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Integration Identities for Some Iterated Exponentials of Order Three

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Abstract

Inspired by the well-known elegant result for the integral of the power tower of length two x^x, namely $\int_0^1 1/x^x dx = \sum_{n=1}^{\infty} 1/n^n$ (first attributed to Johann Bernoulli, and commonly referred to as the "sophomore's dream" in present day literature [1]), we present a number of identities for power towers of length three.

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In the present communication, we shall be primarily concerned with integrals of the type $\mathrm{I} = \int_0^1 F(x)dx$, where the integrand take the form $F(x) = T(f(x); g(x); h(x))$. Here, $T(\alpha; \beta; \gamma) = \alpha^{\beta^\gamma}$ is a general power tower of length three and f, g and h are sufficiently well behaved functions. Making use of successive exponential transformations, one may write

$$\int_0^1 (f(x))^{a(g(x))^{bh(x)}} dx = \sum_{n=0}^{\infty} \frac{a^n}{n!} \sum_{m=0}^{\infty} \frac{b^n n^m}{m!} \int_0^1 (h(x))^m (\ln g(x))^m (\ln f(x))^n dx \,. \tag{1}$$

Applying such a transformation, and making use of the known identity [3]

$$\int_0^1 x^p (\ln x)^q dx = \frac{(-1)^q q!}{(p+1)^{q+1}} \quad p > -1 \,, q = 0, 1, 2, \ldots \,, \tag{2}$$

when $a = -1$, $b = 1$, $f(x) = g(x) = x$ and $h(x) = 1$, we recover the identity of Bernoulli as given in the abstract. Similarly, taking $a = b = 1$, $f(x) = g(x) = x$ and $h(x) = 1$, one obtains $\int_0^1 x^x dx = \sum_{n=1}^{\infty} (-1)^{n+1}/n^n$. One may obtain even

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more ambitious formulae; here we have restricted our attention to the case in which the functions f, g and h in (1) are either constant or linear. To our knowledge, such integral formulas are not common in the literature. Employing the methods above, we find the following six cases.

Case 1: For constants $a > 0$, $b \in \mathbb{R}$ and $c > 0$,

$$\int_0^1 (ax)^{bx^c} dx = \sum_{n=1}^{\infty} b^{n-1} \sum_{i=0}^{n-1} \frac{(-1)^i (\ln a)^{n-i-1}}{(n-i-1)!(c(n-1)+1)^{i+1}} \,. \tag{3}$$

Case 2: For constant $a > 0$,

$$\int_0^1 a^{x^x} dx = a \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^n} B_{n-1}(\ln a) \,, \tag{4}$$

where $B_k(x)$ denotes the kth Bell polynomial (see [4] and references therein).

Case 3: For constant $a > 0$,

$$\int_0^1 x^{a^x} dx = \frac{\ln(2a)}{\ln a} + \frac{1}{\ln a} \sum_{n=1}^{\infty} \frac{(-1)^n a^n}{n} B_{-n}(n \ln a) \,, \tag{5}$$

where computation of Bell polynomials of negative index may be carried out as in [4].

Case 4: For constants $a > 0$ and $b > 0$,

$$\int_0^1 a^{\pm b^x} dx = \frac{1}{\ln a} + \frac{1}{\ln b} \sum_{n=1}^{\infty} \frac{(\pm 1)^n (\ln a)^{n-1}}{n \; n!} (b^n - 1) \,. \tag{6}$$

Case 5: For constants $a > 0$ and $b > 0$,

$$\int_0^1 x^{ax^{(bx)}} dx = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{n+m} a^n b^m}{n! m!} \frac{n^m (m+n)!}{(m+1)^{m+n+1}} \,. \tag{7}$$

Case 6: For constants $a > 0$, $b > 0$ and $c > 0$,

$$\int_0^1 (ax)^{bx^{(cx)}} dx = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\ln a)^n (\ln b)^m c^m n^m}{n! m!} \sum_{r=0}^{n} \frac{(-1)^r r!}{(m+1)^{r+1} (\ln a)^r} \binom{n}{r}$$

$$\times\ {}_2\mathrm{F}_1\left(-m,-r;n-r+1;\frac{\ln a}{\ln b}\right), \tag{8}$$

where ${}_2\mathrm{F}_1$ is the hypergeometric function of Gauss [2]. We remark that in obtaining the identity (8), one must compute the integral $\int_0^1 x^m(\ln(bx))^m(\ln(ax))^n dx$ (in analogy to the integral (2)). This integral, unlike (2), is not standard in reference books. We find that, after a lengthy computation,

$$\int_0^1 x^m(\ln(bx))^m(\ln(ax))^n dx$$

$$= (\ln b)^m(\ln a)^n \sum_{r=0}^{n} \frac{(-1)^r r!}{(m+1)^{r+1}(\ln a)^r}\binom{n}{r}\ {}_2\mathrm{F}_1\left(-m,-r;n-r+1;\frac{\ln a}{\ln b}\right). \tag{9}$$

One may consider both nonlinear functions f, g and h in (1), or perhaps even towers of order greater than three. The results will become increasingly more complicated, in either direction of generalization.

References

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