Integration Identities for Some Iterated Exponentials of Order Three

Robert A. Van Gorder

Department of Mathematics, University of Central Florida
Orlando, FL 32816 USA

Abstract

Inspired by the well-known elegant result for the integral of the power tower of length two \( x^x \), namely \( \int_0^1 \frac{1}{x^x} \, dx = \sum_{n=1}^{\infty} 1/n^n \) (first attributed to Johann Bernoulli, and commonly referred to as the “sophomore’s dream” in present day literature [1]), we present a number of identities for power towers of length three.

Mathematics Subject Classification: 26A36, 40B05, 40A10

Keywords: iterated exponentials; tetrations; power tower; Sophomore’s dream

In the present communication, we shall be primarily concerned with integrals of the type \( I = \int_0^1 F(x) \, dx \), where the integrand take the form \( F(x) = T(f(x); g(x); h(x)) \). Here, \( T(\alpha; \beta; \gamma) = \alpha^\beta^\gamma \) is a general power tower of length three and \( f \), \( g \) and \( h \) are sufficiently well behaved functions. Making use of successive exponential transformations, one may write

\[
\int_0^1 (f(x))^a(g(x))^b(h(x))^c \, dx = \sum_{n=0}^{\infty} \frac{a^n}{n!} \sum_{m=0}^{\infty} \frac{b^n c^m}{m!} \int_0^1 (h(x))^m (\ln g(x))^m (\ln f(x))^n \, dx.
\]  

(1)

Applying such a transformation, and making use of the known identity [3]

\[
\int_0^1 x^p (\ln x)^q \, dx = \frac{(-1)^q q!}{(p+1)^{q+1}} \quad p > -1, \, q = 0, 1, 2, \ldots ,
\]  

(2)

when \( a = -1, \, b = 1, \, f(x) = g(x) = x \) and \( h(x) = 1 \), we recover the identity of Bernoulli as given in the abstract. Similarly, taking \( a = b = 1, \, f(x) = g(x) = x \) and \( h(x) = 1 \), one obtains \( \int_0^1 x^2 \, dx = \sum_{n=1}^{\infty} (-1)^{n+1}/n^n \). One may obtain even

\[1\]Corresponding author. Email: rav@knights.ucf.edu (R. A. Van Gorder)
more ambitious formulae; here we have restricted our attention to the case in which the functions $f$, $g$ and $h$ in (1) are either constant or linear. To our knowledge, such integral formulas are not common in the literature. Employing the methods above, we find the following six cases.

**Case 1:** For constants $a > 0$, $b \in \mathbb{R}$ and $c > 0$,

$$
\int_0^1 (ax)^{bx} \, dx = \sum_{n=1}^{\infty} b^{n-1} \sum_{i=0}^{n-1} \frac{(-1)^i (\ln a)^{n-i-1}}{(n-i-1)! (c(n-1)+1)^{i+1}}.
$$

**Case 2:** For constant $a > 0$,

$$
\int_0^1 ax^a \, dx = a \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^n} B_{n-1}(\ln a),
$$

where $B_k(x)$ denotes the $k$th Bell polynomial (see [4] and references therein).

**Case 3:** For constant $a > 0$,

$$
\int_0^1 x^{ax} \, dx = \ln(2a) \ln a + \frac{1}{\ln a} \sum_{n=1}^{\infty} \frac{(-1)^n a^n}{n} B_{-n}(n \ln a),
$$

where computation of Bell polynomials of negative index may be carried out as in [4].

**Case 4:** For constants $a > 0$ and $b > 0$,

$$
\int_0^1 a^{bx} \, dx = \ln a + \frac{1}{\ln a} \sum_{n=1}^{\infty} \frac{(-1)^n (\ln a)^{n-1}}{n^n} (b^n - 1).
$$

**Case 5:** For constants $a > 0$ and $b > 0$,

$$
\int_0^1 x^{ax(bx)} \, dx = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{n+m} a^n b^m}{n! m!} \frac{n^m (m+n)!}{(m+1)^{m+n+1}}.
$$

**Case 6:** For constants $a > 0$, $b > 0$ and $c > 0$,

$$
\int_0^1 (ax)^{bx(cx)} \, dx = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\ln a)^n (\ln b)^m c^m n^m}{n! m!} \sum_{r=0}^{n} \frac{(-1)^r r!}{(m+1)^{r+1}(\ln a)^r} \binom{n}{r}.
$$
Integration identities for some iterated exponentials of order three

\[ \times \, _2F_1 \left( -m, -r; n - r + 1; \frac{\ln a}{\ln b} \right), \quad (8) \]

where \(_2F_1\) is the hypergeometric function of Gauss [2]. We remark that in obtaining the identity (8), one must compute the integral \( \int_0^1 x^m (\ln(bx))^m (\ln(ax))^n \, dx \) (in analogy to the integral (2)). This integral, unlike (2), is not standard in reference books. We find that, after a lengthy computation,

\[ \int_0^1 x^m (\ln(bx))^m (\ln(ax))^n \, dx \]

\[ = (\ln b)^m (\ln a)^n \sum_{r=0}^{n} \frac{(-1)^r r!}{(m+1)^{r+1} (\ln a)^r} \binom{n}{r} \, _2F_1 \left( -m, -r; n - r + 1; \frac{\ln a}{\ln b} \right). \quad (9) \]

One may consider both nonlinear functions \( f, g \) and \( h \) in (1), or perhaps even towers of order greater than three. The results will become increasingly more complicated, in either direction of generalization.

References


[2] C.F. Gauss, *Disquisitiones Generales Circa Seriem Infinitam \([\alpha \beta]/(1\gamma) \cdot x + ((\alpha+1)\beta(\beta+1))/(12\gamma(\gamma+1)) \cdot x^2 + ((\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2))/(123\gamma(\gamma+1)(\gamma+2)) \cdot x^3 + \ldots\)* Pars Prior, Commentationes Societiones Regiae Scientiarum Gottingensis Recentiores, Vol. II. 1812.


Received: January 2010