*_{sc}$ consisting of functions starlike with respect to symmetric conjugate points.

Further, let $f, g \in \mathcal{U}$. Then we say that $f$ is subordinate to $g$, and we write $f \prec g$, if there exists a function $w \in \mathcal{U}$ such that $f(z) = g(w(z))$ for all $z \in \mathcal{D}$. Specially, if $g$ is univalent in $\mathcal{D}$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathcal{D}) \subseteq g(\mathcal{D})$.

In terms of subordination, Goel and Mehrok in 1982 introduced a subclass of $S^*_s$ denoted by $S^*_s(A, B)$. Let $S^*_s(A, B)$ denote the class of functions of the form (1) and satisfying the condition

$$\frac{2zf'(z)}{f(z) - f(-z)} \prec \frac{1 + Az}{1 + Bz}, \quad -1 \leq B < A \leq 1, \quad z \in \mathcal{D}.$$  

However, for this paper, we consider a subclass of $\mathcal{T}$ where $\mathcal{T}$ denotes the class consisting of functions $f$ of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathcal{D}. \quad (2)$$

Now, let consider $S^*_c T(A, B)$ be the class of functions of the form (2) and satisfying the condition

$$\frac{2zf'(z)}{f(z) + f(z)} \prec \frac{1 + Az}{1 + Bz}, \quad -1 \leq B < A \leq 1, \quad z \in \mathcal{D}.$$  

By definition of subordination it follows that $f \in S^*_c T(A, B)$ if and only if

$$\frac{2zf'(z)}{f(z) + f(z)} = \frac{1 + Aw(z)}{1 + Bw(z)} = P(z), \quad w \in \mathcal{U} \quad (3)$$

where

$$P(z) = 1 + \sum_{n=1}^{\infty} p_n z^n. \quad (4)$$

We study the class $S^*_c T(A, B)$ and obtain coefficient estimates.

2 Preliminary Result

We need the following preliminary lemma, required for proving our result.
Lemma 2.1 ([2]) If $P(z)$ is given by (4) then

$$|p_n| \leq (A - B).$$ \hfill (5)

3 Main Result

We give the coefficient inequalities for the class $S_c^*(A, B)$.

Theorem 3.1 Let $f \in S_c^*(A, B)$, then for $n \geq 1$,

$$|a_{2n}| \leq \frac{(A - B)^{2n-2}}{(2n - 1)!} \prod_{j=1}^{2n-1} (A - B + j),$$ \hfill (6)

$$|a_{2n+1}| \leq \frac{(A - B)^{2n-1}}{(2n)!} \prod_{j=1}^{2n} (A - B + j).$$ \hfill (7)

Proof.

For (3) and (4), we have

$$2(z - 2a_2z^2 - 3a_3z^3 - ... - 2na_{2n}z^{2n} - (2n + 1)a_{2n+1}z^{2n+1} - ...)
\begin{align*}
= 2(z &- a_2z^2 - a_3z^3 - a_4z^4 - ... - a_{2n}z^{2n} - a_{2n+1}z^{2n+1} - ...) \\
&\cdot (1 + p_1z + p_2z^2 + ... + p_{2n}z^{2n} + p_{2n+1}z^{2n+1} + ...)
\end{align*}
$$

Equating the coefficients of like powers of $z$, we have

$$-a_2 = p_1, \quad -2a_3 = p_2 - a_2p_1$$ \hfill (8)

$$-3a_4 = p_3 - a_2p_2 - a_3p_1, \quad -4a_5 = p_4 - a_2p_3 - a_3p_2 - a_4p_1$$ \hfill (9)

$$-(2n - 1)a_{2n} = p_{2n-1} - a_2p_{2n-1} - a_3p_{2n-3} - ... - a_{2n-1}p_1$$ \hfill (10)

$$-(2n)a_{2n+1} = p_{2n} - a_2p_{2n-1} - a_3p_{2n-2} - ... - a_{2n}p_1.$$ \hfill (11)

Easily using Lemma 2.1 and (8), we get

$$|a_2| \leq A - B, \quad |a_3| \leq \frac{(A - B)(A - B + 1)}{2}.$$ \hfill (12)

Again by applying (12) and followed by Lemma 2.1, we get from (9)

$$|a_4| \leq \frac{(A - B)(A - B + 1)(A - B + 2)}{2(3)}.$$
It follows that (6) and (7) hold for \( n=1,2 \). We now prove (6) using induction. Equation (10) in conjunction with Lemma 2.1 yield

\[
|a_{2n}| \leq \frac{A-B}{2n-1} \left[ 1 + \sum_{k=1}^{n-1} |a_{2k}| + \sum_{k=1}^{n-1} |a_{2k+1}| \right].
\]  

We assume that (6) holds for \( k=3,4,\ldots,(n-1) \). Then from (13), we obtain

\[
|a_{2n}| \leq \frac{A-B}{2n-1} \left[ 1 + \sum_{k=1}^{n-1} \frac{A-B}{(2k-1)!} \prod_{j=1}^{2k-2} (A-B+j) + \sum_{k=1}^{n-1} \frac{A-B}{(2k)!} \prod_{j=1}^{2k-1} (A-B+j) \right].
\]  

In order to complete the proof, it is sufficient to show that

\[
\frac{A-B}{2m-1} \left[ 1 + \sum_{k=1}^{m-1} \frac{A-B}{(2k-1)!} \prod_{j=1}^{2k-2} (A-B+j) + \sum_{k=1}^{m-1} \frac{A-B}{(2k)!} \prod_{j=1}^{2k-1} (A-B+j) \right]
\]

\[
= \frac{A-B}{(2m-1)!} \prod_{j=1}^{2m-2} (A-B+j), \quad (m = 3, 4, \ldots, n).
\]  

(15) is valid for \( m = 3, 4, \ldots, n \).

Let us suppose that (15) is true for all \( m, \ 3 < m \leq (n - 1) \). Then from (14)

\[
\frac{A-B}{2n-1} \left[ 1 + \sum_{k=1}^{n-1} \frac{A-B}{(2k-1)!} \prod_{j=1}^{2k-2} (A-B+j) + \sum_{k=1}^{n-1} \frac{A-B}{(2k)!} \prod_{j=1}^{2k-1} (A-B+j) \right]
\]

\[
= \left( \frac{2n-3}{2n-1} \right) \left( \frac{A-B}{2(n-1)-1} \right) \left[ 1 + \sum_{k=1}^{n-2} \frac{A-B}{(2k-1)!} \prod_{j=1}^{2k-2} (A-B+j) + \sum_{k=1}^{n-2} \frac{A-B}{(2k)!} \prod_{j=1}^{2k-1} (A-B+j) \right]
\]

\[
+ \frac{A-B}{2n-1} \frac{A-B}{(2(n-1)-1)!} \prod_{j=1}^{2n-4} (A-B+j) + \frac{A-B}{2n-1} \frac{A-B}{(2(n-1)-1)!} \prod_{j=1}^{2n-3} (A-B+j)
\]

\[
= \frac{2n-3}{2n-1} \frac{A-B}{(2(n-1)-1)!} \prod_{j=1}^{2n-4} (A-B+j)
\]

\[
+ \frac{A-B}{2n-1} \frac{A-B}{(2(n-1)-1)!} \prod_{j=1}^{2n-4} (A-B+j) + \frac{A-B}{2n-1} \frac{A-B}{(2(n-1)-1)!} \prod_{j=1}^{2n-3} (A-B+j)
\]

\[
= \frac{A-B}{(2n-1)(2(n-1)-1)!} \prod_{j=1}^{2n-4} (A-B+j)(A-B+2n-3)
\]
Thus, (15) holds for $m = n$ and hence (6) follows. Similarly, we can prove (7).

References


Received: April, 2009