A Study on Generalised Aluthge Transformation

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Abstract

In this paper, various properties of $T(s, t) = |T|^s U |T|^t$ defined more generally for any $s$ and $t$ such as $s \geq 0$ and $t \geq 0$ the Aluthge transform of an operator $T$ are studied.

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1 Introduction

In [1] A. Aluthge introduced the operator $\tilde{T} = |T|^{1/2} U |T|^{1/2}$ for an operator $T$ with its polar decomposition $T = U |T| = |T^*| U$. [9] Takashi Yoshino defined more generally for any $s$ and $t$ such as $s \geq 0$ and $t \geq 0$ $T(s, t) = |T|^s U |T|^t$ the $p$-hyponormality of the Aluthge transform of $T$. [5] introduced a very interesting class of bounded linear Hilbert space operators Class A. Class A operators have been studied by many researchers for example [2, 3, 6, 10, 11, 12]. Recently Jeon and Kim [7] introduced quasiclass A operators as an extension of the notion of class A operators. In [8] Tanahashi, Jeon, Kim and Uchiyama considered an extension of quasiclass A operators, quasiclass $(A, k)$ operators. [4] H. Crawford Rhaly introduced
posinormal operators in Hilbert space. In this article we are interested in some of the properties of the Althuge transform $T(s, t)$.

## 2 Preliminary Notes

Let $T$ be a bounded linear operator on a Hilbert space $H$. In [1], A. Aluthge introduced the operator $\tilde{T}$ for an operator $T$ with its polar decomposition $T = U \, |T| = |T^*| \, U$ and [9] has introduced $T(s, t) = |T|^s \, U \, |T|^t$.

**Definition 2.1**: [5] An operator $T$ belongs to class A iff
\[(T^* |T|^2 T)^{1/2} \geq T^* T.\]

**Definition 2.2**: [7] An operator $T$ belongs to quasiclass A iff
\[T^* \big( |T|^2 - |T|^2 \big) T \geq 0.\]

**Definition 2.3**: [8] $T$ in $B(H)$ is called a $k$- quasiclass A operator for a positive integer $k$ if $T^{*k} \big( |T|^2 - |T|^2 \big) T^k \geq 0$.

**Definition 2.4**: [4] An operator $T$ in a Hilbert space $H$ is called posinormal if $TT^* \leq c^2 T^* T$ for some $c > 0$.

**Definition 2.5**: [4] An operator $T$ in a Hilbert space $H$ is quasiposinormal if $(TT^*)^2 \leq c^2 T^* T^2$.

**Definition 2.6**: [1] A bounded linear operator on a Hilbert space $H$ is $p$ - hyponormal if $(T^* T)^p \geq (TT^*)^p$, $p > 0$.

## 3 Posinormal Operators

**Theorem 3.1**:
Let $T$ be $p$ - hyponormal for some $p$ such that $p > 0$. Then for any $s, t$ such as max $(s, t) \leq p$, $T(s, t)$ is posinormal.

**Proof**:
Since $T$ is $p$ - hyponormal $|T|^{2p} \geq |T^*|^{2p}$. $T$ is posinormal if $TT^* \leq c^2 T^* T$ for some $c > 0$.

Now,
\[ T(s,t)T^*(s,t) - c^2 T^* (s,t) T(s,t) \]

\[ = |T|^s U |T|^t (|T|^s U |T|^t)^* - c^2 (|T|^s U |T|^t)^* (|T|^s U |T|^t) \]

\[ = |T|^s U |T|^t |T|^t U^* |T|^s - c^2 |T|^t U^* |T|^s U |T|^t \]

\[ = |T|^s |T^*|^2 |T|^s - c^2 |T|^t |T^*|^2 |T|^t \]

\[ = |T|^s (|T^*|^{2p})^{t/p} |T|^s - c^2 |T|^t (|T^*|^{2p})^{s/p} |T|^t \]

\[ \leq |T|^s (|T|^t)^{2} |T|^s - c^2 |T|^t (|T|^t)^{2} s/p |T|^t \]

\[ \leq |T|^{2(s+t)} - c^2 |T|^{2(s+t)} \]

\[ = (1 - c^2) |T|^{2(s+t)} \]

\[ \leq 0, c > 0. \]

\[ \Rightarrow T(s,t) \text{ is quasiposinormal}. \]

**Theorem 3.2:**

If \( T \) is a p-hyponormal operator then \( T(s,t) \) is quasiposinormal.

**Proof:**

\( T \) is quasiposinormal if

\[ (TT^*)^2 \leq c^2 T^{*2} T^2. \]

\[ (T(s,t)T^*(s,t))^2 - c^2 T^{*2}(s,t) T^2(s,t) \]

\[ = (|T|^s (|T^*|^{2p})^{t/p} |T|^s)^2 - c^2 |T|^t (|T^*|^{2p})^{s/p} |T|^t \]

\[ \leq |T|^{2(s+t)} - c^2 |T|^t (|T|^t)^{2} s/p |T|^t \]

\[ = |T|^{4(s+t)} - c^2 |T|^{4(s+t)} \]

\[ = (1 - c^2) |T|^{4(s+t)} \]

\[ \leq 0, c > 0 \]

\[ \Rightarrow T(s,t) \text{ is quasiposinormal}. \]

**4 Class A Operators**

**Theorem 4.1:**

If \( T \) is a p-hyponormal operator in a Hilbert space \( H \), then \( T(s,t) \) is of class \( A \).

**Proof:**

An operator \( T \) is in class \( A \) if

\[ (T^* T)^{1/2} \geq T^* T \]

or \( (T^* T)^{1/2} \geq (T^*)^2 \)
Now,

\[ T^*(s, t)|T(s, t)|^2 T(s, t) \]

\[ = T^*(s, t)T^*(s, t)T(s, t)T(s, t) \]

\[ = T^*(s, t)T^2(s, t) \]

\[ = |T|^2 U^* |T|^2 U |T|^2 \]

\[ = |T|^2 U^* |T|^4 U |T|^2 \]

\[ \geq |T|^2 |T|^4 |T|^2 \]

\[ \geq |T|^{4(s+t)} \]

\[ = (T^*(s, t)T(s, t))^2 \]

\[ \Rightarrow T(s, t) \text{ is in class } A. \]

**Theorem 4.2 :**

If \( T \) is \( p \)-hyponormal then \( T^*(s, t) \) is in quasiclass \( A. \)

**Proof:**

An operator \( T \) belongs to quasi class \( A \) if \( T^* \left( |T|^2 - |T|^2 \right) T \geq 0. \)

\[ T^*(s, t)|T^2(s, t)|T(s, t) - T^*(s, t)|T(s, t)|^2 T(s, t) \]

\[ = T^*(s, t)(T^*(s, t)T^2(s, t))^{1/2}T(s, t) - T^{2}(s, t)T^2(s, t) \]

\[ \geq T^*(s, t)(|T|^{4(s+t)})^{1/2}T(s, t) - |T|^{4(s+t)} \]

\[ \geq T^*(s, t)T^*(s, t)T(s, t)T(s, t) - |T|^{4(s+t)} \]

\[ \geq |T|^{4(s+t)} - |T|^{4(s+t)} \]

\[ = 0 \]

\[ \Rightarrow T(s, t) \text{ belongs to quasiclass } A. \]

**Theorem 4.3 :**

If \( T \) is \( p \)-hyponormal then \( T^*(s, t) \) belongs to \( k \)-quasiclass \( A. \)

**Proof:**

An operator \( T \) is \( k \) quasiclass \( A \) if \( T^k \left( |T|^2 - |T|^2 \right) T^k \geq 0. \)

\[ T^k(s, t)|T^2(s, t)|T(s, t) - T^*(s, t)|T(s, t)|^2 T^k(s, t) \]

\[ = T^k(s, t)(T^2(s, t)T^2(s, t))^{1/2}T(s, t) - T^k(s, t)T^k(s, t)T^k(s, t) \]

\[ \geq T^k(s, t)(|T|^{4(s+t)})^{1/2}T^k(s, t) - T^*(k+1)(s, t)T^*(k+1)(s, t) \]

\[ \geq T^k(s, t)T^*(s, t)T(s, t)T^k(s, t) - |T|^{(k+1)(s+t)} \]

\[ \geq |T|^{(k+1)(s+t)} - |T|^{(k+1)(s+t)} \]

\[ = 0 \]

\[ \Rightarrow T(s, t) \text{ belongs to } k \text{ quasiclass } A. \]
References


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