Int. Journal of Math. Analysis, Vol. 4, 2010, no. 35, 1697 - 1706

Study of an Oxygenation Process in Capillary in the Presence of Magnetic Field

Rekha Bali* and Usha Awasthi **

Harcourt Butler Technological Institute, Kanpur, India
**usha_awasthi@rediffmail.com
*dr.rekhabali@rediffmail.com, **usha_hbti@rediffmail.com

Abstract. A mathematical model for the transport of oxygen in the systemic capillaries and the surrounding tissue in presence of magnetic field is presented in this paper. We have modeled the capillary by a circular cylinder surrounded by tissue of uniform thickness. The model takes into account the transport mechanisms of molecular diffusion, convection and diffusion due to the presence of hemoglobin as a carrier of the gases (oxygen).The resulting system of differential equations have been solved analytically by the method of separation of variable and Picard's method. We have obtained the result for partial pressure of oxygen in capillary and tissue region. The effect of Hartmann number (H) and others parameters have been obtained and discussed through graphs.

Mathematics Subject Classification: 76Z05

Keywords: Blood, Oxygen, Capillary and Tissue Cylinder, Hartmann number, Partial Pressure of Oxygen

Introduction

One of the main functions of the blood is to supply adequate amounts of oxygen to the tissues. It is well known that oxygen is supplied to living tissues through microcirculation of blood. One of the primary function of microcirculation is to ensure adequate oxygen delivery to meet the oxygen demands of every cell with in an organ [5]. This oxygen delivery may be affected in presence of magnetic field. It was found that blood can be considered as a magnetic fluid because the red blood cell contains the hemoglobin molecule, which is a form of iron oxides present in a high concentration in the mature red blood cells.

An attempt has been made in this direction by Deshikachar and Rao [7] who studied the flow and oxygenation of blood in channel of variable cross section but they have not considered the effect of magnetic field in cylindrical capillary surrounded by the tissue, while the determination of oxygen concentration profiles in a single capillary and a surrounding co-axial cylinder of tissue is a fundamental problem in the mathematical study of oxygen transport to tissue [4].It also seems that no other attempt has been made in this direction till now .The work done by Vardayan [13] , Sud et. al. [14], Suri and Suri [11] and Chandrashekar and Rudraih [1] are related to effect of magnetic field on blood flow only. Also the studies related to oxygenation process [2,3,6,8,10,12] have not been discussed in presence of magnetic field. Motivated by above facts, we have presented here a mathematical model for the oxygenation process in capillary tissue exchange system in presence of magnetic field.

Mathematical Analysis

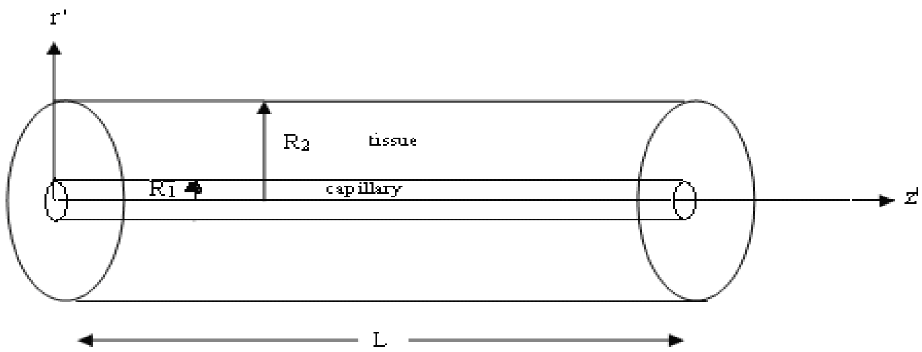

Fig.(1). Geometry of the problem

We have considered a model consisting of a circular cylindrical capillary surrounded by a co-axial circular cylindrical region of the tissue as shown in Fig.(1), where r', z' are radial and axial co-ordinate, R_1 and R_2 are the radius of vessel in capillary and tissue regions, L is the length of the capillary and u is blood velocity in capillary.

We have taken some assumptions for formulating the mathematical model.

(1) Capillary wall is permeable.

(2) Zero and first order metabolic oxygen consumption rate in tissue.

(3) Blood is considered as viscous, incompressible and electrically conducting fluid. Fluid flow is steady and laminar.

(4) Magnetic field (B_0) is constant in transverse direction.

(5) Diffusion of oxygen in both axial and radial direction in the capillary and tissue regions has been taken.

Governing Equations

Introducing the assumptions mentioned above the governing equation for the fluid flow is given as:

$$-\frac{\partial P'}{\partial z'}+\frac{1}{r'}\frac{\partial}{\partial r'}(\mu' r'\frac{\partial u'}{\partial r'})+J'\times B'=0 \quad (1)$$

where, $J'=\sigma_e(E'+u'\times B')$ (1a)

$\mu(r)$ is the coefficient of viscosity of blood proposed by Einstein as:

$$\mu' = \mu_0\left[1+\gamma h'(r')\right] \quad (1b)$$

$h'(r')$ is the hematocrit described by an empirical formula [9].

$$h'(r') = h_m\left[1-(\frac{r'}{R_2})^p\right] \quad (1c)$$

In which h_m is the maximum hematocrit at the centre of the tube and p (>=2) is a parameter determining the exact shape of the profile. The relation 1(c) is valid for a very dilute suspension of red cells which are supposed to be spherical in shape [2, 15]

where

μ' = Viscosity of the blood, $h'(r')$ = Hematocrit of the red blood cell

J' = Magnetic flux, B' = Magnetic field , σ'_e = Electrical conductivity

γ' = Constant , μ'_0 = Plasma viscosity, E' = Electric field

The equation of continuity is given as.

$$\frac{\partial u'}{\partial z'}+\frac{1}{r'}\frac{\partial}{\partial r'}(r'v')=0 \quad (2)$$

Where u', v' are the axial and normal components of velocity

Equation for the pressure distribution ($\overline{P'}$) in the tissue region using of Darcy law can be written as

$$\nabla^2\overline{P'}=0 \quad (3)$$

Partial differential equations for oxygen partial pressure in capillary and tissue region are given as:

$$D'_{1r}\frac{1}{r'}\frac{\partial}{\partial r'}(r'\frac{\partial P'_1}{\partial r'})+D'_{1z}\frac{\partial^2 P'_1}{\partial z'^2}=u'(1+\kappa_1\frac{N}{\alpha_1})\frac{\partial P'_1}{\partial z'} \quad (4)$$

and

$$D'_{2r}\frac{1}{r'}\frac{\partial}{\partial r'}\left(r'\frac{\partial P'_2}{\partial r'}\right)+D'_{2z}\frac{\partial^2 P'_2}{\partial z'^2}=\frac{g_0}{\alpha_2}+g_1P'_2 \tag{5}$$

Here P_1 and P_2 are oxygen partial pressures in the capillary and tissue regions. D_r, D_z are the diffusion coefficient of oxygen in the radial and axial direction. N is the oxygen carrying capacity of the blood. α_1 and α_2 are the oxygen solubility coefficient in the blood. κ_1 represents the saturation constant, g_0 and g_1 are the zero and first order metabolic consumption rate.

Boundary conditions

The physically realistic and mathematically consistent boundary conditions are given below.

$$\left.\begin{aligned}&\frac{\partial \overline{P'}}{\partial r'}=0 && at\ r'=R_2\\ &\overline{P'}=P' && at\ r'=R_1\\ &\overline{P'}=0 && at\ z'=0\ and\ z'=L\end{aligned}\right\}\ldots 6(a),\qquad \left.\begin{aligned}&P'=P'_A && at\ z'=0\\ &P'=P'_V && at\ z'=L\end{aligned}\right\}\ldots 6(b)$$

$$\left.\begin{aligned}&\frac{\partial u'}{\partial r'}=0 && at\ r'=0\\ &u'+\frac{\kappa_0}{\mu_0}\frac{\partial \overline{P'}}{\partial z'}=-\sigma'\frac{\partial u'}{\partial r'} && at\ r'=R_1\end{aligned}\right\}\ldots 6(c),\qquad \left.\begin{aligned}&v'=0 && at\ r'=0\\ &v'=-\frac{\kappa_0}{\mu_0}\frac{\partial \overline{P'}}{\partial r'} && at\ r'=R_1\end{aligned}\right\}\ldots 6(d)$$

Where σ = Slip parameter , κ_0 = Permeability of blood , P_A =Arterial pressure , P_V =Venous pressure

Condition for oxygen partial pressure

$$\left.\begin{aligned}&\frac{\partial P'_1}{\partial r'}=0 && at\ r'=0\\ &\frac{\partial P'_1}{\partial z'}=0 && at\ z'=L\\ &P'_1=P'_{art} && at\ z'=0\end{aligned}\right\}\ldots 6(e)$$

$$\left.\begin{aligned}&\frac{\partial P'_2}{\partial r'}=0 && at\ r'=R_2\\ &\frac{\partial P'_2}{\partial z'}=0 && at\ z'=0\ and\ z'=L\end{aligned}\right\}\ldots\ldots 6(f)$$

Interface conditions

We assume that the resistance offered by the capillary and tissue interface to mass transfer is negligibly small. Accordingly the pressure and the flux across the interface are continuous.

$$\left.\begin{aligned} &P_1'(R_1, z') = P_2'(R_1, z') \\ &D_{1r}'\alpha_1 \frac{\partial P_2'}{\partial r'} = D_{2r}'\alpha_2 \frac{\partial P_2'}{\partial r'} \end{aligned}\right\} \quad \text{....6(g)}$$

Solution of the problem

Introducing the following non-dimensional scheme.

$$\left.\begin{aligned} &z = \frac{z'}{L}, r = \frac{r'}{R_2}, u = \frac{u'}{U_0}, v = \frac{v'}{U_0}, P = \frac{P'}{\rho U_0^2}, \mathrm{Re} = \frac{\rho U_0 R_2}{\mu_0}, h = \frac{h'}{R_2} \\ &\varepsilon = \frac{R_2}{L}, Pe = \frac{U_0 R_2}{D_0}, \mu = \frac{\mu'}{\mu_0}, \sigma = \frac{\sigma'}{R_2}, D = \frac{D'}{D_0}, R = \frac{R_1}{R_2} \end{aligned}\right\} \quad \text{.....(7)}$$

Where D_0= Initial diffusivity coefficient , Pe= Peclet number , ρ = Blood density

U_0= Initial velocity of blood , $H^2 = \dfrac{B_0^2 R_2^2 \sigma_e}{\mu_0}$ (Hartmann number), $\delta = \dfrac{\alpha_2 D_{2r}}{\alpha_1 D_{1r}}$

$$\beta^2 = \frac{R_2^2 D_{1z}}{L^2 D_{1r}}, Q = (1 + \kappa_1 \frac{N}{\alpha_1}) \frac{\rho U_0 R_2^2}{D_0 D_{1r} L}, \lambda^2 = \frac{D_{2z} R_2^2}{D_{2r} L^2}, \xi = \frac{g_1 R_2^2}{D_{1r} D_0}, \eta = \frac{g_0 R_2^2}{\alpha_2 D_{2r} D_0 \rho U_0^2}$$

Solution

By solving the equation (2),(3),(4) and (5) by the method of separation of variable, Picard method and using the boundary and matching conditions (6c) to (6g) .We get the solution for axial velocity, pressure distribution in tissue region and capillary region and an oxygen partial pressure in capillary and tissue region.

Results and Discussions

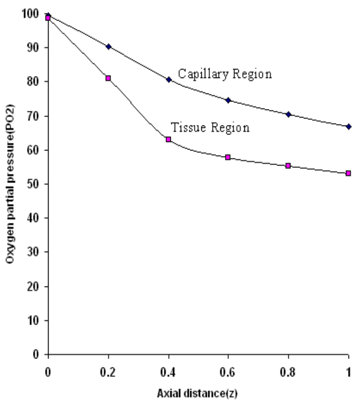

Fig (2a).Axial variation of oxygen partial pressure in capillary and tissue Region for zero-order metabolic consumption rate (g_0),P_0=100 mm Hg

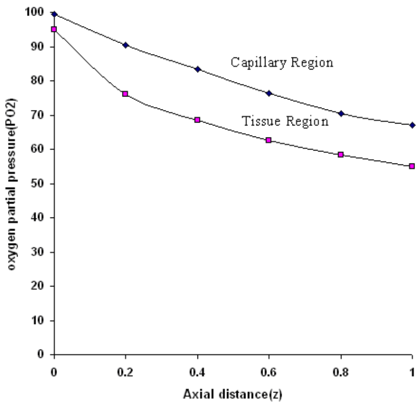

Fig (2b).Axial variation of oxygen partial pressure in capillary and tissue Region for first-order metabolic consumption rate (g_1),P_0=100 mm Hg

Fig.2(a,b) represent the axial variation of oxygen partial pressure in the capillary and tissue with the zero and first order metabolism in the absence of magnetic field. For zero-order metabolism the value of constant metabolic consumption rate (g_o) is taken from [10]. It is concluded from the figure 2(a) and 2(b) that the oxygen partial pressure in the tissue is lower for zero order metabolic consumption rate in comparison to the first order consumption rate. It has also been concluded from the figure that maximum oxygen pressure is near the arterial end and decreases towards venous end.

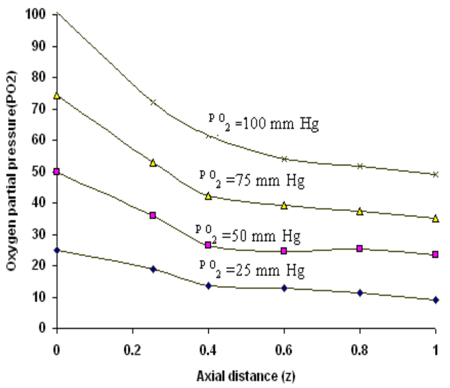

Fig (3a).The variation of oxygen partial pressure in capillary region Along the length of the capillary for different oxygen partial pressure in the arterial blood H=0 ,$R_1=3x10^{-4}$

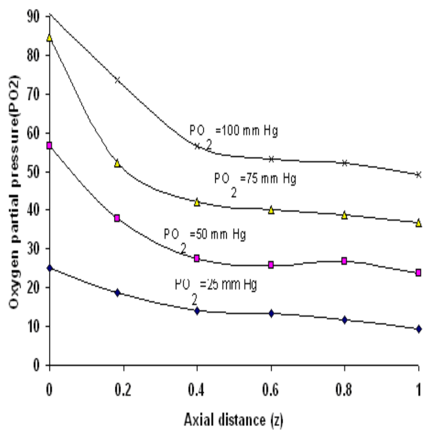

Fig (3b).The variation of oxygen partial pressure in tissue region along The length of the capillary for different oxygen partial pressure in the arterial blood H=0 ,$R_1=3x10^{-4}$

Fig.3(a,b) shows axial variation of oxygen partial pressure in capillary and tissue system with the variation of oxygen partial pressure in the arterial blood.The oxygen partial pressure increases in the capillary and tissue as the oxygen partial pressure in the arterial blood increases. For a fixed value of oxygen partial pressure at the entry, the amount of oxygen decreases continuously as the blood flows from the arterial to the venous end of the capillary. The fall becomes more rapid as the entry oxygen partial pressure increases. The difference in the levels of oxygen partial pressure between the axis of the capillary and the periphery of the tissue increases with the increase in arterial oxygen partial pressure, and the net supply of oxygen by the capillary is found to increase.

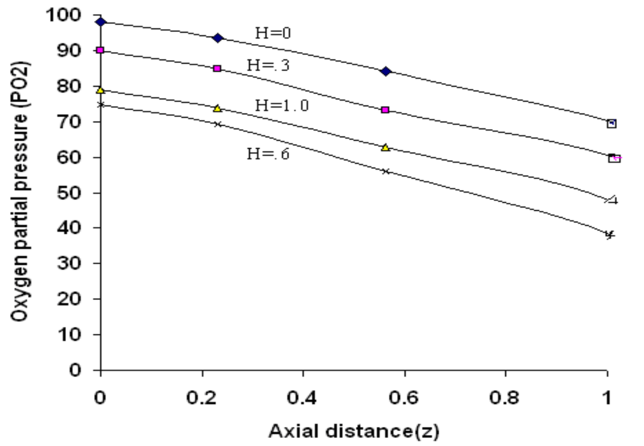

Fig (4).The variation of oxygen partial pressure along the length of the capillary with Hartmann no.(H) in capillary.

Fig.4 represents the axial variation of oxygen partial pressure in the capillary with the variation of Hartmann number (H). From figure 4 it is found the as the axial distance (or length of capillary) increases, the partial pressure of O_2 in the capillary decreases .The partial pressure of oxygen is decreases in magnetic case as compare to the non-magnetic case. We know that the concentration of oxygen is directly proportional to partial pressure of oxygen .In our last paper we have seen the similar effect on oxygen concentration.

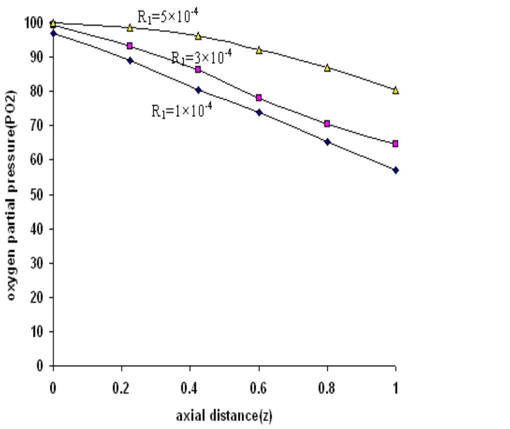

Fig (5a).The variation of oxygen partial pressure in capillary region along the length of the capillary with different capillary radius (R_1), $H=0, P_0=100$ mm Hg

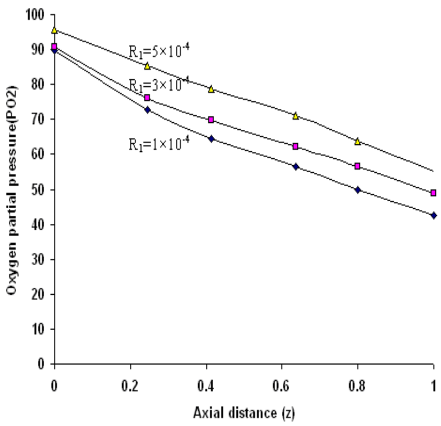

Fig (5b).The variation of oxygen partial pressure in tissue region along the length of the capillary with different capillary radius (R_1), $H=0, P_0=100$ mm Hg

Fig.5(a,b) shows the axial variation of oxygen partial pressure in the capillary and tissue for capillaries of different radii. From figure was observed that oxygen partial pressure in larger capillaries is greater in comparison to smaller capillaries.

Conclusion

The results of analysis deviate from the experimental work with increasing diameter of blood vessel and also with increasing hematocrit.The present theoretical model suitably describes blood flow in small vessels and at low concentration of red cells (≤40%).The reason behind this is the formula used for suspension viscosity based on the Einstein's theory of particular suspension , therefore it is applicable only for low particle concentration [2,15].A modification may be done to increase the range of usefulness of the present theoretical model.

References

[1] B .C .Chandrasekhara and N. Rudraih, "MHD Flow through a Channel of Varying Capillary,"Indian J Pure Appl. Math., Vol.11 (8), 1980, 1105-1123.

[2] D. A .Drew, "Low Concentration of Two Phase Flow near a Stagnant Point," Physics Fluid, Vol.17, 1974, 1688-1691.
[3] E. P .Salathe and T. C. Wang, "Substrate Concentrations in Tissue Surrounding Single Capillaries", Math. BioSci., Vol.49, 1980, 239-247.
[4] E. P .Salathe, T .C. Wang, J. F. Gross, "A Mathematical Analysis of Oxygen Transport to Tissue", Math. BioSci.,Vol.57, 1980, 89-115.
[5] G. Ellis Christopher, Jagger Justin and S .Michael, " The Microcirculation as a functional System, "Critical Care, Vol.9(Suppl.4), 2005, S3-S8.
[6] J .P .Whiteley, D. J .Gavaghan and C.W .Hahn, " Mathematical Modelling of Oxygen Transport to Tissue," J.Math.Biol.,2002,Vol.44,503-522.
[7] K. S .Deshikachar and A .Rao Ramachandra, "Effect of a Static Magnetic Field on the Flow and Blood Oxygenation in Channels of Cross-Section", Int.J.Engng. Sci., Vol.23 (10), 1985, 1121-1133.
[8] M .L. Elsworth, C. G. Ellis, A. S. Popel, R N Pittman, "Role of Microvesseles in Oxygen Supply to Tissue," New Physio Sci. Vol.9, 1994, 119-123.
[9] M. M. Lih , " Transport phenomena in Medicine and Biology",John Wiley & Sons.,Inc.,1974.
[10] M. Sharan, M .P. Singh and B. Singh, "An Analytical Model for Oxygen Transport in Tissue Capillaries in a Hyperbaric Environment with First Order Metabolic Consumption,"Math.Comput. Modelling, Vol.22, 1995, 99-111.
[11] P. K. Suri and P. R. Suri, "Effect of Static Magnetic Field on Blood Flow in a Branch", Indian J Pure and Appl. Math., Vol.12 (7), 1981, 907-918.
[12] T .M. Secomb and R .Hsu, "Simulation of Oxygen Transport in Skeletal Muscle Diffusive and Exchange between Arterioles and Capillaries", Am. J. Physiol., Vol.267, 1994, H1214-H1221.
[13] V. A. Vardanyan, Biophysics, Vol.18, 1973, 515.
[14] V. K. Sud, P. K. Suri and R. K. Mishra, "Effect of Magnetic Field on Oscillating Blood Flow in Arteries", Studia Biophysica, Vol.46, 197, 4163.
[15] V. P. Srivastava, "A Theoretical Model for Blood Flow in Small Vessels", Application and Applied Mathematics: An International Journal, Vol.2 (1), 2007, 51-65.

Received: March, 2010