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On a Nearly Sasakian Manifold with a Semi-Symmetric Semi-Metric Connection

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Abstract

We consider a nearly Sasakian manifold admitting a semi-symmetric semi-metric connection. We study semi-invariant submanifolds of nearly Sasakian manifold endowed with a semi-symmetric semi-metric connection. We also discuss the integrability of distributions on semi-invariant submanifolds.

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1. Introduction

Let $\bar{M}$ be $(2m+1)$-dimensional almost contact metric manifold [5] with a metric tensor g, a tensor field ϕ of type (1,1), a vector field ξ, a 1-form η which satisfies

$$\phi^2 = -I + \eta \otimes \xi, \phi\xi = 0, \eta\phi = 0, \eta(\xi) = 1, \tag{1.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{1.2}$$

for any vector fields X, Y on $\bar{M}$. If in addition to the condition for an almost contact metric structure we have $d\eta(X, Y) = g(X, \phi Y)$, the structure is said to be a contact metric structure.

The almost contact metric manifold $\bar{M}$ is called a nearly Sasakian manifold if it satisfies the condition [6]

$$(\bar{\bar{\nabla}}_X \phi)(Y) + (\bar{\bar{\nabla}}_Y \phi)(X) = \eta(Y)X + \eta(X)Y - 2g(X, Y)\xi, \tag{1.3}$$

where $\bar{\bar{\nabla}}$ denotes the Riemannian connection with respect to g. If, moreover, M satisfies

$$(\bar{\bar{\nabla}}_X\phi)(Y) = -g(X,Y)\xi + \eta(Y)X, \bar{\bar{\nabla}}_X\xi = \phi X \tag{1.4}$$

then it is called Sasakian manifold [5]. Thus every Sasakian manifold is nearly-Sasakian. The converse statement fails in general [6].

In ([3], [7]), A. Friedmann and J. A. Schouten introduced the idea of a semi-symmetric linear connection. A linear connection ∇ is said to be semi-symmetric connection if its torsion tensor T is of the form

$$T(X,Y) = \eta(Y)X - \eta(X)Y,$$

where η is a 1-form. Some properties of semi-symmetric connection are studied in ([12], [4], [8], [9]).

The notion of nearly Sasakian manifold was introduced by Blair et al. in [6]. A. Bejancu and N. Papaghuic introduced and studied semi-invariant submanifolds in Sasakian manifolds in [2]. M. Hasan Shahid [14] and The authors, S. Rahman [13] studied properties of semi-invariant submanifolds of a nearly Sasakian manifold. Motivated by studies of authors in ([11], [13], [10]), in this paper we study semi-invariant submanifolds of a nearly Sasakian manifold with a semi-symmetric semi-metric connection.

The paper is organized as follows : In section 1, we give a brief introduction of nearly Sasakian manifolds. In section 2, we define a semi-symmetric semi-metric connection in a nearly Sasakian manifold and study semi-invariant submanifolds of a nearly Sasakian manifold with a semi-symmetric semi-metric connection. In section 3, we discuss the integrability conditions of distributions of semi-invariant submanifolds of a nearly Sasakian manifold.

2. Semi-invariant submanifolds

Definition : An n-dimensional Riemannian submanifold M of a nearly Sasakian manifold $\bar{M}$ is called a semi-invariant submanifold if ξ is tangent to M and there exists on M a pair of orthogonal distribution $(D, D^{\perp})$ such that [2]
$(i)TM = D \oplus D^{\perp} \oplus \{\xi,\}$
$(ii)$the distribution D is invariant under ϕ, that is $\phi D_x = D_x$, for all $x \in M$,
$(iii)$the distribution $D^{\perp}$ is anti-invariant under ϕ, that is $\phi D_x^{\perp} \subset T_x^{\perp}M$, for all $x \in M$, where T_xM and $T_x^{\perp}M$ are the tangent space and normal space of M at x respectively.

We remark that owing to the existence of the $1-$form η, we can define a semi-symmetric semi-metric connection $\bar{\nabla}$ on $\bar{M}$ by

$$\bar{\nabla}_X Y = \bar{\bar{\nabla}}_X Y - \eta(X)Y + g(X,Y)\xi \tag{2.1}$$

such that $(\bar{\nabla}_X g)(Y,Z) = 2\eta(X)g(Y,Z) - \eta(Y)g(X,Z) - \eta(Z)g(X,Y)$ for any $X,Y,Z \in TM$, where $\bar{\bar{\nabla}}$ is induced connction on M.
From (1.4) and (2.1), we have

$$(\bar{\nabla}_X \phi)(Y) = -g(X,Y)\xi + \eta(Y)X + g(X,\phi Y)\xi, \tag{2.2}$$

$$(\bar{\nabla}_X \phi)(Y) + (\bar{\nabla}_Y \phi)(X) = -2g(Y,X)\xi + \eta(Y)X + \eta(X)Y, \tag{2.3}$$

$$\bar{\nabla}_X \xi = \phi X. \tag{2.4}$$

The Gauss and Weingarten formulas for a semi-symmetric semi-metric connection are respectively given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{2.5}$$

and

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N - \eta(X)N \tag{2.6}$$

for $X,Y \in TM, N \in T^{\perp}M$, where h (resp. A_N) is the second fundamental form (resp. tensor) of M in $\bar{M}$ and $\nabla^{\perp}$ denotes the operator of the normal connection. Moreover, we have

$$g(h(X,Y),N) = g(A_N X, Y). \tag{2.7}$$

For any vector X tangent to M is given as

$$X = PX + QX + \eta(X)\xi, \tag{2.8}$$

where PX and QX belong to the distribution D and $D^{\perp}$ respectively.
For any vector field N normal to M, we put

$$\phi N = BN + CN, \tag{2.9}$$

where BN (resp. CN) denotes the tangential (resp. normal) component of ϕN.

Definition : A semi-invariant submanifold is said to be mixed totally geodesic if $h(X,Z) = 0$ for all $X \in D$ and $Z \in D^{\perp}$.

The Nijenhuis tensor $N(X,Y)$ for semi-symmetric semi-metric connection is defined as

$$N(X,Y) = (\bar{\nabla}_{\phi X}\phi)(Y) - (\bar{\nabla}_{\phi Y}\phi)(X) - \phi(\bar{\nabla}_X \phi)(Y) + \phi(\bar{\nabla}_Y \phi)(X) \tag{2.10}$$

for any $X, Y \in TM$.
Using (2.3), we have

$$(\bar{\nabla}_{\phi X}\phi)(Y) = -2g(\phi X, Y)\xi + \eta(Y)\phi X - (\bar{\nabla}_Y\eta)(X)\xi - \eta(X)\phi Y + \phi(\bar{\nabla}_Y\phi)X. \tag{2.11}$$

By virtue of (2.11) and (2.10), we get

$$N(X, Y) = 4\phi(\bar{\nabla}_Y\phi)X - 4\eta(X)\phi Y - 2g(\phi X, Y)\xi \tag{2.12}$$

for any $X, Y \in TM$.
Lemma 2.2. Let M be a semi-invariant submanifold of a nearly-Sasakian manifold $\bar{M}$ with semi-symmetric semi-metric connection, then

$$2(\bar{\nabla}_X\phi)Y = \nabla_X\phi Y - \nabla_Y\phi X + h(X, \phi Y) - h(Y, \phi X) - \phi[X, Y] - 2g(X, Y)\xi$$

for all $X, Y \in D$.
Proof : After computations similar to lemma 2.1 in [14], lemma follows.
Similar computations also yields
Lemma 2.3. Let M be a semi-invariant submanifold of a nearly Sasakian manifold with semi-symmetric semi-metric connection, thensubmanifold

$$2(\bar{\nabla}_X\phi)Y = -A_{\phi Y}X + \nabla^{\perp}_X\phi Y - \nabla_Y\phi X - h(Y, \phi X) - \phi[X, Y]$$

for any $X \in D$ and $Y \in D^{\perp}$.

Lemma 2.4. Let M be a semi-invariant submanifold of a nearly Sasakian manifold $\bar{M}$ with semi-symmetric semi-metric connection, then

$$P\nabla_X\phi PY + P\nabla_Y\phi PX - PA_{\phi QY}X - PA_{\phi QX}Y = \eta(Y)PX + \eta(X)PY + \phi P\nabla_X Y + \phi P\nabla_Y X, \tag{2.13}$$

$$Q\nabla_X\phi PY + Q\nabla_Y\phi PX - QA_{\phi QY}X - QA_{\phi QX}Y = \eta(Y)QX + \eta(X)QY + 2Bh(X, Y), \tag{2.14}$$

$$h(X, \phi PY) + h(Y, \phi PX) + \nabla^{\perp}_X\phi QY + \nabla^{\perp}_Y\phi QX = 2Ch(X, Y) + \phi Q\nabla_X Y + \phi Q\nabla_Y X + \eta(Y)\phi QX + \eta(X)\phi QY, \tag{2.15}$$

$$\eta(\nabla_X\phi PY + \nabla_Y\phi PX - A_{\phi QY}X - A_{\phi QX}Y) = -2g(\phi X, \phi Y) \tag{2.16}$$

for all $X, Y \in TM$.

Proof : Differentiating (2.9) covariantly and using (2.6) and (2.7), we have

$$(\bar{\nabla}_X\phi)Y + \phi(\nabla_X Y) + \phi h(X,Y) = P\nabla_X(\phi PY) + Q\nabla_X(\phi PY) \tag{2.17}$$
$$+\eta(\nabla_X\phi PY)\xi - PA_{\phi QY}X - QA_{\phi QY}X + \eta(X)\phi QY$$
$$-\eta(A_{\phi QY}X)\xi + \nabla^{\perp}_X\phi QY + h(X,\phi PY).$$

Using (2.3), (2.10) and (2.17), we get

$$\eta(Y)PX + \eta(Y)QX + \eta(X)PY + \eta(X)QY - \eta(Y)\phi QX \tag{2.18}$$
$$-\eta(X)\phi QY + \phi P\nabla_X Y + \phi Q\nabla_X Y + \phi P\nabla_Y X + \phi Q\nabla_Y X$$
$$+2Bh(Y,X) + 2Ch(Y,X) + 2\eta(X)\eta(Y)\xi - 2g(X,Y)\xi = P\nabla_X(\phi PY)$$
$$+P\nabla_Y(\phi PX) + Q\nabla_Y(\phi PX) - PA_{\phi QY}X + Q\nabla_X(\phi PY) - QA_{\phi QY}X$$
$$-QA_{\phi QX}Y + \nabla^{\perp}_X\phi QY - PA_{\phi QX}Y + \nabla^{\perp}_Y\phi QX + h(Y,\phi PX) + h(X,\phi PY)$$
$$+\eta(\nabla_X\phi PY)\xi + \eta(\nabla_Y\phi PX)\xi - \eta(A_{\phi QX}Y)\xi - \eta(A_{\phi QY}X)\xi.$$

Equations (2.13)-(2.16) follows by comparision of tangential, normal and vertical parts of (2.18).

Definition : The horizontal distribution D is said to be parallel with respect to the connection ∇ on M if $\nabla_X Y \in D$ for all vector fields $X, Y \in D$.

Proposition 2.5. Let M be a semi-invariant submanifold of a nearly Sasakian manifold $\bar{M}$ with semi-symmetric semi-metric connection. If the horizontal distribution D is parallel then $h(X,\phi Y) = h(Y,\phi X)$ for all $X, Y \in D$.

Proof : The proposition follows similarly proposition 2.4 in [14].
Lemma 2.6. Let M be a semi-invariant submanifold of a nearly Sasakian manifold $\bar{M}$ with semi-symmetric semi-metric connection, then M is mixed totally geodesic if and only if $A_N X \in D$ for all $X \in D$.
Proof : Proof is similar to lemma 2.5 in [14].

3. Integrability conditions of distributions

Theorem 3.1. Let M be a semi-invariant submanifold of a nearly Sasakian manifold $\bar{M}$ with semi-symmetric semi-metric connection, then the distribution $D \oplus \langle\xi\rangle$ is integrable if the following conditions are satisfied

$$S(X,Y) \in (D \oplus \langle\xi\rangle) \tag{3.1}$$

$$h(X,\phi Y) = h(\phi X, Y) \tag{3.2}$$

for all $X, Y \in D \oplus \langle\xi\rangle$.
Proof : The torsion tensor $S(X, Y)$ of almost contact structure is given by

$$S(X, Y) = N(X, Y) + 2d\eta(X, Y)\xi,$$

where $N(X, Y)$ is Nijenhuis tensor.

$$S(X, Y) = [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + 2d\eta(X, Y)\xi. \tag{3.3}$$

Suppose that $D\oplus\langle\xi\rangle$ is integrable so for $X, Y \in D\oplus\langle\xi\rangle$, $N[X, Y] = 0$, then $S(X, Y) = 2d\eta(X, Y)\xi \in D\oplus\langle\xi\rangle$.Since $(\bar{\nabla}_Y \phi)X = \bar{\nabla}_Y \phi X - \phi\bar{\nabla}_Y X$. By virtue of (2.6) and (2.7), we have

$$\phi(\bar{\nabla}_Y \phi)X = \phi(P\nabla_Y \phi X + Q\nabla_Y \phi X) + Bh(Y, \phi X) + Ch(Y, \phi X)$$
$$+(\nabla_Y X + h(Y, X)) - \eta(\nabla_Y X + h(Y, X))\xi.$$

Now, equation (2.12) gives,

$$N(X, Y) = 4\phi P(\nabla_Y \phi X) + 4\phi Q(\nabla_Y \phi X) + 4Bh(Y, \phi X) \tag{3.4}$$
$$+4Ch(Y, \phi X) - 4h(Y, X) - 4\eta(\nabla_Y X) + 4\nabla_Y X - 2g(\phi X, Y)\xi$$

for $X, Y \in D$.
From (3.3) and (3.4), we get

$$\phi Q(\nabla_Y \phi X) + Ch(Y, \phi X) - h(Y, X) = 0$$

for all $X, Y \in D$.
Replacing Y by ϕZ, where $Z \in D$, we have

$$\phi Q(\nabla_{\phi Z} \phi X) + Ch(\phi Z, \phi X) - h(\phi Z, X) = 0.$$

Consequently, we have

$$\phi Q[\phi X, \phi Z] + h(X, \phi Z) - h(Z, \phi X) = 0$$

from which the assertion follows.

Lemma 3.2. Let M be a semi-invariant submanifold of a nearly Sasakian manifold $\bar{M}$ with semi-symmetric semi-metric connection, then

$$2(\bar{\nabla}_Y)\phi)Z = A_{\phi Y} Z - A_{\phi Z} Y + \nabla^{\perp}_Y \phi Z - \nabla^{\perp}_Z \phi Y - \phi[Y, Z]$$

for all $Y, Z \in D^{\perp}$.

Proof : Lemma follows after similar computations as lemma 3.2 in [14].

Proposition 3.3. Let M be a semi-invariant submanifold of a nearly Sasakian manifold $\bar{M}$ with semi-symmetric semi-metric connection, then

$$A_{\phi Y}Z - A_{\phi Z}Y = \frac{1}{3}\phi P[Y, Z] \tag{3.5}$$

for all $Y, Z \in D^{\perp}$.

Proof : Let $Y, Z \in D^{\perp}$ and $X \in T(M)$ then from (2.6) and (2.8), we have

$$2g(A_{\phi Z}Y, X) = -g(\phi\bar{\nabla}_Y Z, X) + g(A_{\phi Y}Z, X) - \eta(X)g(\phi Y, Z) + \eta(X)g(Y, Z).$$

Transvecting X from both sides, we get

$$2A_{\phi Z}Y = -\phi\bar{\nabla}_Y Z + A_{\phi Y}Z - g(\phi Y, Z)\xi + g(Y, Z)\xi.$$

Thus we have

$$2(A_{\phi Y}Z - A_{\phi Z}Y) = \phi(\bar{\nabla}_Y Z - \bar{\nabla}_Z Y) + (-A_{\phi Y}Z + A_{\phi Z}Y) + 2g(\phi Y, Z)\xi$$

$$3(A_{\phi Y}Z - A_{\phi Z}Y) = \phi[Y, Z] + 2g(\phi Y, Z)\xi$$

from which the assertion follows.

Theorem 3.4. Let M be a semi-invariant submanifold of a nearly-Sasakian manifold $\bar{M}$ with semi-symmetric semi-metric connection. Then the distribution $D^{\perp}$ is integrable if and only if

$$A_{\phi Y}Z - A_{\phi Z}Y = 0$$

for all $Y, Z \in D^{\perp}$.

Proof : Suppose that the distribution $D^{\perp}$ is integrable. Then $[Y, Z] \in D^{\perp}$ for any $Y, Z \in D^{\perp}$. Therefore, $P[Y, Z] = 0$ and from (3.5), we get

$$A_{\phi Y}Z - A_{\phi Z}Y = 0. \tag{3.6}$$

Conversely, let (3.6) holds. Then by virtue of (3.5) we have $\phi P[Y, Z] = 0$ for all $Y, Z \in D^{\perp}$. Since rank $\phi = 2n$, therefore we have either $P[Y, Z] = 0$ or $P[Y, Z] = K\xi$. But $P[Y, Z] = K\xi$ is not possible as P being a projection operator on D. Hence $P[Y, Z] = 0$, which is equivalent to [Y,Z]$\in D^{\perp}$ for all $Z \in D^{\perp}$ and $D^{\perp}$ is integrable.

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