Generalized Fixed Point Theorems in Two Metric Spaces under Implicit Relations

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Abstract. A fixed point theorem in two metric spaces is proved. The theorem generalizes the results obtained in [1,2,3]. A several corollaries are obtained according as the forms of implicit relations.

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1. Introduction

In [1], [2] and [3] the following theorems are proved:

**Theorem 1.1. (Fisher [1]).** Let $(X,d)$ and $(Y,\rho)$ be complete metric spaces. If $T$ is a mapping of $X$ into $Y$ and $S$ is a mapping of $Y$ into $X$ satisfying the inequalities

$$d(Sy, STx) \leq c \max \{d(x, Sy), d(x, STx), \rho(y, Tx)\}$$

$$\rho(Tx, TSy) \leq c \max \{\rho(y, Tx), \rho(y, TSy), d(x, Sy)\}$$

for all $x$ in $X$ and $y$ in $Y$, where $0 \leq c < 1$, then $ST$ has a unique fixed point $z$ in $X$ and $TS$ has a unique fixed point $w$ in $Y$. Further, $Tz=w$ and $Sw=z$.

**Theorem 1.2 (Popa [3])** Let $(X,d)$ and $(Y,\rho)$ be complete metric spaces. If $T$ is a mapping of $X$ into $Y$ and $S$ is a mapping of $Y$ into $X$ satisfying the inequalities
\[ d^2(Sy, STx) \leq c_1 \max \{ \rho(y, Tx)d(x, Sy), \rho(y, Tx)d(x, STx), d(x, Sy) d(x, STx) \} \]
\[ \rho^p(Tx, TSy) \leq c_2 \max \{ d(x, Sy), \rho(y, Tx)d(x, Sy), \rho(y, Ty), \rho(y, Ty)d(x, Sy) \} \]
for all \(x \in X\) and \(y \in Y\), where \(0 \leq c_1, c_2 < 1\), then \(ST\) has a unique fixed point \(z\) in \(X\) and \(TS\) has a unique fixed point \(w\) in \(Y\). Further, \(Tz=w\) and \(Sw=z\).

**Theorem 1.3.** (Nešić [2]). Let \((X, d)\) and \((Y, \rho)\) complete metric spaces. Let \(T\) be a mapping of \(X\) into \(Y\) and \(S\) a mapping of \(Y\) into \(X\). Denote
\[
M_1(x, y) = \left\{ d^p(x, Sy), d^p(x, STx), \rho^p(y, Tx) \right\}
\]
\[
M_2(x, y) = \left\{ \rho^p(y, Tx), \rho^p(y, TSy), d^p(x, Sy) \right\}
\]
for all \(x \in X, y \in Y\) and \(p=1,2,3,\ldots\)

Let \(R^+\) be the set of nonnegative real numbers, and let \(F_i : R^+ \to R^+\) be a mapping such that \(F_i(0)=0\) and \(F_i\) is continuous at 0 for \(i=1,2\). If \(T\) and \(S\) satisfying the inequalities
\[ d^p(Sy, STx) \leq c_1 \max M_1(x, y) + F_1(\min M_1(x, y)) \]
\[ \rho^p(Tx, TSy) \leq c_2 \max M_2(x, y) + F_2(\min M_2(x, y)) \]
for all \(x \in X, y \in Y\), where \(0 \leq c_1, c_2 < 1\), then \(ST\) has a unique fixed point \(z\) in \(X\) and \(TS\) has a unique fixed point \(w\) in \(Y\). Further, \(Tz=w\) and \(Sw=z\).

2. Main results.

The theorem that we are attempting to prove generalizes the Fisher [1], Nešić[2], Popa [3] theorems using an implicit relations.

Let \(\Phi^m\) be the set of continuous functions with 3 variables \(\varphi : [0, \infty)^3 \to [0, \infty)\) satisfying the properties:
(a) \(\varphi\) is non decreasing in \(t_1, t_2, t_3\).
(b) \(\varphi(t, t, t) \leq t^m, \ m \in N\)

some examples of such functions are as follows:
Example 2.1. \(\varphi(t_1, t_2, t_3) = \max \{t_1, t_2, t_3\}\), with \(m=1\).
Example 2.2. \(\varphi(t_1, t_2, t_3) = \max \{t_1 t_2, t_1 t_3, t_2 t_3\}\), with \(m=2\)
Example 2.3. \(\varphi(t_1, t_2, t_3) = \max \{t_1^p, t_2^p, t_3^p\}\), with \(m=p\)
Example 2.4. \(\varphi(t_1, t_2, t_3) = \frac{t_1 + t_2 + t_3}{3}\) or \(\frac{t_1 + t_2}{2}\), with \(m=1\) etc.

Let \(\Psi\) be the set of continuous functions with 3 variables
\(\Psi : [0, \infty)^3 \to [0, \infty)\).
satisfying the property
\[ t_1 t_2 t_3 = 0 \Rightarrow \Psi(t_1, t_2, t_3) = 0 \]

Example 2.5. \( \psi(t_1, t_2, t_3) = \min\{t_1, t_2, t_3\} \)

Example 2.6. \( \psi(t_1, t_2, t_3) = \min\{t_1 t_2 t_3, t_2 t_3\} \)

Example 2.7. \( \psi(t_1, t_2, t_3) = \min\{t_1^3, t_2^3, t_3^3\} \) etc.

Let \( F \) be the set of continuous functions \( F : [0, \infty) \to [0, \infty) \) with \( F(0) = 0 \) (for example \( F(t) = t^q, q > 0 \)).

Theorem 3.8. Let \( (X, d), (Y, \rho) \) be two complete metric spaces and \( T : X \to Y, S : Y \to X \) two mappings. Let \( \phi_i \in \Phi_i^{(m)}, \psi_i \in \Psi_i, F_i \in F \) for \( i = 1, 2 \). If for some \( k \in [0, 1) \), the following inequalities are satisfied:

\[
\begin{align*}
d^m(Sy, STx) &\leq k \phi_1(d(x, Sy), d(x, STx), \rho(y, Tx)) + \\
&\quad + F_1(\psi_1(d(x, Sy), d(x, STx), \rho(y, Tx))) \\
\rho^m(Tx, TSy) &\leq k \phi_2(\rho(y, Tx), \rho(y, TSy), d(x, Sy)) + \\
&\quad + F_2(\psi_2(\rho(y, Tx), \rho(y, TSy), d(x, Sy)))
\end{align*}
\]

for all \( x \in X, y \in Y \) and some \( m = 1, 2, \ldots \), then \( ST \) has a unique fixed point \( \alpha \in X \) and \( TS \) has a unique fixed point \( \beta \in Y \). Further \( T_\alpha = \beta \) and \( S_\beta = \alpha \).

Proof. Let \( x_0 \in X \) be an arbitrary point. We define the sequences \( (x_n) \) and \( (y_n) \) in \( X \) and \( Y \) respectively as follows:

\[
x_n = (ST)^n x_0, \quad y_n = T x_{n-1}, \quad n = 1, 2, \ldots
\]

Let us prove that the sequences \( (x_n) \) and \( (y_n) \) are Cauchy sequences. We assume that \( x_n \neq x_{n+1} \) and \( y_n \neq y_{n+1} \) \( \forall n \in N \), because otherwise if \( x_n = x_{n+1} \) and \( y_n = y_{n+1} \) for some \( n \), we could put \( \alpha = x_n \) and \( \beta = y_n \).

Denote

\[
d_n = d(x_n, x_{n+1}), \quad \rho_n = \rho(y_n, y_{n+1}), \quad n = 1, 2, \ldots
\]

By the inequality (2) for \( x = x_{n-1} \) and \( y = y_n \) we get:

\[
\begin{align*}
\rho^m(y_n, y_{n+1}) &= \rho^m(T x_{n-1}, TASy_n) \\
&\leq k \phi_2(\rho(y_n, y_{n+1}), d(x_{n-1}, x_n)) + \\
&\quad + F_2(\psi_2(\rho(y_n, y_{n+1}), d(x_{n-1}, x_n)))
\end{align*}
\]

or

\[
\begin{align*}
\rho_n^m &\leq k \phi_2(0, \rho_n, d_n) + F_2(\psi_2(0, \rho_n, d_n)) = \\
&= k \phi_2(0, \rho_n, d_n)
\end{align*}
\]
For the coordinates of the point \( (0, \rho_n, d_{n-1}) \) we have:
\[
\rho_n \leq d_{n-1}, \ \forall n \in \mathbb{N},
\]
because, in case that \( \rho_n > d_{n-1} \) for some \( n \), if we replace the coordinates with \( \rho_n \) and apply the property (a) and (b) of \( \phi_2 \) we get:
\[
\rho_n^m \leq k \phi_2(\rho_n, \rho_n, \rho_n) \leq k \rho_n^m.
\]
This is impossible since \( 0 \leq k < 1 \).
Replacing on the right hand side of (3), the coordinates with \( \rho_{n-1} \) and applying properties (a) and (b) of \( \phi_2 \) we get:
\[
\rho_n^m \leq k \phi_2(d_{n-1}, d_{n-1}, d_{n-1}) \leq kd_{n-1}^m.
\]
Thus
\[
\rho_n \leq \sqrt[k]{k} d_{n-1}
\]
By the inequality (1) for \( x = x_n \) and \( y = y_n \) we get:
\[
d^m(x_n, x_{n+1}) = d^m(Sy_n, STx_n) \leq
\]
\[
\leq k \phi_1(d(x_n, x_n), d(x_n, x_{n+1}), \rho(y_n, y_{n+1})) +
\]
\[
+ F_1(\phi_1(d(x_n, x_n), d(x_n, x_{n+1}), \rho(y_n, y_{n+1})))
\]
or
\[
d_n^m \leq k \phi_1(0, d_n, \rho_n) + F_1(\psi_1(0, d_n, \rho_n)) =
\]
\[
= k \phi_1(0, d_n, \rho_n)
\]
In similar, we get:
\[
d_n^m \leq k \phi_1(\rho_n, \rho_n, \rho_n) \leq k \rho_n^m.
\]
Thus we will obtain:
\[
d_n \leq \sqrt[k]{k} \rho_n
\]
By this inequality and (4) we get:
\[
d_n \leq \sqrt[k]{k} \left( \sqrt[k]{k} \cdot d_{n-1} \right) \leq \sqrt[k]{k} d_{n-1}
\]
By the inequalities (4) and (5), using the mathematical induction, we obtain:
\[
d(x_n, x_{n+1}) = d_n \leq q^{n-1} d_i
\]
\[
\rho(y_n, y_{n+1}) = \rho_n \leq q^{n-1} d_i
\]
where \( q = \sqrt[k]{k} < 1 \).
Thus the sequences \( (x_n) \) and \( (y_n) \) are Cauchy sequences. Since the metric spaces \((X, d)\) and \((Y, \rho)\) are complete metric spaces we have:
\[
\lim_{n \to \infty} x_n = \alpha \in X, \quad \lim_{n \to \infty} y_n = \beta \in Y
\]
By (1) for \( y = \beta \) and \( x = x_n \) we get:
\[
d^m(S\beta, x_{n+1}) \leq k \phi_1(d(x_n, S\beta), d(x_n, x_{n+1}), \rho(\beta, y_{n+1})) +
\]
\[
+ F_1(\psi_1(d(x_n, S\beta), d(x_n, x_{n+1}), \rho(\beta, y_{n+1})))
\]
Letting \( n \) tend to infinity, we get:
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\[ d^n(S\beta, \alpha) \leq k\varphi_1(d(\alpha, S\beta), 0, 0) + F(0) = \]
\[ = k\varphi_1(d(\alpha, S\beta), 0, 0) \leq kd^n(\alpha, S\beta) \]

or
\[ d^n(S\beta, \alpha) = 0 \iff S\beta = \alpha \quad (6) \]

By (2) for \( x = x_n \) and \( y = \beta \) we get:
\[ \rho^n(y_{n+1}, TS\beta) \leq k\varphi_2(\rho(\beta, y_{n+1}), \rho(\beta, TS\beta), d(x_n, S\beta)) + F_1(\varphi_2(\rho(\beta, y_{n+1}), \rho(\beta, TS\beta), d(x_n, S\beta))) \]

Letting \( n \) tend to infinity and using (6) we get:
\[ \rho^n(\beta, TS\beta) \leq k\varphi_2(0, \rho(\beta, TS\beta), 0) + F(0) \]

or
\[ \rho^n(\beta, TS\beta) \leq k\rho^n(\beta, TS\beta) \iff TS\beta = \beta \quad (7) \]

By (6) and (7) it follows:
\[ TS\beta = T\alpha = \beta \]
\[ ST\alpha = S\beta = \alpha \]

Thus, we proved that the points \( \alpha, \beta \) are fixed points of ST and TS respectively.

To prove uniqueness, supposes that ST has a second distinct fixed point \( \alpha' \) in X. By (1) for \( x = \alpha' \) and \( y = T\alpha \) we get:
\[ d^n(ST\alpha, ST\alpha') \leq k\varphi_1(d(\alpha', ST\alpha), d(\alpha', ST\alpha'), \rho(T\alpha, T\alpha')) + \]
\[ + F_1(\varphi_1(d(\alpha', \alpha), 0, \rho(T\alpha, T\alpha'))) \]

or
\[ d^n(\alpha, \alpha') \leq k\varphi_1(d(\alpha', \alpha), 0, \rho(T\alpha, T\alpha')) + F_1(0) \leq k\rho^n(T\alpha, T\alpha') \quad (8) \]

By (2) for \( x = \alpha' \) and \( y = T\alpha \) we obtain:
\[ \rho^n(T\alpha', T\alpha) \leq k\varphi_2(\rho(T\alpha, T\alpha'), \rho(T\alpha, T\alpha), d(\alpha', \alpha)) + \]
\[ + F_2(\varphi_2(\rho(T\alpha, T\alpha'), 0, d(\alpha', \alpha))) \]

or
\[ \rho^n(T\alpha', T\alpha) \leq k\varphi_2(\rho(T\alpha, T\alpha'), 0, d(\alpha', \alpha)) \leq kd^n(\alpha', \alpha) \quad (9) \]

By (8) and (9) we get:
\[ d^n(\alpha, \alpha') \leq k^2 d^n(\alpha, \alpha') \]

It follows
\[ d(\alpha, \alpha') = 0 \]

Thus, we have again \( \alpha = \alpha' \). In the same way, it is proved the iniquity of \( \beta \).
3. Corollaries

**Corollary 3.1.** (Theorem Nešić [2]). Let \((X,d)\) and \((Y,\rho)\) be complete metric spaces and \(T : X \to Y, S : Y \to X\) two mapping. Let \(F : [0, \infty) \to [0, \infty)\) be continuous functions with \(F(0) = 0\). If for some \(k \in [0,1)\), the following inequalities are satisfied:

\[
d^m(Sy,STx) \leq k \max \{d(x,Sy),d(x,STx),\rho(y,Tx)\} + F(\min \{d(x,Sy),d(x,STx),\rho(y,Tx)\})
\]

\[
\rho^m(Tx,TSy) \leq k \max \{\rho(y,Tx),\rho(y,TRy),d(x,5y)\} + F(\min \{\rho(y,Tx),\rho(y,TRy),d(x,5y)\})
\]

for all \(x \in X, \ y \in Y\), then \(ST\) has a unique fixed point \(\alpha \in X\) and \(TS\) has a unique fixed point \(\beta \in Y\). Further \(T\alpha = \beta\) and \(S\beta = \alpha\).

**Proof.** The proof follows by Theorem 3.8 in the case \(F_1 = F_2 = F\),

\[\varphi_1 = \varphi_2 = \varphi \in \Phi_3^{(m)}\] such that \(\varphi(t_1,t_2,t_3) = \max \{t_1^m,t_2^m,t_3^m\}\) and \(\psi_1 = \psi_2 = \psi \in \Psi_3\), where \(\psi(t_1,t_2,t_3) = \min \{t_1^m,t_2^m,t_3^m\}\).

We emphasize the fact that in the Theorem 1.3, the mappings \(F_1\) and \(F_2\) can be replaced by \(F : F(t) = \max \{F_1(t),F_2(t)\}\) and \(c_1, c_2\) can be replaced by \(k = \max \{c_1,c_2\}\).

**Corollary 3.2.** In conditions of corollary 3.1 for \(m=1\) and \(F=0\) we get the theorem 1.1 (Fisher [1])

**Corollary 3.3.** By theorem 3.8 for case \(F_1=F_2=0\), \(\varphi_1 = \varphi_2 = \varphi \in \Phi_3^{(2)}\), which that \(\varphi(t_1,t_2,t_3) = \max \{t_1,t_2,t_3\}\) obtain the theorem 1.2 (Popa [3])

**Corollary 3.4** (Popa [3]). Let \((X,d)\) and \((Y,\rho)\) be complete metric spaces if \(T\) is mapping of \(X\) into \(Y\) and \(S\) is a mapping of \(Y\) into \(X\) satisfying the inequalities

\[
d^2(Sy,STx) \leq a_1 \rho(y,Tx)\rho(x,Sy) + b_1 \rho(y,Tx)d(x,STx) + c_1 d(x,STx) + d(x,STx)d(x,STx)
\]

\[
\rho^2(Tx,TSy) \leq a_2 \rho(y,Tx)\rho(y,TSy) + b_2 d(x,STx) + c_2 \rho(y,Tx)d(x,STx)
\]

for all \(x \in X\) and \(y \in Y\), where \(a_1,b_1,c_1 \geq 0\) and \(a_i+b_i+c_i < 1\), \(i = 1,2\) then \(ST\) has a unique fixed point \(\alpha\) in \(X\) and \(TS\) has a unique fixed point \(\beta\) in \(Y\). Further, \(T\alpha = \beta\) and \(S\beta = \alpha\).

**Proof.** The proof follows by Theorem 3.8. in the case \(F_1=F_2=0\); \(\varphi_1, \varphi_2 \in \Phi_3^{(1)}\) such that

\[\varphi(t_1,t_2,t_3) = \frac{a_it_1+b_it_2+c_it_3}{a_i+b_i+c_i}, \quad i = 1,2\]

and \(k = \max \{a_1+b_1+c_1,a_2+b_2+c_2\}\).
Corollary 3.5. Let $(X,d)$ be a complete metric space if $S$ and $T$ are mappings of $X$ into itself. Let $\varphi_i \in \Phi^m_3$, $\psi_i \in \Psi_3$, $F_i \in \mathcal{F}$ for $i=1,2$. If for some $k \in [0,1)$, the following inequalities are satisfied:

\[
d^m(Sy,STx) \leq k\varphi_1(d(x,Sy),d(x,STx),d(y,Tx)) + F_1(\psi_1(d(x,Sy),d(x,STx),d(y,Tx)))
\]

\[
d^m(Tx,TSy) \leq k\varphi_2(d(y,Tx),d(y,TSy),d(x,Sy)) + F_2(\psi_2(d(y,Tx),d(y,TSy),d(x,Sy)))
\]

for all $x,y \in X$. Then $ST$ has a unique fixed point $\alpha$ and $TS$ has a unique fixed point $\beta$. Further, $T\alpha = \beta$ and $S\beta = \alpha$ and if $\alpha = \beta$, $\alpha$ is the unique fixed point of $S$ and $T$.

Proof. The existence of $\alpha$ and $\beta$ follows from theorem 3.8. If $\alpha = \beta$ then $\alpha$ is of course a common fixed point of $S$ and $T$.

Now suppose that $T$ has a second fixed point $\alpha'$. Then on using inequality $(2')$ for $x=\alpha'$ and $y = \alpha$ we have:

\[
d^m(\alpha',\alpha) = d(T\alpha',TS\alpha) \leq k\varphi_2(d(\alpha,T\alpha'),d(\alpha,TS\alpha),d(\alpha',S\alpha)) + F_2(\psi_2(d(\alpha,T\alpha'),d(\alpha,TS\alpha),d(\alpha',S\alpha)))
\]

Since $k \in [0,1)$, the uniqueness of $\alpha$ follows.

Remark. We give many corollaries, which also depend on the form of implicit relations.

References


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