On Some Generalized Difference Sequence Spaces

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Abstract

The main aim of this article is to introduce a new class of difference sequence spaces associated with a multiplier sequence which are isomorphic with the classical spaces \( c_0 \), \( c \) and \( \ell_\infty \) respectively and investigate some algebraic and topological structures of the spaces.

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1 Introduction

Let \( w \) denote the space of all scalar sequences and any subspace of \( w \) is called a sequence space. Let \( \ell_\infty \), \( c \) and \( c_0 \) be the spaces of bounded, convergent and null sequences \( x = (x_k) \) with complex terms respectively, normed by

\[ \|x\|_\infty = \sup_k |x_k| \] (1.1)

Let \( \Lambda = (\lambda_k) \) be a sequence of non-zero scalars. Then for \( E \) a sequence space, the multiplier sequence space \( E(\Lambda) \), associated with the multiplier sequence \( \Lambda \) is defined as

\[ E(\Lambda) = \{(x_k) \in w : (\lambda_k x_k) \in E\} . \]

The scope for the studies on sequence spaces was extended by using the notion of associated multiplier sequences. Goes and Goes [3] defined the differentiated sequence space \( dE \) and integrated sequence space \( \int E \) for a given sequence space \( E \), using the multiplier sequences \( (k^{-1}) \) and \( (k) \) respectively. A multiplier sequence can be used to accelerate the convergence of the sequences in some spaces. In some sense, it can be viewed as a catalyst, which is used to
accelerate the process of chemical reaction. Sometimes the associated multiplier sequence delays the rate of convergence of a sequence. Thus it also covers a larger class of sequences for study. In the present article we shall consider a general multiplier sequence \( \Lambda = (\lambda_k) \) of non-zero scalars.

The notion of difference sequence spaces was introduced by Kizmaz [4]. The notion was further generalized by Et and Colak [1] by introducing the spaces \( \ell_\infty(\Delta^s), c(\Delta^s) \) and \( c_0(\Delta^s) \). Another type of generalization of the difference sequence spaces is due to Tripathy and Esi [6], who studied the spaces \( \ell_\infty(\Delta_r), c(\Delta_r) \) and \( c_0(\Delta_r) \). Tripathy, Esi and Tripathy [7] generalized the above notions and unified these as follows:

Let \( r, s \) be non-negative integers, then for \( Z \) a given sequence space we have

\[
Z(\Delta^s_r, \Lambda) = \{x = (x_k) \in w : (\Delta^s_r x_k) \in Z\},
\]

where \( \Delta^s_r x = (\Delta^s_r x_k) = (\Delta^s_r x_k - \Delta^s_r x_{k+r}) \) and \( \Delta^0_r x_k = x_k \) for all \( k \in N \) and which is equivalent to the binomial representation

\[
\Delta^s_r x_k = \sum_{v=0}^{s} (-1)^v \binom{s}{v} x_{k+rv}.
\]

Let \( r, s \) be non-negative integers and \( \Lambda = (\lambda_k) \) be a sequence of non-zero scalars. Then for \( Z \), a given sequence space we define the following sequence spaces:

\[
Z(\Delta^s_r, \Lambda) = \{x = (x_k) \in w : (\Delta^s_r \lambda_k x_k) \in Z\}, \text{ for } Z = \ell_\infty, c \text{ and } c_0
\]

where \( \Delta^s_r \lambda_k x_k = (\Delta^s_r \lambda_k x_k - \Delta^s_r \lambda_{k-r} x_{k-r}) \) and \( \Delta^0_r \lambda_k x_k = \lambda_k x_k \) for all \( k \in N \) and which is equivalent to the binomial representation

\[
\Delta^s_r \lambda_k x_k = \sum_{v=0}^{s} (-1)^v \binom{s}{v} \lambda_{k-rv} x_{k-rv}.
\]

In this expansion it is important to note that we take \( \lambda_{k-rv} = 0 \) and \( x_{k-rv} = 0 \) for non-positive values of \( k - rv \).

For \( s = 1 \) and \( \lambda_k = 1 \) for all \( k \in N \), we get the spaces \( \ell_\infty(\Delta_r), c(\Delta_r) \) and \( c_0(\Delta_r) \). For \( r = 1 \) and \( \lambda_k = 1 \) for all \( k \in N \), we get the spaces \( \ell_\infty(\Delta^s), c(\Delta^s) \) and \( c_0(\Delta^s) \). For \( r = s = 1 \) and \( \lambda_k = 1 \) for all \( k \in N \), we get the spaces \( \ell_\infty(\Delta), c(\Delta) \) and \( c_0(\Delta) \).

Similarly we can define the spaces \( Z(\Delta^s_r, \Lambda) \), for \( Z = \ell_\infty, c \) and \( c_0 \).

2 Main Results

In this section we study the spaces \( Z(\Delta^s_r, \Lambda) \) and \( Z(\Delta^s, \Lambda) \), for \( Z = \ell_\infty, c \) and \( c_0 \) for some linear algebraic and topological structure.
Proposition 2.1. (i) The spaces \(Z(\Delta_{(r)}^s, \Lambda)\) and \(Z(\Delta_{(r)}^s, \Lambda)\), for \(Z = \ell_\infty, c\) and \(c_0\) are linear.

(ii) \(c_0(\Delta_{(r)}^s, \Lambda) \subset c(\Delta_{(r)}^s, \Lambda) \subset \ell_\infty(\Delta_{(r)}^s, \Lambda)\).

(iii) \(c_0(\Delta_{(r)}^s, \Lambda) \subset c(\Delta_{(r)}^s, \Lambda) \subset \ell_\infty(\Delta_{(r)}^s, \Lambda)\).

Proof. Proofs are routine verification and thus omitted.

Proposition 2.2. For \(Z = \ell_\infty, c\) and \(c_0\)

(i) \(Z(\Delta_{(r)}^s, \Lambda)\) are normed linear spaces, normed by

\[ \|x\| = \sum_{k=1}^{r,s} |\lambda_k x_k| + \sup_k |\Delta_{(r)}^s \lambda_k x_k| \quad (2.1) \]

(ii) \(Z(\Delta_{(r)}^s, \Lambda)\) are normed linear spaces, normed by

\[ \|x\|' = \sup_k |\Delta_{(r)}^s \lambda_k x_k| \quad (2.2) \]

Proof. (i) For \(x = \theta\), we have \(\|x\| = 0\). Conversely, let \(\|x\| = 0\). Then using (2.1), we have

\[ \sum_{k=1}^{r,s} |\lambda_k x_k| + \sup_k |\Delta_{(r)}^s \lambda_k x_k| = 0 \quad (2.3) \]

It follows that \(\sum_{k=1}^{r,s} |\lambda_k x_k| = 0\). Hence \(x_k = 0\), for \(k = 1, 2, \ldots, r,s\), since \((\lambda_k)\) is a sequence of non-zero scalars.

Again from (2.3) we have \(\sup_k |\Delta_{(r)}^s \lambda_k x_k| = 0\). It follows that \(\Delta_{(r)}^s \lambda_k x_k = 0\), for all \(k \geq 1\). Let \(k = 1\), then \(\Delta_{(r)}^s \lambda_1 x_1 = \sum_{v=0}^{s} (-1)^v \binom{s}{v} \lambda_1 x_1 + v x_1 = 0\) and so \(x_{1+r} = 0\) using (2.4). Similarly taking \(k = 2\), we have \(x_{2+r} = 0\). Proceeding in this way \(x_k = 0\) for all \(k \geq 1\). Hence \(x = \theta\). Again it is easy to show that \(\|x + y\| \leq \|x\| + \|y\|\) and for any scalar \(\alpha\), \(\|\alpha x\| = |\alpha| \|x\|\). This completes the proof.

(ii) For this part we only prove that \(\|x\|' = 0\) implies \(x = \theta\). Proof of other properties are similar with part (i).

Let \(\|x\|' = 0\). Then using (2.2), we have \(\sup_k |\Delta_{(r)}^s \lambda_k x_k| = 0\). It follows that \(\Delta_{(r)}^s \lambda_k x_k = 0\) for all \(k \geq 1\). Let \(k = 1\), then \(\Delta_{(r)}^s \lambda_1 x_1 = \sum_{v=0}^{s} (-1)^v \binom{s}{v} \lambda_1 x_1 - v x_1 = 0\) and so \(x_1 = 0\), since \((\lambda_k)\) is a sequence of non-zero scalars and by putting \(x_{1-r} = 0\) for \(v = 1, 2, \ldots, s\). Similarly taking \(k = 2\) we have \(x_2 = 0\). Proceeding in this way \(x_k = 0\) for all \(k \geq 1\). Hence \(x = \theta\). This completes the proof.
Proposition 2.3. For $Z = \ell_\infty$, $c$ and $c_0$

(i) $Z(\Delta^i_r, \Lambda) \subset Z(\Delta^s_r, \Lambda)$, ($i = 0, 1, 2, \ldots, s - 1$) and the inclusions are proper.

(ii) $Z(\Delta^i_r, \Lambda) \subset Z(\Delta^s_r, \Lambda)$, ($i = 0, 1, 2, \ldots, s - 1$) and the inclusions are proper.

Proof. Proofs are easy and so omitted.

Remark 2.4. It is obvious that for any sequence $x = (x_k)$, $x \in Z(\Delta^s_r, \Lambda)$ if and only if $x \in Z(\Delta^s_r, \Lambda)$. Hence from the above Proposition we can conclude that $Z$ is a subspace of both $Z(\Delta^s_r)$ and $Z(\Delta^s_r)$. Now if we compare norm (1.1) with norms (2.1) and (2.2), (2.2) looks quite natural as norm on a generalized space of $Z$. Keeping these in mind this new operator $\Delta^s_r$ is introduced. The fruitfulness of introducing this operator will be more visible in Proposition 2.8. and Proposition 2.9. Again it is clear that norms $\|\cdot\|$ and $\|\cdot\|$ are equivalent.

Theorem 2.5. For $Z = \ell_\infty$, $c$ and $c_0$

(i) $Z(\Delta^s_r, \Lambda)$ are Banach spaces, normed by $\|\cdot\|$.

(ii) $Z(\Delta^s_r, \Lambda)$ are Banach spaces, normed by $\|\cdot\|$.

Proof. We give the proof of part (i) only. Proof of part (ii) follows on applying similar arguments.

Let $(x^i)$ be a Cauchy sequence in $Z(\Delta^s_r, \Lambda)$, where $(x^i) = (x_k^i) = (x_1^i, x_2^i, \ldots)$ for each $i \geq 1$. Then for a given $\epsilon > 0$, there exists a positive integer $n_0$ such that

$$\|x^i - x^j\| = \sup_k |\Delta^s_r(\lambda_k)(x_k^i - x_k^j)| < \epsilon,$$

for all $i, j \geq n_0$. It follows that $|\Delta^s_r(\lambda_k)(x_k^i - x_k^j)| < \epsilon$, for all $i, j \geq n_0$ and for all $k \geq 1$. This implies that $(\Delta^s_r(\lambda_k)x_k^i)$ is a Cauchy sequence in $C$ for all $k \geq 1$ and so it is convergent in $C$ for all $k \geq 1$.

Let $\lim_{i \to \infty} \Delta^s_r(\lambda_k)x_k^i = y_k$, say for each $k \geq 1$. Considering $k = 1, 2, \ldots, rs, \ldots$, we can easily conclude that $\lim_{i \to \infty} x_k^i = x_k$, exists for each $k \geq 1$. Now we can have

$$\lim_{j \to \infty} |\Delta^s_r(\lambda_k)(x_k^i - x_k^j)| < \epsilon$$

for all $i \geq n_0$ and $k \geq 1$ and hence

$$\sup_k |\Delta^s_r(\lambda_k)(x_k^i - x_k)| < \epsilon$$

for all $i \geq n_0$. This implies that $(x^i - x) \in Z(\Delta^s_r, \Lambda)$. Since $Z(\Delta^s_r, \Lambda)$ is a linear space, $x = x^i - (x^i - x) \in Z(\Delta^s_r, \Lambda)$. Hence $Z(\Delta^s_r, \Lambda)$ is complete.

From the above proof we can easily conclude that $\|x^i - x\| \to 0$ implies $|x_k^i - x_k| \to 0$ as $i \to \infty$, for each $k \geq 1$. Hence we have the following Proposition.
Proposition 2.6. For $Z = \ell_\infty$, $c$ and $c_0$, $Z(\Delta_s(\tau), \Lambda)$ and $Z(\Delta_s(\tau), \Lambda)$ are BK-spaces.

Proposition 2.7. (i) The spaces $c_0(\Delta_s(\tau), \Lambda)$ and $c(\Delta_s(\tau), \Lambda)$ are nowhere dense subsets of $\ell_\infty(\Delta_s(\tau), \Lambda)$.
(ii) The spaces $c_0(\Delta_s(\tau), \Lambda)$ and $c(\Delta_s(\tau), \Lambda)$ are nowhere dense subsets of $\ell_\infty(\Delta_s(\tau), \Lambda)$.

Proof. We give the proof of part (i). Proof of part (ii) follows on applying similar arguments.

From proposition 2.1 (ii), we have $c_0(\Delta_s(\tau), \Lambda)$ and $c(\Delta_s(\tau), \Lambda)$ are proper subspaces of $\ell_\infty(\Delta_s(\tau), \Lambda)$. Again from Theorem 2.5 (i), it follows that $c_0(\Delta_s(\tau), \Lambda)$ and $c(\Delta_s(\tau), \Lambda)$ are closed subspaces of $\ell_\infty(\Delta_s(\tau), \Lambda)$. Hence the proof follows.

Proposition 2.8. (i) The spaces $c_0(\Delta_s(\tau), \Lambda)$, $c(\Delta_s(\tau), \Lambda)$ and $\ell_\infty(\Delta_s(\tau), \Lambda)$ are topologically isomorphic to the spaces $c_0$, $c$ and $\ell_\infty$ respectively.
(ii) The spaces $S c_0(\Delta_s(\tau), \Lambda)$, $S c(\Delta_s(\tau), \Lambda)$ and $S \ell_\infty(\Delta_s(\tau), \Lambda)$ are topologically isomorphic to the spaces $c_0$, $c$ and $\ell_\infty$ respectively, where $SZ(\Delta_s, \Lambda)$ is a subspace of $Z(\Delta_s, \Lambda)$ defined by

$$SZ(\Delta_s, \Lambda) = \{x = (x_k) : x \in Z(\Delta_s, \Lambda), \ x_1 = x_2 = \cdots = x_{rs} = 0\}$$

and normed by

$$\|x\| = \sup_k |\Delta_s^x \lambda_k x_k|.$$

Proof. We give the proof of part (i) and proof of part (ii) follows on applying similar arguments. For $Z = \ell_\infty$, $c$ and $c_0$, consider the mapping

$$T : Z(\Delta_s(\tau), \Lambda) \longrightarrow Z,$$

defined by $Tx = y = (\Delta_s^x \lambda_k x_k)$, for every $x \in Z(\Delta_s(\tau), \Lambda)$ (2.5)

Then clearly $T$ is a linear homeomorphism.

Proposition 2.9. For $Z = \ell_\infty$, $c$ and $c_0$, $Z(\Delta_s(\tau), \Lambda)$ and $Z(\Delta_s(\tau), \Lambda)$ are isometrically isomorphic with the spaces $c_0$, $c$ and $\ell_\infty$ respectively.

Proof. In view of Remark 2.4, we can define a mapping exactly similar with (2.5) on both the spaces $Z(\Delta_s(\tau), \Lambda)$ and $Z(\Delta_s(\tau), \Lambda)$. Then it is obvious that this mapping will be an isomorphic and isometry. This completes the proof.

Theorem 2.10. The continuous dual of $Z(\Delta_s(\tau), \Lambda)$ and $Z(\Delta_s(\tau), \Lambda)$, for $Z = c$, $c_0$ is $\ell_1$.

Proof. Since continuous dual of $c_0$ and $c$ is $\ell_1$, proof follows from Proposition 2.9.
Theorem 2.11. The spaces $Z(\Delta^{s}_{(r)}, \Lambda)$ and $Z(\Delta^{s}_{r}, \Lambda)$, for $Z = c, c_{0}$ are separable.

Proof. Since $\ell_{1}$ is separable, the proof follows from the fact that if the dual of a normed space is separable, then the space itself is separable.

Theorem 2.12. (i) The spaces $Z(\Delta^{s}_{(r)}, \Lambda)$ and $Z(\Delta^{s}_{r}, \Lambda)$ for $Z = c, c_{0}$ are not reflexive.
(ii) The spaces $Z(\Delta^{s}_{(r)}, \Lambda)$ and $Z(\Delta^{s}_{r}, \Lambda)$ for $Z = c, c_{0}$ are not Hilbert spaces.
(iii) The spaces $\ell_{\infty}(\Delta^{s}_{(r)}, \Lambda)$ and $\ell_{\infty}(\Delta^{s}_{r}, \Lambda)$ are not Hilbert spaces.
(iv) The spaces $\ell_{\infty}(\Delta^{s}_{(r)}, \Lambda)$ and $\ell_{\infty}(\Delta^{s}_{r}, \Lambda)$ are not reflexive.

Proof. (i) Since $\ell_{1}$ is not reflexive, the proof follows from the fact that if a normed space is reflexive then its dual is also reflexive.
(ii) Proof follows from the fact that every Hilbert space is reflexive.
(iii) We know that a closed subspace of a Hilbert space is Hilbert space. Here $Z(\Delta^{s}_{(r)}, \Lambda)$, $Z = c, c_{0}$ are closed subspaces of $\ell_{\infty}(\Delta^{s}_{(r)}, \Lambda)$ but both of them are not Hilbert spaces. So $\ell_{\infty}(\Delta^{s}_{(r)}, \Lambda)$ is not a Hilbert space.
By applying similar arguments we can argue that $\ell_{\infty}(\Delta^{s}_{r}, \Lambda)$ is not a Hilbert space.
(iv) We know that a closed subspace of a reflexive Banach space is reflexive. Here $Z(\Delta^{s}_{(r)}, \Lambda)$, $Z = c, c_{0}$ are closed subspaces of $\ell_{\infty}(\Delta^{s}_{(r)}, \Lambda)$ but both of them are not reflexive. So $\ell_{\infty}(\Delta^{s}_{(r)}, \Lambda)$ is not reflexive.
By applying similar arguments we can argue that $\ell_{\infty}(\Delta^{s}_{r}, \Lambda)$ is not reflexive.

Theorem 2.13. The spaces $\ell_{\infty}(\Delta^{s}_{(r)}, \Lambda)$ and $\ell_{\infty}(\Delta^{s}_{r}, \Lambda)$ are not separable.

Proof. We give the proof for the space $\ell_{\infty}(\Delta^{s}_{(r)}, \Lambda)$ only. For the other space $\ell_{\infty}(\Delta^{s}_{r}, \Lambda)$ it will follow on applying similar arguments. We can associate for every $y' \in [0, 1]$, a sequence $y = (y_{k}) \in \ell_{\infty}(\Delta^{s}_{(r)}, \Lambda)$ of zeros and ones, where $y' = \frac{y_{1}}{2} + \frac{y_{2}}{2^{2}} + \frac{y_{3}}{2^{3}} + \ldots$. Since $[0, 1]$ is uncountable, so there are uncountably many sequences of zeros and ones. For any two different sequences $x$ and $y$ of $\ell_{\infty}(\Delta^{s}_{(r)}, \Lambda)$ we have

$$\|x - y\| = \sup_{k} |\Delta^{s}_{(r)}\lambda_{k}(x_{k} - y_{k})|$$

$$= \sup_{k} |(x_{k} - y_{k})| = 1 \quad \text{by Proposition 2.9}$$

If we let each of these sequences be the centers of neighbourhoods, say, of radius $\frac{1}{2}$, these neighbourhoods do not intersect and we have uncountably many of them. If $D$ is any dense set in $\ell_{\infty}(\Delta^{s}_{(r)}, \Lambda)$, each of these non intersecting neighbourhoods must contain an element of $D$. Hence $D$ can not be countable. Since $D$ was an arbitrary dense set, this shows that $\ell_{\infty}(\Delta^{s}_{(r)}, \Lambda)$ can not have countable dense subset. Consequently, $\ell_{\infty}(\Delta^{s}_{(r)}, \Lambda)$ is not separable.
Theorem 2.14. (i) Let \( A \subset Z \). If \( A \) is convex, then \( A(\Delta^*_r, \Lambda) \) is convex in \( Z(\Delta^*_r, \Lambda) \).

(ii) Let \( A \subset Z \). If \( A \) is convex, then \( A(\Delta^*_r, \Lambda) \) is convex in \( Z(\Delta^*_r, \Lambda) \).

Proof. We give proof of part (i) only. The proof of part (ii) follows on applying similar arguments.
Let \( x, y \in A(\Delta^*_r, \Lambda) \), then \((\Delta^*_r \lambda_k x_k), (\Delta^*_r \lambda_k y_k) \in A \). Since \( \Delta^*_r \) is linear, we have
\[
\delta(\Delta^*_r \lambda_k x_k) + (1 - \delta)(\Delta^*_r \lambda_k y_k) = \Delta^*_r(\delta \lambda_k x_k + (1 - \delta) \lambda_k y_k)), \quad 0 \leq \delta \leq 1
\]
Since \( A \) is convex, \( \delta(\Delta^*_r \lambda_k x_k) + (1 - \delta)(\Delta^*_r \lambda_k y_k) \in A \).
Hence \( \delta x + (1 - \delta) y \in A(\Delta^*_r, \Lambda), \quad 0 \leq \delta \leq 1 \). This completes the proof.

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