A Unique Common Fixed Point Theorem
for Four Maps in Cone Metric Spaces

Amit Singh¹, R.C. Dimri² and Sandeep Bhatt³

Department of Mathematics, H. N. B. Garhwal University
Srinagar (Garhwal), Uttarakhand-246174, India
¹singhamit841@gmail.com
²dimrirc@gmail.com
³bhattsandeep1982@gmail.com

Abstract
In this paper we prove a unique common fixed point theorem for four weakly compatible self maps in complete cone metric spaces without using the notion of continuity. Our result generalizes and extends the results of Abbas and Rhoades [2] and others.

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1 Introduction and preliminaries
Recently Huang and Zhang [8] generalized the concept of a metric space, replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mappings satisfying different contractive conditions. Subsequently Abbas and Jungck [1] and Abbas and Rhoades [2] studied common fixed point theorems in cone metric spaces (see also, [8],[14] and the references mentioned therein). Jungck [10] defined a pair of self-mappings to be weakly if they commute at their coincidence points. In recent years, several authors have obtained coincidence point results for various classes of mappings on a metric space, utilizing these concepts. For a survey of coincidence point theory, its applications, comparison of different contractive conditions and related results, we refer to [3], [6], etc. and references contained therein. In this paper we prove a unique common fixed point theorem for four maps satisfying the notion of weakly compatibility without using the notion of continuity which generalizes and extends the results of Abbas and Rhoades [2] and others.
In this section we recall the definition of cone metric space and some of their properties (see c.f., [8]). The following notions will be used in order to prove the main result.

**Definition 1.1** Let $E$ be a real Banach Space and $P$ a subset of $E$. The set $P$ is called a cone if and only if

(i) $P$ is closed, non-empty and $P \neq 0$;
(ii) $a, b \in \mathbb{R}, a, b \geq 0, x, y \in P \Rightarrow ax + by \in P$;
(iii) $P \cap (-P) = 0$.

For a given cone $P \subset E$, we define a partial ordering $\leq$ with respect to $P$ by $x \leq y$ if and only if $y - x \in P$. We write $x < y$ to indicate that $x \to y$ but $x \neq y$, while $x << y$ will stand for $y - x \in \text{Int } P$, where $\text{Int } P$ denotes the interior of the set $P$.

**Definition 1.2** Let $E$ be a Banach Space and $P \subset E$ a cone. The cone $P$ is called normal if there is a number $K > 0$ such that for all $x, y \in E$,

\[ 0 \leq x \leq y \text{ implies } \|x\| \leq K \|y\|. \quad (1) \]

The least positive number $K$ satisfying the above inequality is called the normal constant of $P$.

In the following we always suppose that $E$ is a Banach space, $P$ is a cone in $E$ with is partial ordering with respect to $P$.

**Definition 1.3** Let $X$ be a non-empty set. Suppose that the mapping $d : X \times X \to E$ satisfies:

(a) $0 < d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(b) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(c) $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in X$.

Then $d$ is called a cone metric on $X$ and $(X, d)$ is called a cone metric space.

Notice that the notion of cone metric space is more general than the corresponding metric space followed by an example:

**Example 1.4** Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\} \subset \mathbb{R}^2, x = R$ and $d : X \times X \to E$ such that $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant. Then $(X, d)$ is a cone metric space.

**Definition 1.5** Let $(X, d)$ be a cone metric space. Let $\{x_n\}$ be a sequence in $X$ and $x \in X$.

(i) $\{x_n\}$ converges to $x$ if for every $c \in E$ with $0 < c$, there is an $n_0$ such
that for all \( n \geq n_0, d(x_n, x) << c \). We denote this by \( \lim_{n \to \infty} x_n = x \) or \( x_n \to x, \ (n \to \infty) \).

(ii) If for any \( c \in E \) with \( 0 << c \), there is an \( n_0 \) such that for all \( n, m \geq n_0, d(x_n, x_m) << c \), then \( \{x_n\} \) is called a Cauchy sequence in \( X \).

\( (X, d) \) is called a complete cone metric space, if every Cauchy sequence in \( X \) is convergent in \( X \).

**Definition 1.6** Let \( f \) and \( G \) be self-maps on a set \( X \). if \( w = fx = gx, \) for some \( x \) in \( X \), then \( x \) is called coincidence point of \( f \) and \( g \), where \( w \) is called a point of coincidence of \( f \) and \( g \).

**Definition 1.7** Let \( f \) and \( g \) be two self-maps defined on a set \( X \). then \( f \) and \( g \) are said to be weakly compatible if they commute at coincidence points.

**2 Main Result**

**Theorem 2.1** Let \( (X, d) \) be a complete cone metric space and \( P \) a normal cone with normal constant \( K \). Suppose that the mappings \( f, g, S \) and \( T \) are four self-maps of \( X \) such that \( T(X) \subseteq f(X) \) and \( S(X) \subseteq g(X) \) and satisfying

\[
d(Sx, Ty) \leq ad(fx, gy) + b[d(fx, Sx) + d(gy, Ty)] + c[d(fx, Ty) + d(gy, Sx)] \tag{2}
\]

for all \( x, y \in X \), where \( a, b, c \geq 0 \) and \( a + 2b + 2c < 1 \). Suppose that the pairs \( \{f, S\} \) and \( \{g, T\} \) are weakly compatible, then \( f, g, S \) and \( T \) have a unique common fixed point.

*Proof. Suppose \( x_0 \) is an arbitrary point of \( X \) and define the sequence \( \{y_n\} \) in \( X \) such that

\[
y_{2n} = Sx_{2n} = gx_{2n+1} \\
y_{2n+1} = Tx_{2n+1} = fx_{2n+2}
\]

By (2), we have

\[
d(y_{2n}, y_{2n+1}) = d(Sx_{2n}, Tx_{2n+1}) \\
\leq ad(fx_{2n}, gx_{2n+1}) + b[d(fx_{2n}, Sx_{2n}) + d(gx_{2n+1}, Tx_{2n+1})] \\
\quad + c[d(fx_{2n}, Tx_{2n+1}) + d(gx_{2n+1}, Sx_{2n})] \\
\leq ad(y_{2n-1}, y_{2n}) + b[d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})] \\
\quad + c[d(y_{2n-1}, y_{2n+1}) + d(y_{2n}, y_{2n})] \\
\leq (a + b + c)d(y_{2n-1}, y_{2n}) + (b + c)d(y_{2n}, y_{2n+1})
\]
which implies that
\[
\begin{align*}
d(y_{2n}, y_{2n+1}) & \leq \frac{a + b + c}{1 - (b + c)} d(y_{2n-1}, y_{2n}) \\
d(y_{2n}, y_{2n+1}) & \leq \delta d(y_{2n-1}, y_{2n})
\end{align*}
\]
where \( \delta = \frac{a + b + c}{1 - (b + c)} < 1 \). Similarly it can be shown that
\[
\begin{align*}
d(y_{2n+1}, y_{2n+2}) & \leq \delta d(y_{2n}, y_{2n+1})
\end{align*}
\]
Therefore, for all \( n \),
\[
\begin{align*}
d(y_{n+1}, y_{n+2}) & \leq \delta d(y_{n}, y_{n+1}) \\
& \leq \ldots \leq (\delta)^{n+1} d(y_0, y_1).
\end{align*}
\]
Now, for any \( m > n \),
\[
\begin{align*}
d(y_n, y_m) & \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \ldots + d(y_{m-1}, y_m) \\
& \leq [(\delta)^n + (\delta)^{n+1} + \ldots + (\delta)^{m-1}] d(y_1, y_0).
\end{align*}
\]
From (1), we have
\[
\begin{align*}
\|d(y_n, y_m)\| & \leq \frac{\delta^n}{1 - \delta} K \|d(y_1, y_0)\|.
\end{align*}
\]
which implies that \( d(y_n, y_m) \to 0 \) as \( n, m \to \infty \). Hence \( \{y_n\} \) is a Cauchy sequence. Since \( X \) is complete, there exists a point \( z \) in \( X \) such that \( \lim_{n \to \infty} \{y_n\} = z \), \( \lim_{n \to \infty} Sx_{2n} = \lim_{n \to \infty} gx_{2n+1} = z \) and \( \lim_{n \to \infty} Tx_{2n+1} = \lim_{n \to \infty} f x_{2n+2} = z \) i.e.,
\[
\lim_{n \to \infty} Sx_{2n} = \lim_{n \to \infty} gx_{2n+1} = \lim_{n \to \infty} Tx_{2n+1} = \lim_{n \to \infty} f x_{2n+2} = z.
\]
Since \( T(X) \subseteq f(X) \), there exists a point \( u \in X \) such that \( z = fu \). Then, by (2), we have
\[
\begin{align*}
d(Su, z) & \leq d(Su, Tx_{2n-1}) + d(Tx_{2n-1}, z) \\
& \leq ad(fu, gx_{2n-1}) + b[d(fu, Su) + d(gx_{2n-1}, Tx_{2n-1})] \\
& \quad + c[d(fu, Tx_{2n-1}) + d(gx_{2n-1}, Su)] + d(Tx_{2n-1}, z)
\end{align*}
\]
By (1), we have
\[
\begin{align*}
\|d(Su, z)\| & \leq aK \|d(fu, gx_{2n-1})\| \\
& \quad + bK \|d(fu, Su) + d(gx_{2n-1}, Tx_{2n-1})\| \\
& \quad + cK \|d(fu, Tx_{2n-1}) + d(gx_{2n-1}, Su)\| \\
& \quad + K \|d(Tx_{2n-1}, z)\|
\end{align*}
\]
Taking the limit as \( n \to \infty \) yields
\[
\begin{align*}
d(Su, z) & \leq ad(z, z) + b[d(z, Su) + d(z, z)]
\end{align*}
\]
\[ c[\dist(z,z) + \dist(z, Su)] + \dist(z, z) \leq (b + c)\dist(Su, z) \]

which is a contradiction since \( a + 2b + 2c < 1 \). Therefore \( Su = fu = z \). Since \( S(X) \subseteq g(X) \), there exists a point \( v \in X \) such that \( z = gv \). Then, by (2), we have

\[
d(z, Tv) \leq d(Su, Tv) \\
\leq ad(fu, gv) + b[d(fu, Su) + d(gv, Tv)] \\
+ c[d(fu, Tv) + d(gv, Su)] \\
\leq ad(z, z) + b[d(z, z) + d(z, Tv)] \\
+ c[d(z, Tv) + d(z, z)] \\
\leq (b + c)d(z, Tv)
\]

which is a contradiction since \( a + 2b + 2c < 1 \). Therefore \( Tv = gv = z \). Thus \( Su = fu = Tv = gv = z \).

Since \( f \) and \( S \) are weakly compatible maps, then \( Sfu = fSu \) i.e., \( Sz = fz \). Now we show that \( z \) is a fixed point of \( S \). If \( Sz \neq z \), then by (2), we have

\[
d(Sz, z) \leq d(Sz, Tv) \\
\leq ad(fz, gz) + b[d(fz, Sz) + d(gz, Tv)] \\
+ c[d(fz, Tv) + d(gz, Sz)] \\
\leq ad(Sz, z) + b[d(Sz, Sz) + d(z, z)] \\
+ c[d(Sz, z) + d(z, Sz)] \\
\leq (a + 2c)d(Sz, z)
\]

which is a contradiction since \( a + 2b + 2c < 1 \). Therefore \( Sz = z \). Hence \( Sz = fz = z \).

Similarly, \( g \) and \( T \) are weakly compatible maps, we have \( Tz = gz \). Now we show that \( z \) is a fixed point of \( T \). If \( Tz \neq z \), then by (2), we have

\[
d(z, Tz) \leq d(Sz, Tz) \\
\leq ad(fz, gz) + b[d(fz, Sz) + d(gz, Tz)] \\
+ c[d(fz, Tz) + d(gz, Sz)] \\
\leq ad(z, Tz) + b[d(z, z) + d(Tz, Tz)] \\
+ c[d(z, Tz) + d(Tz, z)] \\
\leq (a + 2c)d(z, Tz)
\]
which is a contradiction since $a + 2b + 2c < 1$. Therefore $Tz = z$. Hence $Tz = gz = z$.

Thus $Sz = Tz = fz = gz = z$, i.e. $z$ is a common fixed point of $f$, $g$, $S$ and $T$.

Finally, in order to prove the uniqueness of $z$, suppose that $z$ and $w$, $z \neq w$, are common fixed points of $f$, $g$, $S$ and $T$ respectively. then by (2), we have

\[
d(z, w) \leq d(Sz, Tw) \\
\leq ad(fz, gw) + b[d(fz, Sz) + d(gw, Tw)] \\
+ c[d(fz, Tw) + d(gw, Sz)] \\
\leq ad(z, w) + b[d(z, z) + d(w, w)] \\
+ c[d(z, w) + d(w, z)] \\
\leq (a + 2c)d(z, w)
\]

which is a contradiction since $a + 2b + 2c < 1$. Therefore $z = w$.

Hence $z$ is the unique common fixed point of $f$, $g$, $S$ and $T$ respectively.

**Corollary 2.2** Let $(X, d)$ be a complete cone metric space and $P$ a normal cone with normal constant $K$. Suppose that the mappings $f$, $S$ and $T$ are three self-maps of $X$ such that $T(X) \subseteq f(X)$ and $S(X) \subseteq f(X)$ and satisfying

\[
d(Sx, Ty) \leq ad(fx, fy) + b[d(fx, Sx) + d(fy, Ty)] + c[d(fx, Ty) + d(fy, Sx)]
\]

for all $x, y \in X$, where $a, b, c \geq 0$ and $a + 2b + 2c < 1$. Suppose that the pairs \{f, S\} and \{f, T\} are weakly compatible, then $f$, $S$ and $T$ have a unique common fixed point.

Prof. If we take $f = g$ in Theorem 2.1, then we get the proof of it.

**References**


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