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Several Discrete Inequalities

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Abstract. In this paper, we establish some interesting discrete inequalities and pose some open problems.

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1. Introduction

The following problem was posed by Qi in his article [13]: *"Under what condition does the inequality*

$$\int_a^b \left[f(x)\right]^t dx \geq \left(\int_a^b f(x)dx\right)^{t-1} \tag{1.1}$$

hold for $t > 1$?".

There are numerous answers and extension results to this open problem [1, 2, 3, 4, 5, 6, 7, 8, 11, 12, 14, 15, 16]. These results were obtained by different approaches, such as, e.g. Jensen's inequality, the convexity method [16]; functional inequalities in abstract spaces [1, 2]; probability measures view [4, 7]; Hölder inequality and its reversed variants [2, 12]; analytical methods [11, 15]; Cauchy's mean value theorem [3, 14].

In [9], the authors introduced the following discrete version of (1.1) as follows, *"Under what condition does the inequality*

$$\sum_{i=1}^{n} x_i^{\alpha} a_i \geq \left(\sum_{i=1}^{n} x_i a_i\right)^{\beta} \tag{1.2}$$

hold for $\alpha, \beta > 0$?". (For the infinite series, the same method in the above finite series can be discussed.) Very recently, some similar discrete inequalities were developed (for instance, the reference [10]). In the paper, based on the results in [5], we will establish some discrete type inequalities and pose some open problems.

2. Main results

Theorem 2.1. *Let $\{x_i, i = 1, \dots, n\}$ and $\{y_i, i = 1, \dots, n\}$ be two sequences of nonnegative real numbers such that $x_1 \le x_2 \le \cdots \le x_n$ and $y_1 \ge y_2 \ge \cdots \ge y_n$. Then we have*

$$\frac{\sum_{i=1}^n x_i^\beta}{\sum_{i=1}^n x_i^\gamma} \ge \frac{\sum_{i=1}^n x_i^\beta y_i^\alpha}{\sum_{i=1}^n x_i^\gamma y_i^\alpha} \tag{2.1}$$

for every positive real number $\alpha > 0$ and $\beta \ge \gamma > 0$. If $x_1 \le x_2 \le \cdots \le x_n$ and $y_1 \le y_2 \le \cdots \le y_n$, then the inequality in (2.1) reverses.

Proof. In order to get the inequality (2.1), it is sufficient to show

$$\sum_{i=1}^n x_i^\beta \sum_{i=1}^n x_i^\gamma y_i^\alpha \ge \sum_{i=1}^n x_i^\gamma \sum_{i=1}^n x_i^\beta y_i^\alpha$$

which is equivalent to

$$\sum_{i=1}^n \sum_{j=1}^n x_i^\beta x_j^\gamma y_j^\alpha \ge \sum_{i=1}^n \sum_{j=1}^n x_i^\gamma x_j^\beta y_j^\alpha.$$

That is to prove

$$\sum_{i=1}^n \sum_{j=1}^n x_i^\gamma x_j^\gamma y_j^\alpha \left(x_i^{\beta-\gamma} - x_j^{\beta-\gamma}\right) \ge 0.$$

Putting

$$D := \sum_{i=1}^n \sum_{j=1}^n x_i^\gamma x_j^\gamma y_j^\alpha \left(x_i^{\beta-\gamma} - x_j^{\beta-\gamma}\right)$$

then it is not difficult to check

$$D = \sum_{i=1}^n \sum_{j=1}^n x_j^\gamma x_i^\gamma y_i^\alpha \left(x_j^{\beta-\gamma} - x_i^{\beta-\gamma}\right).$$

Since $x_1 \le x_2 \le \cdots \le x_n$ and $y_1 \ge y_2 \ge \cdots \ge y_n$, we have

$$2D = \sum_{i=1}^n \sum_{j=1}^n x_j^\gamma x_i^\gamma (y_i^\alpha - y_j^\alpha) \left(x_j^{\beta-\gamma} - x_i^{\beta-\gamma}\right) \ge 0,$$

which implies the inequality (2.1). The second desired result can be obtained by the same proof. □

Theorem 2.2. *Let $\{x_i, i = 1, \dots, n\}$ and $\{y_i, i = 1, \dots, n\}$ be two sequences of nonnegative real numbers such that*

$$\left(x_j^{\beta-\gamma} - x_i^{\beta-\gamma}\right) \left(x_j^\alpha y_i^\alpha - x_i^\alpha y_j^\alpha\right) \ge 0, \quad \text{for all } \ 1 \le i, j \le n. \tag{2.2}$$

Then we have

$$\frac{\sum_{i=1}^n x_i^{\alpha+\beta}}{\sum_{i=1}^n x_i^{\alpha+\gamma}} \ge \frac{\sum_{i=1}^n x_i^\beta y_i^\alpha}{\sum_{i=1}^n x_i^\gamma y_i^\alpha} \tag{2.3}$$

for every positive real number $\alpha > 0$ and $\beta \geq \gamma > 0$. If the inequality (2.2) reverses, then the inequality in (2.3) reverses also.

Proof. We only need to prove

$$D := \sum_{i=1}^{n}\sum_{j=1}^{n} x_i^{\alpha+\gamma} x_j^{\gamma} y_j^{\alpha} \left(x_i^{\beta-\gamma} - x_j^{\beta-\gamma}\right) \geq 0.$$

Since it is not difficult to check

$$D = \sum_{i=1}^{n}\sum_{j=1}^{n} x_j^{\alpha+\gamma} x_i^{\gamma} y_i^{\alpha} \left(x_j^{\beta-\gamma} - x_i^{\beta-\gamma}\right)$$

then it follows that

$$2D = \sum_{i=1}^{n}\sum_{j=1}^{n} x_j^{\gamma} x_i^{\gamma} \left(x_j^{\beta-\gamma} - x_i^{\beta-\gamma}\right) \left(x_j^{\alpha} y_i^{\alpha} - x_i^{\alpha} y_j^{\alpha}\right)$$

which implies that, from the condition (2.2), the inequality (2.3) holds. The proof of the other result is similar to (2.3). □

Theorem 2.3. *Let $\{x_i, i = 1, \ldots, n\}$, $\{y_i, i = 1, \ldots, n\}$ and $\{z_i, i = 1, \ldots, n\}$ be three sequences of nonnegative real numbers such that*

$$(z_i - z_j)\left(\frac{x_j}{y_j} - \frac{x_i}{y_i}\right) \geq 0, \quad \text{for all} \ \ 1 \leq i, j \leq n. \tag{2.4}$$

Then we have

$$\frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} y_i} \geq \frac{\sum_{i=1}^{n} x_i z_i}{\sum_{i=1}^{n} y_i z_i}. \tag{2.5}$$

If the inequality (2.4) reverses, then the inequality in (2.5) reverses also.

Proof. We need to prove

$$\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i z_i \geq \sum_{i=1}^{n} y_i \sum_{i=1}^{n} x_i z_i$$

which is equivalent to

$$D := \sum_{i=1}^{n}\sum_{j=1}^{n} z_j (x_i y_j - y_i x_j) \geq 0.$$

Noting

$$D = \sum_{i=1}^{n}\sum_{j=1}^{n} z_i (x_j y_i - y_j x_i)$$

then we have

$$2D = \sum_{i=1}^{n}\sum_{j=1}^{n} (z_i - z_j)(x_j y_i - y_j x_i) = \sum_{i=1}^{n}\sum_{j=1}^{n} y_i y_j (z_i - z_j) \left(\frac{x_j}{y_j} - \frac{y_i}{x_i}\right)$$

which yields the inequality (2.5) by the condition (2.4). The proof of the other result is similar to (2.5). □

At last, we give some open problems as follows.

Open Problem 1. Can the assumption (2.2) in Theorem 2.2 and the assumption (2.4) in Theorem 2.3 be improved?

Open Problem 2. Under what conditions does the inequality

$$\sum_{i=1}^{n} x_i^{\alpha+\beta} \geq \left(\sum_{i=1}^{n} x_i^{\alpha} y_i^{\beta} \right)^{\gamma}$$

hold for α, β, γ?

Open Problem 3. Under what conditions does the inequality

$$\frac{\sum_{i=1}^{n} x_i^{\alpha+\beta}}{\sum_{i=1}^{n} x_i^{\alpha+\gamma}} \geq \frac{\left(\sum_{i=1}^{n} x_i^{\alpha} y_i^{\beta} \right)^{\delta}}{\left(\sum_{i=1}^{n} x_i^{\alpha} y_i^{\gamma} \right)^{\lambda}}$$

hold for $\alpha, \beta, \gamma, \delta, \lambda$?

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