Conditions for Reflexivity on Some Sequence Spaces

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Abstract

In this paper we will give sufficient conditions for the multiplication operator $M_z$ to be reflexive on the weighted Hardy spaces.

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1. Introduction

Let $\{\beta(n)\}_{n=-\infty}^{\infty}$ be a sequence of positive numbers satisfying $\beta(0) = 1$. If $1 < p < \infty$, the space $L^p(\beta)$ consists of all Laurent power series $f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n)z^n$ such that the norm

$$\|f\|_p = \|f\|_{\beta}^p = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^p \beta(n)^p$$

is finite. These are reflexive Banach spaces with the norm $\| \cdot \|_{\beta}$. Let $\hat{f}_k(n) = \delta_k(n)$. So $f_k(z) = z^k$ and then $\{f_k\}_k$ is a basis for $L^p(\beta)$ such that $\|f_k\| = \beta(k)$. We denote the set of multipliers $\{\varphi \in L^p(\beta) : \varphi L^p(\beta) \subseteq L^p(\beta)\}$ by $L^p_{\infty}(\beta)$ and the linear operator of multiplication by $\varphi$ on $L^p(\beta)$ by $M_\varphi$.

We say that a complex number $\lambda$ is a bounded point evaluation on $L^p(\beta)$ if the functional $e(\lambda) : L^p(\beta) \longrightarrow \mathbb{C}$ defined by $e(\lambda)(f) = f(\lambda)$ is bounded.
It is well known that $L^p(\beta)^* = L^q(\beta^*)$, where $\frac{1}{p} + \frac{1}{q} = 1$. Also if $f(z) = \sum_n \hat{f}(n)z^n \in L^p(\beta)$ and $g(z) = \sum_n \hat{g}(n)z^n \in L^q(\beta^*)$, then clearly

$$<f, g> = \sum_n \hat{f}(n)\overline{\hat{g}(n)}\beta(n)^p$$

where the notation $<f, g>$ stands for $g(f)$. For a source in formal series, we refer the reader to [1–12].

Recall that if $E$ is a separable Banach space and $A \in B(E)$, then $Lat(A)$ is by definition the lattice of all invariant subspaces of $A$, and $AlgLat(A)$ is the algebra of all operators $B$ in $B(E)$ such that $Lat(A) \subset Lat(B)$. For the algebra $B(E)$, the weak operator topology is the one induced by the family of seminorms $p_{x^*, x}(A) = |<Ax, x^*>|$ where $x \in E$, $x^* \in E^*$ and $A \in B(E)$. Hence $A_\alpha \rightarrow A$ in the weak operator topology if and only if $A_\alpha x \rightarrow Ax$ weakly. Also similarly $A_\alpha \rightarrow A$ in the strong operator topology if and only if $A_\alpha x \rightarrow Ax$ in the norm topology. An operator $A$ in $B(E)$ is said to be reflexive if $AlgLat(A) = W(A)$, where $W(A)$ is the smallest subalgebra of $B(E)$ that contains $A$ and the identity $I$ and is closed in the weak operator topology.

2. Main Result

In this section we will investigate the reflexivity of the operator $M_z$ acting on $L^p(\beta)$.

From now on we suppose that $\Omega_1 \neq \emptyset$ and $M_z$ is invertible on $L^p(\beta)$. Also, we will use the following notations:

$$r_{01} = \lim_{n \to \infty} \beta(-n)^{-\frac{1}{p}}, \quad \Omega_{01} = \{ z \in \mathbb{C} : |z| > r_{01} \}$$
$$r_{11} = \lim_{n \to \infty} \beta(n)^{-\frac{1}{p}}, \quad \Omega_{11} = \{ z \in \mathbb{C} : |z| < r_{11} \}$$
$$\Omega_1 = \Omega_{01} \cap \Omega_{11}$$

**Theorem.** If $L^p(\beta) = L^p_{\infty}(\beta)$, then $M_z$ is reflexive on $L^p(\beta)$.

**Proof.** First note that since $L^p(\beta) = L^p_{\infty}(\beta)$, $|f(\lambda)| \leq \|f\|$ for all $\lambda \in \Omega_1$, so $\|f\|_{\Omega_1} \leq \|f\|$ where the second norm is that of $L^p(\beta)$.

Put $M = H^\infty(\Omega_{11}) \cap L^p_{\infty}(\beta)$. Then $M \neq \emptyset$, since $1 \in M$. To prove that $M$ is closed, choose a sequence $\{f_n\}$ in $M$ such that $f_n$ converges to $f$ in $L^p(\beta)$. Then $\|f_n\|_{\Omega_1} \leq c$ for a constant $c$. By the continuity of point evaluations, $f_n(\lambda) \to f(\lambda)$ for every $\lambda \in \Omega_1$. Because $\|f_n\|_{\Omega_{11}} = \|f_n\|_{\Omega_1} \leq c$ ([14]), the sequence $\{f_n\}$ is a normal family and by passing to a subsequence, if necessary, we may assume that $f_n \to g$ uniformly on compact subsets of $\Omega_{11}$. Therefore $g$ is bounded and analytic on $\Omega_{11}$. From the pointwise convergence
on $\Omega_1$ of $f_n \to f$ we conclude that $f = g \in H^\infty(\Omega_{11})$. Thus indeed $f \in M$, since $L^p(\beta) = L^p_\infty(\beta)$, so $M$ is closed.

Now we show that $L : (M, \|\cdot\|_{\Omega_{11}}) \to B(L^p(\beta))$ be given by $L(\varphi) = M_\varphi$ is continuous. Suppose that the sequence $\{\varphi_n\}_n$ converges to $\varphi$ in $M$ and $L(\varphi_n) = M_{\varphi_n}$ converges to $A$ in $B(L^p(\beta))$. Then for each $f \in L^p(\beta)$,

$$Af = \lim_n M_{\varphi_n}f = \lim_n \varphi_n f$$

and so $\{\varphi_n f\}_n$ is convergent in $L^p(\beta)$. Note that by the continuity of point evaluations $\varphi_n f$ converges pointwise to $\varphi f$. Thus $Af$ is analytic and agree with $\varphi f$ on $\Omega_1$. Hence $A = M_\varphi$ and so $L$ is continuous. This implies that there is a constant $d > 0$ such that $\|M_\varphi\| \leq d\|\varphi\|_{\Omega_{11}}$ for all $\varphi$ in $M$.

Let $A \in AlgLat(M_\beta)$. Then $A = M_\psi$ for some $\psi \in L^p_\infty(\beta)$. Now clearly $M \in Lat(M_\beta)$, thus $AM \subseteq M$. Since $1 \in M$ we get $A1 = \varphi \in M = H^\infty(\Omega_{11})$. But $\Omega_{11}$ is a Caratheodory domain and so there is a sequence $\{p_n\}_n$ of polynomials converging to $\psi$ such that for all $n$, $\|p_n\|_{\Omega_{11}} \leq e$ for some $e > 0$. So we obtain

$$\|M_{p_n}\| \leq d\|p_n\|_{\Omega_{11}} \leq de$$

for all $n$. Since $L^p(\beta)$ is reflexive, the unit ball of $L^p(\beta)$ is weakly compact. Therefore ball $B(L^p(\beta))$ is compact in the weak operator topology and so by passing to a subsequence if necessary, we may assume that for some $X \in B(L^p(\beta))$, $M_{p_n} \to X$ in the weak operator topology. Using the fact that $M_{p_n}^* \to X^*$ in the weak operator topology and by acting these operators on $e(\lambda)$ we obtain that

$$p_n(\lambda) e(\lambda) = M_{p_n}^* e(\lambda) \to X^* e(\lambda)$$

weakly. Since $p_n(\lambda) \to \psi(\lambda)$ we see that $X^* e(\lambda) = \psi(\lambda) e(\lambda)$. Because the closed linear span of $\{e(\lambda) : \lambda \in \Omega_1\}$ is dense in $L^q(\beta)$, where $\frac{1}{p'} + \frac{1}{q} = 1$, we conclude that $X = M_\psi = A$. This implies that $A \in W(M_\beta)$ and so $M_\beta$ is reflexive. This completes the proof. □

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References


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