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Fuzzy Connected Sets in L-Topological Spaces

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Abstract

The purpose of this paper is to investigate some weaker forms of L-fuzzy connectedness notions, namely c_1, c_2, c_3, c_4 and c_s-connectedness in L-topological spaces. Also, we compare these concepts and find the interrelations between them.

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1 Introduction

U. Höhle and A. P. Sostak [7] presented the general theory of fuzzy topological spaces based on the quadruple $M = (L, \leq, \otimes, *)$, where $(L, \leq)$ is a complete lattice, $\otimes$ and $*$ are binary operations on L as a generalizations of a fuzzy topological space given by C. L. Chang [3]. In their works are given the definitions of a fuzzy topological space, an fuzzy interior operator and an fuzzy continuity as L-topological space, an L-interior operator and an L-continuity, respectively.

The aim of this paper is studies the weaker forms of connectedness as c_i-connectedness $(i = 1, 2, 3, 4)$, c_s-connectedness in [1] and L-continuity on the quadruple $M = (L, \leq, \otimes, *)$. Further we investigate the several types of L-connectedness notions under some weaker forms of L-continuous functions in L-topological spaces.

The layout of this paper can be stated as follows: we introduce the fundamental definitions of the quadruple $M = (L, \leq, \otimes, *)$, L-topological spaces,

L-co-topological spaces, L-interior operators, L-closure operators and its properties in Section 2. In the third section, we shall present the weaker forms of connectedness of L-fuzzy sets in L-topological spaces.

2 Preliminary Notes

The lattice theoretical foundations of L-topological spaces and their properties, presented in [7 − 10]. For preliminary notions and definitions we refer to the paper [2]. In this study, we always assume that $(L, \leq)$ is a complete lattice, where $\wedge$, $\vee$, $\mathbf{1}$, $\mathbf{0}$ respectively denote the meet operation, join operation, the top element of L and the bottom element of L, the quadruple $M = (L, \leq, \otimes, *)$ consists of an integral, commutative cl-monoid $(L, \leq, *)$ and a cl-quasi- monoid $(L, \leq, \otimes)$, unless otherwise stated. In any integral, commutative cl-monoid $(L, \leq, *)$, there exists a further binary operation $\longrightarrow$ on L, called the residuum operation on L , such that

$$\alpha * \beta \leq \gamma \Longleftrightarrow \alpha \leq \beta \longrightarrow \gamma, \forall \alpha, \beta, \gamma \in L. \quad (AD) \tag{1}$$

The adjunction property (AD) uniquely determines the residuum $\longrightarrow$, the residuum operation is explicitly given the formula

$$\alpha \longrightarrow \beta = \vee\{\lambda \in L : \alpha * \lambda \leq \beta\}, \forall \alpha, \beta \in L. \tag{2}$$

Furthermore, the capital letters X and Y always stands for the non empty ordinary sets. A map $f : X \longrightarrow L$ is called an L-fuzzy set of X. The set of all L-fuzzy sets of X is denoted by L^X. For each $\alpha \in L$, the constant map $f : X \longrightarrow L$, $f(x) = \alpha, \forall x \in X$, denoted by $\alpha.1_X$. The mappings $\mathbf{0}.1_X$ and $\mathbf{1}.1_X$ are denoted by $1_\varnothing$ and 1_X , respectively. For a given map $\varphi : X \longrightarrow Y$, and for $f \in L^X, g \in L^Y$ the image of f and the inverse image of g under φ are respectively, L-fuzzy sets $\varphi(f) \in L^Y$ and $\varphi^{-1}(g) \in L^X$ defined by

$$\varphi(f)(y) = \left\{ \begin{array}{c} \vee_{x \in \varphi^{-1}(y)} f(x), \text{if } \varphi^{-1}(y) \neq \varnothing \\ \mathbf{0}, \text{otherwise} \end{array} \right\} \text{and } \varphi^{-1}(g) = (g \circ \varphi)(x). \tag{3}$$

For a given quadruple $M = (L, \leq, \otimes, *)$, where $(L, \leq, *)$ is an integral, commutative cl-monoid and $\otimes$ is any binary operation on L, we will need a further binary operation $\sqcup : L \times L \longrightarrow L$, defined by $\alpha \sqcup \beta = {}^\neg(({}^\neg(\alpha) \otimes {}^\neg(\beta))), \forall \alpha, \beta \in L$, is said to be the dual of $\otimes$.

For all $f, g \in L^X$ and for all family $\{f_i : i \in I\} \subseteq L^X$, the operations $\leq, \vee, \wedge, *, \otimes$ and $\sqcup$ can be pointwisely extended to L^X as follows:

(*i*) $f \leq g \Longleftrightarrow f(x) \leq g(x), \forall x \in X$,
(*ii*) $(\bigvee_{i\in I} f_i)(x) = \bigvee_{i\in I} f_i(x), \forall x \in X$,
(*iii*) $(\bigwedge_{i\in I} f_i)(x) = \bigwedge_{i\in I} f_i(x), \forall x \in X$,
(*iv*) $(f * g)(x) = f(x) * g(x), \forall x \in X$,
(*v*) $(f \otimes g)(x) = f(x) \otimes g(x), \forall x \in X$,
(*vi*) $(f \sqcup g)(x) = f(x) \sqcup g(x), \forall x \in X$.

In any integral, commutative cl-monoid $(L, \leq, *)$, we define the negation as an unary operation in the sence of $^\urcorner : L \longrightarrow L$ by $^\urcorner(\alpha) = \alpha \longrightarrow \mathbf{0}$, $\forall \alpha \in L$. The negation operation is order reversing, but not involutive i.e.$^\urcorner(^\urcorner(\alpha)) \neq \alpha$, $\forall \alpha \in L$, in general. This unary operation can be pointwisely extended to L^X as $(^\urcorner(f))(x) = {}^\urcorner(f(x))$, for all $x \in X$. An integral, commutative cl-monoid $(L, \leq, *)$ is called an integral, commutative Girard quantale iff the negation operation "$^\urcorner$" is involutive, i.e. $^\urcorner(^\urcorner(\alpha)) = (\alpha \longrightarrow \mathbf{0}) \longrightarrow \mathbf{0} = \alpha, \forall \alpha \in L$.

Definition 2.1 [7] *A subset τ of L^X is called an L-topology on X iff τ satisfies the following conditions:*
$(LO1)$ $1_X, 1_\emptyset \in \tau$,
$(LO2)$ $f, g \in \tau \Longrightarrow f \otimes g \in \tau, \forall f, g \in L^X$,
$(LO3)$ $\{f_i : i \in I\} \subseteq \tau \Longrightarrow \bigvee_{i\in I} f_i \in \tau, \forall \{f_i : i \in I\} \subseteq L^X$. *For a given L-topology τ on X, the pair (X, τ) is called an L-topological space.*

Definition 2.2 [5] *A subfamily $\mathcal{C}$ of L^X is called an L-co-topology on X iff $\mathcal{C}$ satisfies the following conditions*
$(LC1)$ $1_X, 1_\emptyset \in \mathcal{C}$,
$(LC2)$ $f, g \in \mathcal{C} \Longrightarrow f \sqcup g \in \tau, \forall f, g \in L^X$,
$(LC3)$ $\{f_i : i \in I\} \subseteq \mathcal{C} \Longrightarrow \bigwedge_{i\in I} f_i \in \mathcal{C}, \forall \{f_i : i \in I\} \subseteq L^X$. *For a given L-co-topology $\mathcal{C}$ on X, the pair $(X, \mathcal{C})$ is called an L-co-topological space.*

Theorem 2.3 [5] *Let $(L, \leq, *)$ be an integral, commutative Girard quantale and $\mathcal{C}$ an L-co-topology on X. Then the family $\tau_C = \{f :^\urcorner (f) \in \mathcal{C}\}$ is an L-topology on X. Conversely, for a given L-topology τ on X, the family $\mathcal{C}_\tau = \{f :^\urcorner (f) \in \tau\}$ is an L-co-topology on X. Furthermore, $\mathcal{C}_{\tau_\mathcal{C}} = \mathcal{C}$ and $\tau_{\mathcal{C}_\tau} = \tau$.*

Remark 2.4 *As a consequence of Theorem 2.3, in any integral, commutative Girard quantale $(L, \leq, *)$, L-co-topologies on X and L-topologies on X are the same meaning.*

Definition 2.5 [7] *A map $\mathcal{I} : L^X \longrightarrow L^X$ is called an L-interior operator on X iff $\mathcal{I}$ satisfies the next conditions:*
$(\mathcal{I}.0)$ $\mathcal{I}(1_X) = 1_X$,
$(\mathcal{I}.1)$ $f \leq g \Longrightarrow \mathcal{I}(f) \leq \mathcal{I}(g)$, $\forall f, g \in L^X$,

($\mathcal{I}$.2) $\mathcal{I}(f) \otimes \mathcal{I}(g) \leq \mathcal{I}(f \otimes g), \forall f, g \in L^X$,
($\mathcal{I}$.3) $\mathcal{I}(f) \leq f$, $\forall f \in L^X$,
($\mathcal{I}$.4) $\mathcal{I}(f) \leq \mathcal{I}(\mathcal{I}(f))$, $\forall f \in L^X$.

Remark 2.6 [7] *Each L-topology τ on X induces an L-interior operator $\mathcal{I}_\tau$ by $\mathcal{I}_\tau(f) = \vee\{g \in \tau : g \leq f\}, \forall f \in L^X$. Conversely, each L-interior operator $\mathcal{I}$ induces an L-topology $\tau_{\mathcal{I}}$ by the $\tau_{\mathcal{I}} = \{g \in L^X : g \leq \mathcal{I}(g)\}$. Furthermore, the equalities $\tau_{\mathcal{I}_\tau} = \tau$ and $\mathcal{I} = \mathcal{I}_{\tau_{\mathcal{I}}}$ are valid. Thus, the L-interior operators and L-topologies are equivalent notions.*

Definition 2.7 [5] *A map $\mathcal{C}l : L^X \longrightarrow L^X$ is called an L-closure operator on X iff $\mathcal{C}l$ satisfies the following five conditions:*
($\mathcal{C}l$.0) $\mathcal{C}l(1_\emptyset) = 1_\emptyset$,
($\mathcal{C}l$.1) $f \leq g \Longrightarrow \mathcal{C}l(f) \leq \mathcal{C}l(g)$, $\forall f, g \in L^X$,
($\mathcal{C}l$.2) $\mathcal{C}l(f \sqcup g) \leq \mathcal{C}l(f) \sqcup \mathcal{C}l(g), \forall f, g \in L^X$,
($\mathcal{C}l$.3) $f \leq \mathcal{C}l(f)$, $\forall f \in L^X$,
($\mathcal{C}l$.4) $\mathcal{C}l(\mathcal{C}l(f)) \leq \mathcal{C}l(f)$, $\forall f \in L^X$.

Theorem 2.8 [5] *If $\mathcal{C}l$ be an L-closure operator on X, then the family $\mathcal{C}_{\mathcal{C}l} = \{g \in L^X : \mathcal{C}l(g) \leq g\}$ is an L-co-topology on X. Conversely, if $\mathcal{C}$ be an L-co-topology on X, then the map $\mathcal{C}l_{\mathcal{C}}(f) = \wedge\{h \in \mathcal{C} : f \leq g\}, \forall f \in L^X$, is an L-closure operator on X. Moreover $\mathcal{C} = \mathcal{C}_{\mathcal{C}l_{\mathcal{C}}}$ and $\mathcal{C}l = \mathcal{C}l_{\mathcal{C}_{\mathcal{C}l}}$.*

Corollary 2.9 [5] *Let $(L, \leq, *)$ be an integral, commutative Girard quantale and $\mathcal{I}$ an L-interior operator on X. Then the map $\mathcal{C}l_{\mathcal{I}} : L^X \longrightarrow L^X$, defined by $\mathcal{C}l_{\mathcal{I}}(f) = {}^\neg(\mathcal{I}({}^\neg(f))), \forall f \in L^X$, is an L-closure operator on X. Conversely, for a given L-closure operator $\mathcal{C}l$ on X, the map $\mathcal{I}_{\mathcal{C}l} : L^X \longrightarrow L^X$, defined by $\mathcal{I}_{\mathcal{C}l}(f) = {}^\neg(\mathcal{C}l\ ({}^\neg(f))), \forall f \in L^X$, is an L-interior operator on X. In addition, we have $\mathcal{C}l = \mathcal{C}l_{I_{\mathcal{C}l}}$ and $\mathcal{I} = \mathcal{I}_{\mathcal{C}l_{\mathcal{I}}}$.*

Remark 2.10 *From the Corollary 2.1 we say that for any integral, commutative Girard quantale $(L, \leq, *)$ L-interior operators and L-closure operators are equivalent concepts.*

Definition 2.11 [5] *Let (X, τ) be an L-topological space and $f \in L^X$. Then,*
(i) f is said to be τ-closed iff ${}^\neg(f) \in \tau$.
(ii) τ-closure $\overline{f}$ of f is defined by $\overline{f} = \wedge\{h \in L^X : {}^\neg(h) \in \tau,\ f \leq h\}$.

Remark 2.12 [5] *Let (X, τ) be an L-topological space. We denote with $\tau^{(\neg)}$ the set of all τ-closed L-fuzzy sets of X. The family $\tau^{(\neg)}$ holds the conditions LC1 and LC2 in Definition 2.2, but not the condition LC3, that is $\tau^{(\neg)} \neq \mathcal{C}_\tau$ in general. Moreover, the map $\mathcal{C}l(f) = \overline{f}$ is not an L-closure operator on X, i.e. $\overline{f} \neq \mathcal{C}l_{\mathcal{C}_\tau}(f)$, in general.*

Proposition 2.13 [5] *If* $(L, \leq, *)$ *is an integral, commutative Girard quantale, then*
(*i*) $\tau^{(\urcorner)} = \mathcal{C}_\tau$,
(*ii*) $\mathcal{C}l_{\mathcal{C}_\tau}(f) = {}^\urcorner(\mathcal{I}_{\ \tau}({}^\urcorner(f))), \forall f \in L^X$.

Proof. The assertions can be obtained from the consequence of Theorem 2.3, Theorem 2.8 and Corollary 2.9. ■

For the sake of simplicity, we always use the notations $\overline{f}$, f° instead of $\mathcal{C}l_{\mathcal{C}_\tau}(f)$ and $\mathcal{I}_\tau(f)$, respectively, whenever $(L, \leq, *)$ is an integral, commutative Girard quantale.

3 Main Results

These are the main results of the paper. Here we shall give several weaker forms of L-fuzzy connectedness in L-topological spaces. These types of connectedness are given for fuzzy topological spaces as $L = [0, 1]$ complete lattice, in the earlier works N. Ajmal [1], A. K. Chaudhuri [4], U. V. Fatteh [6] and N. Turanlı[12]. In this section, we start and adopt the definitions of quasi-coincident and non quasi-coincident of two fuzzy sets as given Ming and Ming [11] to the L-fuzzy sets. Throughout this section $(L, \leq, *)$ or briefly L will always denote an integral, commutative Girard quantale.

Definition 3.1 *An L-fuzzy set* f *is said to be L-quasi-coincident with an L-fuzzy set* g *iff there exists an element* $x \in X$ *such that* $f(x) > ({}^\urcorner g)(x) = ({}^\urcorner(g))(x)$ *and denoted by* $f\ (Lq)\ g$.

Definition 3.2 *Two L-fuzzy sets* f *and* g *in* X *are said to be non L-quasi-coincident iff* $f \leq {}^\urcorner(g)$ *and denoted by* $f\ (nLq) g$.

Definition 3.3 *Let* (X, τ) *be an L-topological space and* $f, g \in L^X$. f *and* g *are said to be L-weakly separated iff* $\overline{f}\ (nLq)\ g$ *and* $f\ (nLq)\ \overline{g}$ *or equivalently* $\overline{f} \leq {}^\urcorner(g)$ *and* $\overline{g} \leq {}^\urcorner(f)$.

Remark 3.4 f *and* g *are L-weakly separated iff there exist* $u, v \in \tau$ *such that* $f \leq u$, $g \leq v$, $f\ (nLq)\ u$ *and* $g\ (nLq)\ v$.

Definition 3.5 *Let* (X, τ) *be an L-topological space. Two non-zero L-fuzzy sets* f *and* g *are said to be L-q-separated if* $\overline{f} \wedge g = f \wedge \overline{g} = 1_\emptyset$.

Now we give definitions of some connectedness notion based on [1].

Definition 3.6 *An L-fuzzy set* f *in* (X, τ) *is said to be disconnected if there are two non-zero L-fuzzy weakly separated L-fuzzy sets* g, $h \in \tau$ *such that* $f = g \vee h$. *Furthermore,* f *is called connected (or* c_s*-connected) if* f *is not disconnected.*

Definition 3.7 *Let (X,τ) be L-topological space. An L-fuzzy set f in (X,τ) is said to be c_i-disconnection $(i=1,2,3,4)$ if there exist $g,h\in\tau$ such that*

$$[c_1]\ :\ f\leq g\vee h,\ g\wedge h\leq\ ^{\urcorner}(f)\ \text{and}\ f\wedge g\neq 1_{\emptyset},\ f\wedge h\neq 1_{\emptyset};\qquad(4)$$
$$[c_2]\ :\ f\leq g\vee h,\ f\wedge g\wedge h=1_{\emptyset}\ \text{and}\ f\wedge g\neq 1_{\emptyset},\ f\wedge h\neq 1_{\emptyset};\qquad(5)$$
$$[c_3]\ :\ f\leq g\vee h,\ g\wedge h\leq\ ^{\urcorner}(f)\ \text{and}\ g\nleq ^{\urcorner}(f),\ h\nleq\ ^{\urcorner}(f);\qquad(6)$$
$$[c_4]\ :\ f\leq g\vee h,\ f\wedge g\wedge h=1_{\emptyset}\ \text{and}\ g\nleq ^{\urcorner}(f),\ h\nleq\ ^{\urcorner}(f).\qquad(7)$$

f is said to be c_i-connected (for $i=1,2,3,4$) if there does not exist any c_i-disconnection of f in (X,τ).

Proposition 3.8 *Let (X,τ) be L-topological space and $f\in L^X$ in (X,τ). If f is c_1-connected then f is c_2-connected.*

Proof. For the contrary, we assume that f be not c_2-connected. Then there exist $g,h\in\tau$ such that $f\leq g\vee h$, $f\wedge g\wedge h=1_{\emptyset}$ and $f\wedge g\neq 1_{\emptyset}$, $f\wedge h\neq 1_{\emptyset}$. $f\wedge g\wedge h=1_{\emptyset}$ implies that $g\wedge h\leq\ ^{\urcorner}(f)$. This is a contradiction with the c_1-connectedness of f. ■

Proposition 3.9 *Let (X,τ) be L-topological space and $f\in L^X$ in (X,τ). If f is c_2-connected then f is c_4-connected.*

Proof. Let f be not c_4-connected. Then there exist $g,h\in\tau$ such that $f\leq g\vee h$, $f\wedge g\wedge h=1_{\emptyset}$, $g\nleq\ ^{\urcorner}(f)$ and $h\nleq\ ^{\urcorner}(f)$. $g\nleq\ ^{\urcorner}(f)$ and $h\nleq\ ^{\urcorner}(f)$ give us $f\wedge g\neq 1_{\emptyset}$, $f\wedge h\neq 1_{\emptyset}$, respectively. But, this is a contradiction with the c_2-connectedness of f. ■

Proposition 3.10 *Let (X,τ) be an L-topological space and $f\in L^X$ in (X,τ). Then the following implications are valid for f:*

$$c_1\text{-connectedness}\Longrightarrow c_3\text{-connectedness}\Longrightarrow c_4\text{-connectedness}\qquad(8)$$

Proof. It is clear from the definitions. ■

The following definition is the natural generalization of a continuous function between fuzzy topological spaces to L-topological spaces [7].

Definition 3.11 *Let (X,τ) and (Y,ϑ) be L-topological spaces, $\varphi:(X,\tau)\longrightarrow(Y,\vartheta)$ a function. φ is called L-continuous iff for each $g\in\vartheta$, $\varphi^{-1}(g)\in\tau$.*

The following gives the preservation of c_1-connectedness under L-continuous functions.

Proposition 3.12 *Let (X,τ) and (Y,ϑ) be L-topological spaces. If $\varphi : (X,\tau) \longrightarrow (Y,\vartheta)$ is surjective, L-continuous function and f is c_1-connected L-fuzzy set in (X,τ), then $\varphi(f)$ is a c_1-connected L-fuzzy set in (Y,ϑ).*

Proof. Suppose that $\varphi(f)$ is not c_1-connected. Then there exist $g, h \in \vartheta$ such that $\varphi(f) \leq g \vee h$, $g \wedge h \leq {}^{\neg}(\varphi(f))$, $\varphi(f) \wedge g \neq 1_\emptyset$ and $\varphi(f) \wedge h \neq 1_\emptyset$ which implies $f \leq \varphi^{-1}(g) \vee \varphi^{-1}(h)$ and $\varphi^{-1}(g \wedge h) \leq \varphi^{-1}({}^{\neg}(\varphi(f))) = {}^{\neg}(\varphi^{-1}(\varphi(f))$. The inequality $f \leq \varphi^{-1}(\varphi(f))$ give us ${}^{\neg}(\varphi^{-1}(\varphi(f)) \leq {}^{\neg}(f)$. By hypothesis, the function φ is L-continuous, we have $\varphi^{-1}(g)$, $\varphi^{-1}(h) \in \tau$. Also, there exist $y_1, y_2 \in Y$ such that (a) $(\varphi(f)(y_1)) \wedge g(y_1) \neq 1_\emptyset$ and (b) $(\varphi(f)(y_2)) \wedge g(y_2) \neq 1_\emptyset$. Since φ is onto, $\varphi^{-1}(y_1)$ and $\varphi^{-1}(y_2)$ are non empty subsets of X. Furthermore, we get $\varphi^{-1}(g)(x_i) = g(y_1)$, for every $x_i \in \varphi^{-1}(y_1)$ and $\varphi(f)(y_1) = \bigvee_{x_i \in \varphi^{-1}(y_1)} f(x_i)$ from image and preimage of L-fuzzy sets, respectively. We claim that $f \wedge \varphi^{-1}(g) \neq 1_\emptyset$ and $f \wedge \varphi^{-1}(h) \neq 1_\emptyset$. For the contrary, we suppose that $f \wedge \varphi^{-1}(g) = 1_\emptyset$. Then we get $f(x_i) \wedge \varphi^{-1}(g)(x_i) = 1_\emptyset$, for every $x_i \in \varphi^{-1}(y_1)$. This means that $f(x_i) \wedge g(y_1) = 1_\emptyset$, for every $x_i \in \varphi^{-1}(y_1)$ which is $(\bigvee_{x_i \in \varphi^{-1}(y_1)} f(x_i)) \wedge g(y_1) = 1_\emptyset$ or $(\varphi(f)(y_1)) \wedge g(y_1) \neq 1_\emptyset$. This contradicts with (a). Similarly, it can be shown that (b). Thus the proof is completed. ■

Corollary 3.13 *If f is a c_2-connected L-fuzzy set in X, then $\varphi(f)$ is a c_2-connected in Y.*

Proposition 3.14 *Let (X,τ) and (Y,ϑ) be L-topological spaces. If $\varphi : (X,\tau) \longrightarrow (Y,\vartheta)$ is surjective, L-continuous function and f is c_3-connected L-fuzzy set in (X,τ), then $\varphi(f)$ is a c_3-connected L-fuzzy set in (Y,ϑ).*

Proof. Similar to the proof Proposition 3.2. ■

Corollary 3.15 *If f is a c_4-connected L-fuzzy set in X, then $\varphi(f)$ is a c_4-connected in Y.*

Proposition 3.16 *Let (X,τ) be an L-topological space. If f_1, f_2 are intersecting c_1-connected L-fuzzy sets, then $f_1 \vee f_2$ is a c_1-connected L-fuzzy set.*

Proof. Assume that $f_1$and f_2 are intersecting c_1-connected L-fuzzy sets but $f_1 \vee f_2$ is not c_1-connected L-fuzzy set in X. Then there exist L-fuzzy open sets g and h in X such that $f_1 \vee f_2 \leq g \vee h$ and $g \wedge h \leq {}^{\neg}(f_1 \vee f_2)$. From these inequalities and assumption give us

$$f_1 \wedge g = 1_\emptyset or f_1 \wedge h = 1_\emptyset, f_2 \wedge g = 1_\emptyset or f_2 \wedge h = 1_\emptyset. \tag{9}$$

Now, we consider two cases: In the first case, we suppose that $f_1 \wedge g = 1_\emptyset$. As f_1 and f_2 are intersecting, there exists $x_0 \in X$ such that $f_1(x_0) \neq \mathbf{0} \neq$

$f_2(x_0)$. We claim that $f_2 \wedge h \neq 1_\emptyset$. Suppose, if possible $f_2 \wedge h = 1_\emptyset$. Then we get $(f_2 \wedge h)(x_0) = \mathbf{0}$ but $f_2(x_0) \neq \mathbf{0}$ and $h(x_0) = \mathbf{0}$. From the hypothesis, i.e. $f_1 \wedge g = 1_\emptyset$ we have $g(x_0) = \mathbf{0}$, since $f_1(x_0) \neq \mathbf{0}$. Hence we get $(g \vee h)(x_0) = \mathbf{0}$ which is a contradiction with $(f_1 \vee f_2)(x_0) \neq \mathbf{0}$. Therefore, we have $f_2 \wedge g = 1_\emptyset$, which implies that $(f_1 \vee f_2) \wedge g = 1_\emptyset$. In the last case, we suppose that $f_1 \wedge h = 1_\emptyset$. As in the first case, we can show that $f_2 \wedge g = 1_\emptyset$ is not possible hence $f_2 \wedge h = 1_\emptyset$. Then we have $(f_1 \vee f_2) \wedge g = 1_\emptyset$ and $f_1 \vee f_2$ is c_1-connected. Thus, the proof is completed. ■

Corollary 3.17 *If f_1, f_2 are intersecting c_2-connected L-fuzzy sets, then $f_1 \vee f_2$ is a c_2-connected L-fuzzy set.*

Proposition 3.18 *Let $\{f_i : i \in I\} \subseteq L^X$ be a family of c_1-connected L-fuzzy sets in (X, τ), such that for $i \neq j$, the L-fuzzy sets f_i and f_j are intersecting. Then $\bigvee_{i \in I} f_i$ is a c_1-connected L-fuzzy set in X.*

Proof. Suppose $g, h \in \tau$ such that $f \leq g \vee h$ and $g \wedge h \leq \urcorner(f)$, where $f = \bigvee_{i \in I} f_i$. We choose any L-fuzzy set $f_{i_0} \in \{f_i : i \in I\}$. Then we get $f_{i_0} \leq g \vee h$ and $g \wedge h \leq \urcorner(f_{i_0})$. From the hypothesis $g \wedge h \leq \urcorner(f)$ give us $g \wedge h \leq \bigwedge_{i \in I} \urcorner(f_i)$ and furthermore $g \wedge h \leq \urcorner(f_{i_0})$. Since f_{i_0} is c_1-connected L-fuzzy set, we have $f_{i_0} \wedge g = 1_\emptyset$ or $f_{i_0} \wedge h = 1_\emptyset$. The result follows in view of the facts that if $f_{i_0} \wedge g = 1_\emptyset$, then from the Proposition 3.6, we can prove that $f_i \wedge g = 1_\emptyset$, for each $i \in I \setminus \{i_0\}$, and $\bigvee_{i \in I} (f_{i_0} \wedge g) = 1_\emptyset$ implies that $f \wedge g = 1_\emptyset$. ■

Corollary 3.19 *Let $\{f_i : i \in I\} \subseteq L^X$ be a family of c_2-connected L-fuzzy sets in (X, τ), such that for $i \neq j$, the L-fuzzy sets f_i and f_j are intersecting. Then $\bigvee_{i \in I} f_i$ is a c_2-connected L-fuzzy set in X.*

Proposition 3.20 *Let (X, τ) be an L-topological space. If f_1, f_2 are L-quasi-coincident c_3-connected L-fuzzy sets, then $f_1 \vee f_2$ is a c_3-connected L-fuzzy set.*

Proof. Suppose that $g, h \in \tau$ such that $f_1 \vee f_2 \leq g \vee h$ and $g \wedge h \leq \urcorner(f_1 \vee f_2)$. Then since f_1 and f_2 are c_3-connected L-fuzzy sets, we have

$$\left[g \leq \urcorner(f_1) \text{ or } h \leq \urcorner(f_1)\right] \text{ and } \left[g \leq \urcorner(f_2) \text{ or } h \leq \urcorner(f_2)\right] . \tag{10}$$

Now by assumption, f_1, f_2 are L-quasi-coincident. This means that $f_1(x_0) > \urcorner(f_2(x_0))$. Again, as in Proposition 3.16, two cases are considered:

Case 1: Suppose $g \leq \urcorner(f_1)$. Then we have $g(x_0) < f_2(x_0)$. For the contrary, we claim that $g \not\leq \urcorner(f_2)$. For, if not, then

$$g(x_0) \leq^{\urcorner} (f_2(x_0)) < f_1(x_0).$$

Now, we get $(g \vee h)(x_0) < (f_1 \vee f_2)(x_0)$, which implies $f_1 \vee f_2 \nleq g \vee h$. But this is a contradiction. Thus $g \leq {}^\neg(f_2)$. On the other hand, we conclude that

$$g \leq {}^\neg(f_1) \wedge {}^\neg(f_2) = {}^\neg(f_1 \vee f_2).$$

Case 2: Suppose $h \leq {}^\neg(f_1)$. It can be shown that $g \nleq {}^\neg(f_2)$ as in Case I, similarly. Therefore, $h \leq {}^\neg(f_2)$. Hence, we obtain that $h \leq {}^\neg(f_1 \vee f_2)$. Thus the proof is completed. ■

Corollary 3.21 *If f_1, f_2 are L-quasi-coincident c_4-connected L-fuzzy sets, then $f_1 \vee f_2$ is a c_4-connected L-fuzzy set.*

Proposition 3.22 *Let $\{f_i : i \in I\} \subseteq L^X$ be a family of c_3-connected L-fuzzy sets in (X, τ), such that for $i \neq j$, the L-fuzzy sets f_i and f_j are L-quasi-coincident. Then $\bigvee_{i \in I} f_i$ is a c_3-connected L-fuzzy set in X.*

Proof. As for the proof of Proposition 3.18. ■

Corollary 3.23 *Let $\{f_i : i \in I\} \subseteq L^X$ be a family of c_4-connected L-fuzzy sets in (X, τ), such that for $i \neq j$, the L-fuzzy sets f_i and f_j are L-quasi-coincident. Then $\bigvee_{i \in I} f_i$ is a c_4-connected L-fuzzy set in X.*

Proposition 3.24 *Let (X, τ) be an L-topological space. If $f \in L^X$ is a c_3-connected and $f \leq u \leq \overline{f}$, then u is c_3-connected L-fuzzy set in (X, τ).*

Proof. Suppose that g and h L-fuzzy open sets in (X, τ) such that $u \leq g \vee h$ and $g \wedge h \leq {}^\neg(u)$. By considering the c_3-connectedness of f, we have $f \leq {}^\neg(g)$ or $f \leq {}^\neg(h)$. The inequality $f \leq {}^\neg(g)$ give us

$$\overline{f} \leq \overline{{}^\neg(g)} = {}^\neg((g)^o) = {}^\neg(g).$$

Because of the inequality $f \leq {}^\neg(h)$ and from Proposition 2.13, it is clear that

$$\overline{f} \leq \overline{{}^\neg(h)} = {}^\neg((h)^o) = {}^\neg(h),$$

Hence we find that $u \leq \overline{f} \leq {}^\neg(g)$ or $u \leq \overline{f} \leq {}^\neg(h)$, i.e. u is c_3-connected L-fuzzy set in X. ■

Corollary 3.25 *If $f \in L^X$ is a c_4-connected and $f \leq u \leq \overline{f}$, then u is c_4-connected L-fuzzy set in (X, τ).*

Proposition 3.26 *If $f \in L^X$ is a c_1-connected, then f is a c_s-connected.*

Proof. Let f be c_1-connected and not c_s-connected L-fuzzy set in X. Then there exist two non-zero L-fuzzy weakly separated L-fuzzy sets g, $h \in \tau$ such that $f = g \vee h$. By Using the Remark 3.4, we get $u, v \in \tau$ such that $f \leq u$, $g \leq v$, f (nLq) u and g (nLq) v . Then we may write $f \leq u \vee v$. Therefore $f \wedge u \neq 1_\emptyset$. For, if $f \wedge u = 1_\emptyset$ then $f \wedge g \neq 1_\emptyset$ so that $g = 1_\emptyset$, which contradicts that g is non zero. It can be shown that $f \wedge v \neq 1_\emptyset$ similarly. Now $u \wedge v \leq {}^\urcorner(h)$. Conversely, if $u \wedge v \nleq {}^\urcorner(h)$ then there exists $x_0 \in X$ such that $(u \wedge v)(x_0) > {}^\urcorner(f)(x_0)$. Then we observe that $u(x_0) > {}^\urcorner(f)(x_0)$ and $v(x_0) > {}^\urcorner(f)(x_0)$. Therefore we have $u(x_0) > {}^\urcorner(h)(x_0)$ or $v(x_0) > {}^\urcorner(g)(x_0)$. Furthermore, we may write u (Lq) h or v (Lq) g which is a contradiction. Hence the assertion follows. ■

Proposition 3.27 *If $f \in L^X$ is a c_s-connected, then f is a c_3-connected.*

Proof. It follows immediately from definitions. ■

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