Properties of Certain Analytic Functions 
Associated with Two Boundary Points

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Abstract
For analytic functions \( f(z) \) in the closed unit disk \( U \), two boundary points \( z_1 \) and \( z_2 \) such that \( \alpha = (f'(z_1) + f'(z_2))/2 \in f'(U) \) are considered. The object of the present paper is to discuss some interesting conditions for \( f(z) \) to be \( |f'(z) - 1| < \rho |1 - \alpha| \) in \( U \) with some examples.

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1 Introduction

Let \( \mathcal{A}_n \) denote the class of functions
\[
f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \ldots \quad (n = 1, 2, 3, \ldots)
\]
that are analytic in the closed unit disk \( \overline{U} = \{ z \in \mathbb{C} : |z| \leq 1 \} \) and \( \mathcal{A} = \mathcal{A}_1 \).

Also, the open unit disk is denoted by \( U = \{ z \in \mathbb{C} : |z| < 1 \} \).

For two boundary points \( z_1 \) and \( z_2 \), let us consider
\[
\alpha = \frac{f'(z_1) + f'(z_2)}{2} \in f'(U).
\]

For such \( \alpha \) \( (\alpha \neq 1) \), if \( f(z) \in \mathcal{A}_n \) satisfies
\[
\left| \frac{f'(z) - \alpha}{1 - \alpha} - 1 \right| < \rho \quad (z \in U)
\]
for some real \( \rho > 0 \), then
\[
|f'(z) - 1| < \rho |1 - \alpha| \quad (z \in U).
\]
Therefore, if $0 < \rho|1 - \alpha| < 1$, then $f(z)$ is close-to-convex (univalent) in $\mathbb{U}$. In the present paper, we use such a technique for $f(z)$.

The basic tool in proving our results is the following lemma due to Jack [1] (also, due to Miller and Mocanu [2]).

**Lemma 1.** Let the function $w(z)$ defined by

$$w(z) = a_n z^n + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \ldots \quad (n = 1, 2, 3, \ldots)$$

be analytic in $\mathbb{U}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r$ at a point $z_0 \in \mathbb{U}$, then there exists a real number $k \geq n$ such that

$$\frac{z_0 w'(z_0)}{w(z_0)} = k$$

and

$$\Re \left( \frac{z_0 w''(z_0)}{w'(z_0)} \right) + 1 \geq k.$$

**2 Main results**

Applying Lemma 1, we drive the following results.

**Theorem 1.** If $f(z) \in A_n$ satisfies

$$\left| \frac{zf''(z)}{f'(z)} \right| < \frac{|1 - \alpha| n \rho}{1 + |1 - \alpha| \rho} \quad (z \in \mathbb{U})$$

for some complex $\alpha = \frac{f'(z_1) + f'(z_2)}{2} \in f'(\mathbb{U})$ and $\alpha \neq 1$ such that $z_1, z_2 \in \partial \mathbb{U}$, and for some real $\rho > 1$, then

$$|f'(z) - 1| < \rho|1 - \alpha| \quad (z \in \mathbb{U}).$$

**Proof.** Let us define $w(z)$ by

$$w(z) = \frac{f'(z) - \alpha}{1 - \alpha} - 1 \quad (z \in \mathbb{U})$$

$$= \frac{(n + 1)a_{n+1}}{1 - \alpha} z^n + \frac{(n + 2)a_{n+2}}{1 - \alpha} z^{n+1} + \ldots.$$
Then, clearly, \( w(z) \) is analytic in \( U \) and \( w(0) = 0 \). Differentiating both sides in (1), we obtain
\[
\frac{zf''(z)}{f'(z)} = \frac{(1 - \alpha)zw'(z)}{(1 - \alpha)w(z) + 1},
\]
and therefore,
\[
\left| \frac{zf''(z)}{f'(z)} \right| = \left| \frac{(1 - \alpha)zw'(z)}{(1 - \alpha)w(z) + 1} \right| < \frac{1 - \alpha|n\rho}{1 + |1 - \alpha|\rho} \quad (z \in U).
\]

If there exists a point \( z_0 \in U \) such that
\[
\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = \rho,
\]
then Lemma 1 gives us that \( w(z_0) = \rho e^{i\theta} \) and \( z_0 w'(z_0) = kw(z_0) \) (\( k \geq n \)).

For such a point \( z_0 \), we have
\[
\left| \frac{z_0 f''(z_0)}{f'(z_0)} \right| = \left| \frac{(1 - \alpha)z_0w'(z_0)}{(1 - \alpha)w(z_0) + 1} \right| = \frac{|(1 - \alpha)|k\rho}{1 + |1 - \alpha|\rho} \quad (\phi = \theta + \arg(1 - \alpha))
\]
\[
\geq \frac{|1 - \alpha|n\rho}{1 + |1 - \alpha|\rho}.
\]

This contradicts our condition in the theorem. Therefore, there is no \( z_0 \in U \) such that \( |w(z_0)| = \rho \). This means that \( |w(z)| < \rho \) for all \( z \in U \). It follows that
\[
\left| \frac{f'(z) - \alpha}{1 - \alpha} - 1 \right| < \rho \quad (z \in U)
\]
so that \( |f'(z) - 1| < \rho|1 - \alpha| \) in \( U \). \( \square \)
**Example 1.** Let us consider a function

\[ f(z) = z + a_{n+1}z^{n+1} \quad (z \in \mathbb{U}) \]

with \( |a_{n+1}| < \frac{1}{2(n+1)} \). Differentiating the function \( f(z) \), we obtain

\[ \frac{zf''(z)}{f'(z)} = \frac{n(n+1)a_{n+1}z^n}{1 + (n+1)a_{n+1}z^n} \]

and therefore,

\[ \left| \frac{zf''(z)}{f'(z)} \right| = \left| \frac{n(n+1)a_{n+1}z^n}{1 + (n+1)a_{n+1}z^n} \right| < \frac{n(n+1)|a_{n+1}|}{1 - (n+1)|a_{n+1}|} \quad (z \in \mathbb{U}). \]

We choose two boundary points \( z_1 \) and \( z_2 \) such that \( f'(z_1) = 1 + (n+1)|a_{n+1}| \) and \( f'(z_2) = 1 + (n+1)|a_{n+1}i| \). Then we know that \( z_1 = e^{-i\frac{\arg(a_{n+1})}{n}} \) and \( z_2 = e^{i\frac{\pi - 2\arg(a_{n+1})}{2n}} \). For such \( z_1 \) and \( z_2 \), we obtain

\[ \alpha = \frac{f'(z_1) + f'(z_2)}{2} = 1 + \frac{(n+1)|a_{n+1}|(1+i)}{2} \]

so that

\[ 1 - \alpha = -\frac{(n+1)|a_{n+1}|(1+i)}{2}. \]

Now, we consider some \( \rho > 1 \) such that

\[ \frac{n(n+1)|a_{n+1}|}{1 - (n+1)|a_{n+1}|} \leq \frac{|1 - \alpha|n\rho}{1 + |1 - \alpha|\rho} = \frac{n(n+1)|a_{n+1}|\rho}{\sqrt{2} + (n+1)|a_{n+1}|\rho}. \]

This gives us that

\[ \rho \geq \frac{\sqrt{2}}{1 - 2(n+1)|a_{n+1}|}. \]

For such \( \alpha \) and \( \rho \), we know that \( f(z) \) satisfies

\[ |f'(z) - 1| < (n+1)|a_{n+1}| \leq \frac{(n+1)|a_{n+1}|}{1 - 2(n+1)|a_{n+1}|} \leq \rho|1 - \alpha|. \]

If we defined

\[ \max_{z \in \mathbb{U}} |f'(z) - \alpha| = M_\alpha, \tag{2} \]
then we can have
\[
|f'(z) - 1| < \rho|1 - \alpha| \\
= \rho|f'(0) - \alpha| \\
\leq \rho M_\alpha \quad (z \in \mathbb{U}).
\]

So, we get

**Corollary 1.** If \( f(z) \in \mathcal{A}_n \) satisfies
\[
\left| \frac{zf''(z)}{f'(z)} \right| < \frac{|1 - \alpha|n\rho}{1 + |1 - \alpha|\rho} \quad (z \in \mathbb{U})
\]
for some complex \( \alpha = \frac{f'(z_1) + f'(z_2)}{2} \in f'(\mathbb{U}) \) and \( \alpha \neq 1 \) such that \( z_1, z_2 \in \partial\mathbb{U} \), and for some real \( \rho > 1 \), then
\[
|f'(z) - 1| < \rho M_\alpha \quad (z \in \mathbb{U}).
\]

Putting
\[
\alpha = \frac{f'(z_1) + f'(z_2) + \ldots + f'(z_m)}{m}
\]
(3)
for \( z_1, z_2, \ldots, z_m \in \partial\mathbb{U} \), we obtain

**Corollary 2.** If \( f(z) \in \mathcal{A}_n \) satisfies
\[
\left| \frac{zf''(z)}{f'(z)} \right| < \frac{|1 - \alpha|n\rho}{1 + |1 - \alpha|\rho} \quad (z \in \mathbb{U})
\]
for some complex \( \alpha = \frac{f'(z_1) + f'(z_2) + \ldots + f'(z_m)}{m} \in f'(\mathbb{U}) \) such that \( z_1, z_2, \ldots, z_m \in \partial\mathbb{U} \) and \( \alpha \neq 1 \), and for some real \( \rho > 1 \), then
\[
|f'(z) - 1| < \rho|1 - \alpha| \quad (z \in \mathbb{U}).
\]

We also derive
Theorem 2. If \( f(z) \in A_n \) satisfies
\[
\left| zf''(z) - \frac{zf''(z)}{f'(z)} \right| < \frac{|1 - \alpha|^2 n \rho^2}{1 + |1 - \alpha| \rho} \quad (z \in \mathbb{U})
\]
for some complex \( \alpha = \frac{f'(z_1) + f'(z_2)}{2} \in f'(\mathbb{U}) \) and \( \alpha \neq 1 \) such that \( z_1, z_2 \in \partial \mathbb{U} \), and for some real \( \rho > 1 \), then
\[
|f'(z) - 1| < \rho |1 - \alpha| \quad (z \in \mathbb{U}).
\]

Proof. Define \( w(z) \) by
\[
w(z) = \frac{f'(z) - \alpha}{1 - \alpha} - 1 \quad (z \in \mathbb{U}) \tag{4}
\]
\[
= \frac{(n + 1)a_{n+1}}{1 - \alpha} z^n + \frac{(n + 2)a_{n+2}}{1 - \alpha} z^{n+1} + \ldots.
\]

Evidently, \( w(z) \) is analytic in \( \mathbb{U} \) and \( w(0) = 0 \). Differentiating (4) logarithmically and simplifying, we have
\[
z f''(z) - \frac{zf''(z)}{f'(z)} = zf''(z) \left( 1 - \frac{1}{f'(z)} \right)
\]
\[
= \frac{(1 - \alpha)^2 z w'(z) w(z)}{(1 - \alpha) w(z) + 1},
\]
and hence,
\[
\left| zf''(z) - \frac{zf''(z)}{f'(z)} \right| = \left| \frac{(1 - \alpha)^2 z w'(z) w(z)}{(1 - \alpha) w(z) + 1} \right| < \frac{|1 - \alpha|^2 n \rho^2}{1 + |1 - \alpha| \rho} \quad (z \in \mathbb{U}).
\]

If there exists a point \( z_0 \in \mathbb{U} \) such that
\[
\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = \rho,
\]
then Lemma 1 gives us that \( w(z_0) = \rho e^{i\theta} \) and \( z_0 w'(z_0) = kw(z_0) \ (k \geq n) \).
For such a point $z_0$, we have
\[
\left| z_0 f''(z_0) - \frac{z_0 f''(z_0)}{f'(z_0)} \right| = \left| \frac{(1 - \alpha)^2 z_0 w'(z_0) w(z_0)}{(1 - \alpha) w(z_0) + 1} \right| \\
= \frac{|1 - \alpha|^2 \rho^2 k}{|(1 - \alpha) \rho e^{i\theta} + 1|} \\
= \frac{|1 - \alpha|^2 \rho^2 k}{\sqrt{1 + |1 - \alpha|^2 \rho^2 \cos^2 \phi + (|1 - \alpha|^2 \rho^2 \sin^2 \phi)^2}} \\
= \frac{|1 - \alpha|^2 \rho^2 k}{\sqrt{1 + |1 - \alpha|^2 \rho^2 + 2|1 - \alpha| \rho \cos \phi}} \\
\geq \frac{|1 - \alpha|^2 \rho^2 k}{1 + |1 - \alpha| \rho} \\
\geq \frac{|1 - \alpha|^2 \rho^2}{1 + |1 - \alpha| \rho}
\]

(\phi = \theta + \arg(1 - \alpha))

This contradicts our condition in the theorem. Therefore, there is no $z_0 \in \mathbb{U}$ such that $|w(z_0)| = \rho$. This means that $|w(z)| < \rho$ for all $z \in \mathbb{U}$. This implies that
\[
\left| \frac{f'(z) - \alpha}{1 - \alpha} - 1 \right| < \rho \quad (z \in \mathbb{U}).
\]

Example 2. We consider a function $f(z)$ given by
\[
f(z) = z + a_{n+1} z^{n+1} \quad (z \in \mathbb{U})
\]
with $|a_{n+1}| < \frac{1}{n + 1}$. Differentiating $f(z)$, we obtain that
\[
z f''(z) - \frac{z f''(z)}{f'(z)} = \frac{n(n + 1)^2 a_{n+1}^2 z^{2n}}{1 + (n + 1) a_{n+1} z^n},
\]
that is, that
\[
\left| z f''(z) - \frac{z f''(z)}{f'(z)} \right| = \left| \frac{n(n + 1)^2 a_{n+1}^2 z^{2n}}{1 + (n + 1) a_{n+1} z^n} \right| \\
\leq \frac{n(n + 1)^2 |a_{n+1}|^2}{1 - (n + 1) |a_{n+1}|} \quad (z \in \mathbb{U}).
\]

Choosing same points $z_1$ and $z_2$ in Example 1, we see that
\[
1 - \alpha = -\frac{(n + 1) |a_{n+1}| (1 + i)}{2}.
\]
If we consider some $\rho > 1$ such that
\[
\frac{n(n+1)^2|a_{n+1}|^2}{1 - (n+1)|a_{n+1}|} \leq \frac{|1 - \alpha|^2n\rho^2}{1 + |1 - \alpha|\rho},
\]
we have that
\[
\rho \geq \frac{\sqrt{2}}{1 - (n+1)|a_{n+1}|}.
\]
For such $\alpha$ and $\rho$, we know that
\[
|f'(z) - 1| < (n+1)|a_{n+1}| \leq \frac{(n+1)|a_{n+1}|}{1 - (n+1)|a_{n+1}|} \leq \rho|1 - \alpha|.
\]
Further, we obtain

**Corollary 3.** If $f(z) \in A_n$ satisfies
\[
|zf''(z) - zf''(z)| < \frac{|1 - \alpha|^2n\rho^2}{1 + |1 - \alpha|\rho} \quad (z \in U)
\]
for some complex $\alpha = \frac{f'(z_1) + f'(z_2)}{2} \in f'(U)$ and $\alpha \neq 1$ such that $z_1, z_2 \in \partial U$, and for some real $\rho > 1$, then
\[
|f'(z) - 1| < \rho M_\alpha \quad (z \in U).
\]

**Corollary 4.** If $f(z) \in A_n$ satisfies
\[
|zf''(z) - zf''(z)| < \frac{|1 - \alpha|^2n\rho^2}{1 + |1 - \alpha|\rho} \quad (z \in U)
\]
for some complex $\alpha = \frac{f'(z_1) + f'(z_2) + \ldots + f'(z_m)}{m} \in f'(U)$ such that $z_1, z_2, \ldots, z_m \in \partial U$ and $\alpha \neq 1$, and for some real $\rho > 1$, then
\[
|f'(z) - 1| < \rho|1 - \alpha| \quad (z \in U).
\]

Further, we discuss a new application for Lemma 1.
Theorem 3. If \( f(z) \in A_n \) satisfies
\[
\Re \left( \frac{z(zf''(z))'}{f'(z) - 1} \right) < n^2 \quad (z \in \mathbb{U})
\]
for some complex \( \alpha = \frac{f'(z_1) + f'(z_2)}{2} \in f'(\mathbb{U}) \) and \( \alpha \neq 1 \) such that \( z_1, z_2 \in \partial \mathbb{U} \), and for some real \( \rho > 1 \), then
\[
|f'(z) - 1| < \rho|1 - \alpha| \quad (z \in \mathbb{U}).
\]

Proof. Defining the function \( w(z) \) by
\[
w(z) = \frac{f'(z) - \alpha}{1 - \alpha} - 1 \quad (z \in \mathbb{U})
\]
\[
= \frac{(n + 1)a_{n+1}}{1 - \alpha}z^n + \frac{(n + 2)a_{n+2}}{1 - \alpha}z^{n+1} + \ldots,
\]
we have that \( w(z) \) is analytic in \( \mathbb{U} \) with \( w(0) = 0 \). Since,
\[
\frac{z(zf''(z))'}{f'(z) - 1} = \frac{zw'(z) + z^2w''(z)}{w(z)},
\]
we obtain that
\[
\Re \left( \frac{z(zf''(z))'}{f'(z) - 1} \right) = \Re \left( \frac{zw'(z) + z^2w''(z)}{w(z)} \right)
\]
\[
= \Re \left( \frac{zw'(z)}{w(z)} \left( 1 + \frac{zw''(z)}{w'(z)} \right) \right) < n^2 \quad (z \in \mathbb{U}).
\]

If there exists a point \( z_0 \in \mathbb{U} \) such that
\[
\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = \rho,
\]
then Lemma 1 gives us that \( w(z_0) = \rho e^{i\theta} \) and \( z_0w'(z_0) = kw(z_0) \) \( (k \geq n) \).

Thus we have
\[
\Re \left( \frac{z_0(z_0f''(z_0))'}{f'(z_0) - 1} \right) = \Re \left( \frac{z_0w'(z_0)}{w(z_0)} \left( 1 + \frac{z_0w''(z_0)}{w'(z_0)} \right) \right)
\]
\[
= k \left( 1 + \Re \left( \frac{z_0w''(z_0)}{w'(z_0)} \right) \right)
\]
\[
= k^2
\]
\[
\geq n^2.
\]

This contradicts our condition in the theorem. Therefore, there is no \( z_0 \in \mathbb{U} \) such that \( |w(z_0)| = \rho \). This means that \( |w(z)| < \rho \) for all \( z \in \mathbb{U} \). \( \square \)
We also have the following corollaries.

**Corollary 5.** If \( f(z) \in \mathcal{A}_n \) satisfies

\[
\text{Re} \left( \frac{z z f''(z)'}{f'(z) - 1} \right) < n^2 \quad (z \in \mathbb{U})
\]

for some complex \( \alpha = \frac{f'(z_1) + f'(z_2)}{2} \in f'(\mathbb{U}) \) and \( \alpha \neq 1 \) such that \( z_1, z_2 \in \partial \mathbb{U} \), and for some real \( \rho > 1 \), then

\[
|f'(z) - 1| < \rho M_{\alpha} \quad (z \in \mathbb{U}).
\]

**Corollary 6.** If \( f(z) \in \mathcal{A}_n \) satisfies

\[
\text{Re} \left( \frac{z z f''(z)'}{f'(z) - 1} \right) < n^2 \quad (z \in \mathbb{U})
\]

for some complex \( \alpha = \frac{f'(z_1) + f'(z_2) + \ldots + f'(z_m)}{m} \in f'(\mathbb{U}) \) such that \( z_1, z_2, \ldots, z_m \in \partial \mathbb{U} \) and \( \alpha \neq 1 \), and for some real \( \rho > 1 \), then

\[
|f'(z) - 1| < \rho |1 - \alpha| \quad (z \in \mathbb{U}).
\]

Next our result is contained in

**Theorem 4.** If \( f(z) \in \mathcal{A}_n \) satisfies

\[
\frac{z f''(z)}{f'(z) - 1} \neq k \quad (z \in \mathbb{U})
\]

for some complex \( \alpha = \frac{f'(z_1) + f'(z_2)}{2} \in f'(\mathbb{U}) \) and \( \alpha \neq 1 \) such that \( z_1, z_2 \in \partial \mathbb{U} \), some real \( \rho > 1 \), and for all real \( k \geq n \), then

\[
|f'(z) - 1| < \rho |1 - \alpha| \quad (z \in \mathbb{U}).
\]
Proof. Let us define the function \( w(z) \) by
\[
w(z) = \frac{f'(z)}{1 - \alpha} - 1 \quad (z \in \mathbb{U})
\]
\[
= \frac{(n + 1)a_{n+1}}{1 - \alpha} z^n + \frac{(n + 2)a_{n+2}}{1 - \alpha} z^{n+1} + \ldots.
\]

Clearly, \( w(z) \) is analytic in \( \mathbb{U} \) with \( w(0) = 0 \). We want to prove that \( |w(z)| < \rho \) in \( \mathbb{U} \). Note that
\[
\frac{zf''(z)}{f'(z) - 1} = \frac{zw'(z)}{w(z)} \quad (z \in \mathbb{U}).
\]

If there exists a point \( z_0 \in \mathbb{U} \) such that
\[
\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = \rho,
\]
then Lemma 1 gives us that \( w(z_0) = \rho e^{i\theta} \) and \( z_0 w'(z_0) = kw(z_0) \) \((k \geq n)\).

Thus we have
\[
\frac{z_0 f''(z_0)}{f'(z_0) - 1} = \frac{z_0 w'(z_0)}{w(z_0)} = k.
\]

This contradicts the condition in the theorem. Therefore, there is no \( z_0 \in \mathbb{U} \) such that \( |w(z_0)| = \rho \). This means that \( |w(z)| < \rho \) for all \( z \in \mathbb{U} \). We conclude that \( |f'(z) - 1| < \rho |1 - \alpha| \) in \( \mathbb{U} \).

**Corollary 7.** If \( f(z) \in \mathcal{A}_n \) satisfies
\[
\frac{zf''(z)}{f'(z) - 1} \neq k \quad (z \in \mathbb{U})
\]
for some complex \( \alpha = \frac{f'(z_1) + f'(z_2)}{2} \in f'(\mathbb{U}) \) and \( \alpha \neq 1 \) such that \( z_1, z_2 \in \partial \mathbb{U} \), some real \( \rho > 1 \), and for all real \( k \geq n \), then
\[
|f'(z) - 1| < \rho M_\alpha \quad (z \in \mathbb{U}).
\]

**Corollary 8.** If \( f(z) \in \mathcal{A}_n \) satisfies
\[
\frac{zf''(z)}{f'(z) - 1} \neq k \quad (z \in \mathbb{U})
\]
for some complex $\alpha = \frac{f'(z_1) + f'(z_2) + \ldots + f'(z_m)}{m} \in f'(U)$ such that $z_1, z_2, \ldots, z_m \in \partial U$, some real $\rho > 1$, and for all real $k \geq n$, then

$$|f'(z) - 1| < \rho|1 - \alpha| \quad (z \in U).$$

Finally, we derive

**Theorem 5.** If $f(z) \in A_n$ satisfies

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < \frac{|1 - \beta|n\rho}{1 + |1 - \beta|\rho} \quad (z \in U)$$

for some complex $\beta = \frac{F(z_1) + F(z_2)}{2} \in F(U)$ such that $z_1, z_2 \in \partial U$ and $\beta \neq 1$, and for some real $\rho > 1$, where $F(z) = \frac{f(z)}{z}$, then

$$\left| \frac{f(z)}{z} - 1 \right| < \rho|1 - \beta| \quad (z \in U).$$

**Proof.** Let us define $w(z)$ by

$$w(z) = \frac{\frac{f(z)}{z} - \beta}{1 - \beta} - 1 \quad (z \in U) \quad (6)$$

$$= \frac{a_{n+1}}{1 - \beta}z^n + \frac{a_{n+2}}{1 - \beta}z^{n+1} + \ldots.$$

Then, we have that $w(z)$ is analytic in $U$ and $w(0) = 0$. Differentiating (6) in both side logarithmically and simplifying, we obtain

$$zf'(z) = \frac{(1 - \beta)zw'(z)}{(1 - \beta)w(z) + 1},$$

and hence,

$$\left| \frac{zf'(z)}{f(z)} \right| = \left| \frac{(1 - \beta)zw'(z)}{(1 - \beta)w(z) + 1} \right| < \frac{|1 - \beta|n\rho}{1 + |1 - \beta|\rho} \quad (z \in U).$$

Using the same process of the proof in Theorem 1, we complete the proof of the theorem.
Example 3. We consider a function

\[ f(z) = z + a_{n+1}z^{n+1} \quad (z \in \mathbb{U}) \]

with \( |a_{n+1}| < \frac{1}{2} \). Differentiating the function, we obtain

\[ \frac{zf'(z)}{f(z)} - 1 = \frac{n a_{n+1} z^n}{1 + a_{n+1} z^n}, \]

and therefore,

\[ \left| \frac{zf'(z)}{f(z)} - 1 \right| = \left| \frac{n a_{n+1} z^n}{1 + a_{n+1} z^n} \right| < n |a_{n+1}| \quad (z \in \mathbb{U}). \]

Consider two boundary points \( z_1 = e^{-i \frac{\arg(a_{n+1})}{n}} \) and \( z_2 = e^{i \frac{\pi - 2 \arg(a_{n+1})}{2n}} \). Then, since \( F(z_1) = 1 + |a_{n+1}| \) and \( F(z_2) = 1 + |a_{n+1}|i \), we see that

\[ 1 - \beta = - \frac{|a_{n+1}|(1 + i)}{2}. \]

For such \( \beta \), we consider some \( \rho \) such that

\[ \frac{n |a_{n+1}|}{1 - |a_{n+1}|} \leq \frac{|1 - \beta| n \rho}{1 + |1 - \beta| \rho}. \]

This gives us that

\[ \rho \geq \frac{\sqrt{2}}{1 - 2|a_{n+1}|}. \]

Therefore, we have that

\[ |\frac{f(z)}{z} - 1| < |a_{n+1}| \leq \frac{|a_{n+1}|}{1 - 2|a_{n+1}|} \leq \rho |1 - \beta|. \]

Defining \( M_\beta \) by

\[ \max_{z \in \mathbb{U}} \left| \frac{f(z)}{z} - \beta \right| = M_\beta, \]

we have the following corollary.
Corollary 9. If \( f(z) \in \mathcal{A}_n \) satisfies
\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| < \frac{|1-\beta|n\rho}{1+|1-\beta|\rho} \quad (z \in \mathbb{U})
\]
for some complex \( \beta = \frac{F(z_1) + F(z_2)}{2} \in F(\mathbb{U}) \) such that \( z_1, z_2 \in \partial \mathbb{U} \) and \( \beta \neq 1 \), and for some real \( \rho > 1 \), where \( F(z) = \frac{f(z)}{z} \), then
\[
\left| \frac{f(z)}{z} - 1 \right| < \rho M_\beta \quad (z \in \mathbb{U}).
\]

Also considering
\[
\beta = \frac{f(z_1)}{z_1} + \frac{f(z_2)}{z_2} + \ldots + \frac{f(z_m)}{z_m}
\]
for \( z_1, z_2, \ldots, z_m \in \partial \mathbb{U} \), we obtain

Corollary 10. If \( f(z) \in \mathcal{A}_n \) satisfies
\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| < \frac{|1-\beta|n\rho}{1+|1-\beta|\rho} \quad (z \in \mathbb{U})
\]
for some complex \( \beta = \frac{F(z_1) + F(z_2) + \ldots + F(z_m)}{m} \in F(\mathbb{U}) \) such that \( z_1, z_2, \ldots, z_m \in \partial \mathbb{U} \) and \( \beta \neq 1 \), and for some real \( \rho > 1 \) where \( F(z) = \frac{f(z)}{z} \), then
\[
\left| \frac{f(z)}{z} - 1 \right| < \rho |1-\beta| \quad (z \in \mathbb{U}).
\]

References


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