

Class with Negative Coefficients and Quasi-Convex with Respect to Symmetric Points

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Abstract

Let $C_sT(A, B)$ denote the class of functions f which are analytic in an open unit disc $\mathcal{D} = \{z : |z| < 1\}$ and satisfying the condition $\frac{2(zf'(z))'}{(f(z)-f(-z))'} \prec \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, $z \in \mathcal{D}$. In this paper, we consider the class $K_s^*T(A, B)$ consisting of analytic functions f and satisfying $\frac{(zf'(z))'}{(g(z)-g(-z))'} \prec \frac{1+Az}{1+Bz}$, $g \in C_sT(A, B)$, $-1 \leq B < A \leq 1$, $z \in \mathcal{D}$. The aims of paper are to determine coefficient estimates and distortion bounds for the class $K_s^*T(A, B)$.

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1 Introduction

Let \mathcal{U} be the class of functions which are analytic in the open unit disc $\mathcal{D} = \{z : |z| < 1\}$ given by

$$w(z) = \sum_{k=1}^{\infty} b_k z^k$$

and satisfying the conditions

$$w(0) = 0, |w(z)| < 1, z \in \mathcal{D}.$$

Let \mathcal{S} denote the class of functions f which are analytic and univalent in \mathcal{D} of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in \mathcal{D}. \quad (1)$$

Also, let \mathcal{S}_s^* be the subclass of \mathcal{S} consisting of functions given by (1) satisfying

$$\operatorname{Re} \left\{ \frac{z f'(z)}{f(z) - f(-z)} \right\} > 0, \quad z \in \mathcal{D}.$$

These functions are called starlike with respect to symmetric points and were introduced by Sakaguchi in 1959. Then, Das and Singh in 1977 extend the results of Sakaguchi to other class in \mathcal{D} , namely convex functions with respect to symmetric points. Let \mathcal{C}_s denote the class of convex functions with respect to symmetric points and satisfying the condition

$$\operatorname{Re} \left\{ \frac{(z f'(z))'}{(f(z) - f(-z))'} \right\} > 0, \quad z \in \mathcal{D}.$$

In 1982, Goel and Mehrotra introduced a subclass of \mathcal{S}_s^* which were denoted by $\mathcal{S}_s^*(A, B)$.

Let denote $\mathcal{S}_s^*(A, B)$ the class of functions of the form (1) and satisfying the condition

$$\frac{2z f'(z)}{f(z) - f(-z)} \prec \frac{1 + Az}{1 + Bz}, \quad -1 \leq B < A \leq 1, \quad z \in \mathcal{D}.$$

Janteng and Halim in 2008 considered the class of functions of the form (1) denoted by $\mathcal{C}_s(A, B)$ and satisfying the condition

$$\frac{2(z f'(z))'}{(f(z) - f(-z))'} \prec \frac{1 + Az}{1 + Bz}, \quad -1 \leq B < A \leq 1, \quad z \in \mathcal{D}.$$

However, for this paper, we consider a subclass of \mathcal{T} where \mathcal{T} denotes the class consisting of functions f of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathcal{D}. \quad (2)$$

In 2009, Janteng and Abdul Halim considered $\mathcal{C}_s \mathcal{T}(A, B)$ as the class of functions of the form (2) and satisfying the condition

$$\frac{2(z f'(z))'}{(f(z) - f(-z))'} \prec \frac{1 + Az}{1 + Bz}, \quad -1 \leq B < A \leq 1, \quad z \in \mathcal{D}.$$

In this paper, let consider $K_s^*T(A, B)$ be the class of functions of the form (2) and satisfying the condition

$$\frac{2(zf'(z))'}{(g(z) - g(-z))'} \prec \frac{1 + Az}{1 + Bz}, \quad g \in C_sT(A, B), \quad -1 \leq B < A \leq 1, \quad z \in \mathcal{D}.$$

By definition of subordination it follows that $f \in K_s^*T(A, B)$ if and only if

$$\frac{2(zf'(z))'}{(g(z) - g(-z))'} = \frac{1 + Aw(z)}{1 + Bw(z)} = P(z), \quad w \in \mathcal{U} \quad (3)$$

where

$$P(z) = 1 + \sum_{k=1}^{\infty} p_k z^k. \quad (4)$$

We study the class $K_s^*T(A, B)$ and obtain coefficient estimates and distortion bounds.

2 Preliminary Results

We need the following preliminary results, required for proving our results.

Lemma 2.1 ([3]) *If $P(z)$ is given by (4) then*

$$|p_n| \leq (A - B). \quad (5)$$

Theorem 2.1 ([5]) *Let $f \in C_sT(A, B)$, then for $n \geq 1$,*

$$|b_{2n}| \leq \frac{(A - B)}{(2n)n!2^n} \prod_{j=1}^{n-1} (A - B + 2j),$$

$$|b_{2n+1}| \leq \frac{(A - B)}{(2n + 1)n!2^n} \prod_{j=1}^{n-1} (A - B + 2j).$$

3 Main Results

First, we give the coefficient inequalities for the class $K_s^*T(A, B)$.

Theorem 3.1 *Let $f \in K_s^*T(A, B)$, then for $n \geq 1$,*

$$|a_{2n}| \leq \frac{(A - B)}{(2n)n!2^n} \prod_{j=1}^{n-1} (A - B + 2j). \quad (6)$$

and

$$|a_{2n+1}| \leq \frac{(A - B)}{(2n + 1)^2(n - 1)!2^{n-1}} \prod_{j=1}^{n-1} (A - B + 2j). \quad (7)$$

Proof.

Since $g \in \mathcal{C}_sT(A, B)$, it follows that

$$2(zg'(z))' = (g(z) - g(-z))'K(z)$$

for $z \in \mathcal{D}$, with $\operatorname{Re} K(z) > 0$ where $K(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \dots$. Upon equating coefficients, we obtain

$$-2^2b_2 = c_1, \quad -3(2)b_3 = c_2, \quad (8)$$

$$-4^2b_4 = c_3 - 3b_3c_1, \quad -5(4)b_5 = c_4 - 3b_3c_2. \quad (9)$$

For (3) and (4), we have

$$\begin{aligned} & 2(1 - 2^2a_2z - 3^2a_3z^2 - 4^2a_4z^3 - 5^2a_5z^4 - \dots - (2n)^2a_{2n}z^{2n-1} - (2n+1)^2a_{2n+1}z^{2n} - \dots) \\ &= 2(1 - 3b_3z^2 - 5b_5z^4 - 7b_7z^6 - \dots - (2n-1)a_{2n-1}z^{2n-2} - a_{2n+1}z^{2n} - \dots) \\ & \quad \bullet (1 + p_1z + p_2z^2 + \dots + p_{2n}z^{2n} + p_{2n+1}z^{2n+1} + \dots) \end{aligned}$$

Equating the coefficients of like powers of z , we have

$$-2^2a_2 = p_1, \quad -3^2a_3 = p_2 - 3b_3 \quad (10)$$

$$-4^2a_4 = p_3 - 3b_3p_1, \quad -5^2a_5 = p_4 - 3b_3p_2 - 5b_5 \quad (11)$$

$$-(2n)^2a_{2n} = p_{2n-1} - 3b_3p_{2n-3} - 5b_5p_{2n-5} - \dots - (2n-1)b_{2n-1}p_1 \quad (12)$$

$$-(2n+1)^2a_{2n+1} = p_{2n} - 3b_3p_{2n-2} - 5b_5p_{2n-4} - \dots - (2n-1)b_{2n-1}p_2.$$

Easily using Lemma 2.1 and (10), we get

$$|a_2| \leq \frac{A-B}{2(2)}, \quad |a_3| \leq \frac{A-B}{3(2)}.$$

Again by applying (8), (9) and followed by Lemma 2.1, we get from (11)

$$|a_4| \leq \frac{(A-B)(A-B+2)}{4(4)(2)}, \quad |a_5| \leq \frac{(A-B)(A-B+2)}{5(4)(2)}.$$

It follows that (6) hold for $n=1,2$. We now prove (6) using induction. Equation (12) in conjunction with Lemma 2.1 yield

$$|a_{2n}| \leq \frac{A-B}{(2n)^2} \left[1 + \sum_{k=1}^{n-1} (2k+1)|b_{2k+1}| \right] \quad (13)$$

We assume that (6) holds for $k=3,4,\dots,(n-1)$. Then from (13) and Theorem 2.1, we obtain

$$|a_{2n}| \leq \frac{A - B}{(2n)^2} \left[1 + \sum_{k=1}^{n-1} \frac{A - B}{k!2^k} \prod_{j=1}^{k-1} (A - B + 2j) \right]. \tag{14}$$

In order to complete the proof, it is sufficient to show that

$$\frac{A - B}{(2m)^2} \left[1 + \prod_{k=1}^{m-1} \frac{A - B}{k!2^k} \prod_{j=1}^{k-1} (A - B + 2j) \right] = \frac{A - B}{(2m)m!2^m} \prod_{j=1}^{m-1} (A - B + 2j), \quad (m = 3, 4, \dots, n). \tag{15}$$

(15) is valid for $m = 3$.

Let us suppose that (15) is true for all m , $3 < m \leq (n - 1)$. Then from (14)

$$\begin{aligned} & \frac{A - B}{(2n)^2} \left[1 + \prod_{k=1}^{n-1} \frac{A - B}{k!2^k} \prod_{j=1}^{k-1} (A - B + 2j) \right] \\ &= \left(\frac{n - 1}{n} \right)^2 \left(\frac{A - B}{(2(n - 1))^2} \left(1 + \sum_{k=1}^{n-2} \frac{A - B}{k!2^k} \prod_{j=1}^{k-1} (A - B + 2j) \right) \right) \\ & \quad + \frac{A - B}{(2n)^2} \frac{A - B}{(n - 1)!2^{n-1}} \prod_{j=1}^{n-2} (A - B + 2j) \\ &= \left(\frac{n - 1}{n} \right)^2 \frac{A - B}{2(n - 1)(n - 1)!2^{n-1}} \prod_{j=1}^{n-2} (A - B + 2j) + \frac{A - B}{(2n)^2} \frac{A - B}{(n - 1)!2^{n-1}} \prod_{j=1}^{n-2} (A - B + 2j) \\ &= \frac{A - B}{2n^2(n - 1)!2^{n-1}} \prod_{j=1}^{n-2} (A - B + 2j) \left(\frac{A - B + 2(n - 1)}{2} \right) \\ &= \frac{A - B}{(2n)n!2^n} \prod_{j=1}^{n-1} (A - B + 2j) \end{aligned}$$

Thus, (15) holds for $m = n$ and hence (6) follows. Similarly we can prove (7).

Theorem 3.2 Let $f \in K_s^*T(A, B)$, then for $|z| = r$, $0 < r < 1$,

$$\frac{1}{r} \int_0^r \frac{1 - t^2}{(1 + t^2)^2} \frac{(1 - At)}{(1 - Bt)} dt \leq |f'(z)| \leq \frac{1}{r} \int_0^r \frac{1 + t^2}{(1 - t^2)^2} \frac{(1 + At)}{(1 + Bt)} dt. \tag{16}$$

Proof.

Put $h(z) = \frac{g(z)-g(-z)}{2}$. Then from (3), we obtain

$$|(zf'(z))'| = |h'(z)| \left| \frac{1 + Aw(z)}{1 + Bw(z)} \right|. \tag{17}$$

Since h is odd and starlike, it follows that (see [2])

$$\frac{1-r^2}{(1+r^2)^2} \leq |h'(z)| \leq \frac{1+r^2}{(1-r^2)^2}. \quad (18)$$

Furthermore, for $w \in \mathcal{U}$, it can also be easily established that

$$\frac{1-Ar}{1-Br} \leq \left| \frac{1+Aw(z)}{1+Bw(z)} \right| \leq \frac{1+Ar}{1+Br}. \quad (19)$$

Next, applying results (18) and (19) in (17) we obtain

$$\frac{1-r^2}{(1+r^2)^2} \frac{1-Ar}{1-Br} \leq |(zf'(z))'| \leq \frac{1+r^2}{(1-r^2)^2} \frac{1+Ar}{1+Br}. \quad (20)$$

Finally, setting $|z| = r$, and integrating (20) gives our result.

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References

- [1] Das, R.N. and Singh, P. : On subclasses of schlicht mapping, *Indian J. Pure Appl. Math.*, **8**(1977): 864-872.
- [2] Duren, P.L. (1983). *Univalent functions*. New York: Springer-Verlag.
- [3] Goel, R.M. and Mehrok, B.C. : A subclass of starlike functions with respect to symmetric points, *Tamkang J. Math.*, **13**(1)(1982): 11-24.
- [4] Janteng, A. and Abdul Halim, S. : A subclass of convex functions with respect to symmetric points, *Proceedings of The 16th National Symposium on Mathematical Sciences*, **2**(2008): 2-6.
- [5] Janteng, A. and Halim, S.A : Class with negative coefficients and convex with respect to symmetric points, (2009), *submitted*.
- [6] Sakaguchi, K. : On a certain univalent mapping, *J. Math. Soc. Japan*, **11**(1959): 72-75.

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