Matricial Operators which Preserve Schauder Basis in p-Adic Analysis

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Abstract

In this work we give a generalization of the results established by W. Ruckle and L. W. Baric for the matrix transformations which preserve schauder basis in the classical case for a p-adic analysis. We give several characterizations of matricial operators which preserve Schauder bases in non archimedean Barrelled spaces.

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1 Introduction

W. Ruckle ([7], theorem2.1, p. 548) gave a characterization of a matrix $A = (a_{ij})_{i,j}$ which transforms a basis sequence $(x_i)_i$ to a basis sequence $(y_i)_i$ $y_i = \sum_{j=1}^{\infty} a_{ij} x_j$ for all $i \geq 1$ in a Frechet space $E$ on a field $K$ where $K = \mathbb{R}$ or $K = \mathbb{C}$.

By $A^T$, the transpose matrix of $A$, he gave an other characterization in case which $E$ is a Banach space ([7], theorem3.2, p.549). In ([1], theorem1 and theorem2, p.277, 278) L. W. Baric and W. Ruckle enhanced these results. These characterizations are based on the result established by A.Wilansky ([10], Theorem5, p. 211) and generalized by J. R. Retherford and C. W. MC. Arthur to a Hausdorff barrelled space ([6], theorem3.1 and theorem3.2, p. 40, 41). J. T. Marti ([4], III. 2, theorem2 and corollary3, p. 31) gave a standard criterion for a biorthogonal system to be a basis in a $F$-space (Frechet space). In ([11], theorem1, p. 6) N. Zobin gave a characterization of a matrix $A$ which transforms an absolute basis $(e_i)_i$ of a Frechet space to an absolute basis $(f_i)_i$. 
Throughout we consider $\tau$ defined by a family $(\mathcal{P})$ of n.a. seminorms. For fundamentals of locally $K$-convex space we refer to [5], [8] and [9]. A sequence $B = (x_i)_i$ in $X$ is called a (topological) base of $X$ if each $x \in X$ can be written uniquely as $x = \sum_{i=1}^{\infty} \lambda_i x_i$ with $\lambda_i \in K$. If the coefficient functionals $f_n : X \to K$, $x = \sum_{i=1}^{\infty} \lambda_i x_i \mapsto \lambda_n$ for all $n \geq 1$ are continuous then the basis $B$ is called a Schauder basis. If the topology of $X$ can be determined by a family $(\mathcal{P})$ of n.a. seminorms satisfying the condition if $x \in X$, $x = \sum_{i=1}^{\infty} \lambda_i x_i$ then $p(x) = \max_i p(\lambda_i x_i)$ for all $p \in (\mathcal{P})$ the basis is called orthogonal. A sequence $B = (x_i)_i$ is called a basic sequence if it is a Schauder basis of $[x_i]$ (it’s closed linear span) and it is called complete if $[x_i] = X$. A sequence $(x_i, f_i)_i$, $x_i \in X$, $f_i \in X'$, the topological dual of $X$, is said to be biorthogonal if $\langle x_i, f_j \rangle = \delta_{ij}$, $i, j = 1, 2, ...$. For general properties of bases before we refer to [2] and [3]. For every Schauder basis $B = (x_i)_i$ the sequence $F = (f_i)_i$ is a Schauder basis of $(X', \sigma(X', X))$ where $\sigma(X', X)$ is the weak topology on $X'$ ([2], lemma3, p. 402).

$(\omega, \tau_\omega) : = \{x : x \in K \}$ endowed with the product topology $\tau_\omega$ which is generated by the family of n.a. seminorms $(p_n)$ which is defined by $p_n(\lambda) = |\lambda|_n$ for all $\lambda = (\lambda_n)_n \in \omega$. A sequence space on $K$ is a subspace of $\omega$.

A sequence space $(E, \tau_E)$ on $K$, is called a $K$-space if for every $n$ in $\mathbb{N}$ the functional map $\pi_n : (E, \tau_E) \to K$, $\lambda_i \mapsto \lambda_n$ is continuous. It is called a $FK$-space if it is a $F$-space and a $K$-space over $K$.

Let $E$ be a non empty subset of $\omega$, the $\beta$-dual of $E$ is the set noted by $E^\beta$ and defined as follows:

$E^\beta = \{ (\lambda_j)_{1} \in \omega : \lim_{j} \lambda_j \mu_j = 0 \text{ for all } (\mu_j)_{1} \in E \}.$

Throughout $B = (x_i)_i$ is a Schauder basis of $(X, \tau)$ and $F = (f_i)_i$ is a weak Schauder basis associated.

Let $T \in B(X)$, then there exists an infinite matrix $A = (a_{ij})_{i,j}$ such that $a_{ij} \in K$ for all $i, j = 1, 2, ...$ and $T(x_i) = y_i = \sum_{j=1}^{\infty} a_{ij} x_j$ for all $i$ in $\mathbb{N}$. Throughout we consider $T = (a_{ij})_{i,j}$ it is called matricial operator and we denoted by $\omega_T$, $E_T$, $\Lambda$ and $L$ the following spaces:

$\omega_T = \{ (\lambda_i)_i \in \omega : \sum_i \lambda_i x_i \text{ converges in } X \}$

$E_T = \{ y \in X : there \ exists (\lambda_i)_i \in \omega_T : y = \sum_{i=1}^{\infty} \lambda_i y_i \}$

$\Lambda = \{ (\lambda_i)_i \in \omega : \sum_i \lambda_i x_i \text{ converges in } X \}$

$L = \{ y_i \}$

We provide $\omega_T$ and $E_T$ with the topology $\tau_T$ defined by the family of n.a
seminorms \( \{p_T\}_{p \in \mathcal{P}} \) where \( p_T \) is defined as: 
\[
p_T(y) = \sup_{\lambda} p(\lambda y)
\]
for all \((\lambda_i)_i \in \omega_T\), all \(y = \sum_{i=1}^{\infty} \lambda_i y_i \in E_T\) and all \(p \in \mathcal{P}\). Over \(\Lambda\) we define the topology \(\tau\) with the family of n.a seminorms \(\{p\}_p \in \mathcal{P}\) where \(p\) is defined as follows: 
\[
p((\lambda_i)_i) = p(x) \text{ for all } (\lambda_i)_i \in \Lambda, \text{ all } x = \sum_{i=1}^{\infty} \lambda_i x_i \in X \text{ and all } p \in \mathcal{P}.
\]

In §2 we give several notions of preserving and equicontinuous operators and characterize them. We also give an analogue result to ([7], theorem1.21, p. 547) which characterizes the sequence that are a Schauder basis in a n.a barrelled space. In §3 considering a barrelled space which is sequentially complete we give some characterizations of the matrix \(A\) which transforms a Schauder basis \(B\) into a topological basis or into a Schauder basis or into an orthogonal basis.

## 2 Preserving matricial operators

**Definitions 1** Let \(T\) be a matricial operator; \(T\) is called:

(a). Semi-preserving if it transforms the Schauder basis \(B\) into a basic sequence,

(b). Preserving if it transforms the Schauder basis \(B\) into a Schauder basis of \(X\),

(c). Topologically preserving if it transforms the Schauder basis \(B\) into a topological basis of \(X\),

(d). Semi-preserving orthogonally if it transforms the Schauder basis \(B\) into an orthogonal basis of \((E_T, \tau_T)\),

(e). Orthogonally-preserving if it transforms the Schauder basis \(B\) into an orthogonal basis of \(X\).

### Proposition 1

\((\omega_T, \tau_T)\) is a K-space for which \((e^i)_i\) is a topological basis and the mapping \(\Phi_T^{-1}\) is continuous, where \(\Phi_T : (E_T, \tau_{E_T}) \longrightarrow (\omega_T, \tau_T)\) 
\[
y = \sum_{i=1}^{\infty} \lambda_i y_i \longmapsto (\lambda_i)_i \text{ and } e^i = (\delta_{ij})_j \text{ for all } i \geq 1 \text{ (\(\delta_{ij}\) is the Kroneker symbol)}.
\]

**Proof.** Let us prove that for all \(n \in \mathbb{N}\) \(\pi_n : \omega_T \longrightarrow K (\lambda_i)_i \longmapsto \lambda_n\) is continuous. Let \((\lambda_i)_i \in \omega_T\), then for every \(n \geq 1\) there exists \(p \in \mathcal{P}\) such that \(p(y_n) \neq 0\) and \(|\lambda_n| \leq \frac{p_T((\lambda)_i)}{p(y_n)}\), as \(\pi_n\) is linear then it is continuous.

Let \((\lambda_i)_i \in \omega_T\), then \(\sum_i \lambda_i y_i\) converges in \(X\), so for all \(p \in \mathcal{P}\) there exists \(i_0 \in \mathbb{N}\) such that for all \(i > i_0\) \(p(\lambda_i y_i) \leq 1\) then for all \(p \in \mathcal{P}\) there exists \(i_0 \in \mathbb{N}\) such that \(p_T\left(\sum_{i=1}^{\infty} \lambda_i e^i\right) \leq 1\). Therefore \(\lim_{n \to \infty} \sum_{i=1}^{\infty} \lambda_i e^i = \lambda\) in \((\omega_T, \tau_T)\). On the other hand, for all \(y = \sum_{i=1}^{\infty} \lambda_i y_i \in E_T\) and all \(p \in \mathcal{P}\), \(p(y) \leq p_T(y)\), then \(\Phi_T^{-1}\) is continuous.

**Remark 1** \(\Phi_T\) is an isometry from \((E_T, \tau_T)\) towards \((\omega_T, \tau_T)\).
Proposition 2 \((E_T, \mathfrak{T}_T)\) is a Hausdorff space.

Proof. Let \(y = \sum_{i=1}^{\infty} \lambda_i y_i \in E_T\) such that \(y \neq 0\), then there exists \(p \in (\mathcal{P})\) such that \(p(y) \neq 0\), then there exists \(i_0 \in \mathbb{N}\) such that \(p(\lambda_{i_0} y_{i_0}) \neq 0\) and so \(\mathfrak{T}_T(y) \neq 0\). \(\square\)

Proposition 3 If \((X, \tau)\) is sequentially complete, then \((E_T, \mathfrak{T}_T)\) is complete.

Proof. Let \( (y^i)_{i \in I} \) be a Cauchy-net in \((E_T, \mathfrak{T}_T)\) and let \(y^i = \sum_{n=1}^{\infty} \lambda^i_n y_n\) for all \(i \in I\), then for all \(p \in (\mathcal{P})\) and all \(\varepsilon > 0\) there exists \(i_0 \in I\) such that for all \(i, j \geq i_0\) \(\mathfrak{T}_T(y^i - y^j) \leq \varepsilon\); moreover \(\sup_n |\lambda^1_n - \lambda^2_n| p(y_n) \leq \varepsilon\) for all \(i, j \geq i_0\), so for all \(n \geq 1\), \((\lambda^i_n)_{i \in I}\) is a Cauchy-net in \(K\), then for all \(n \geq 1\), there exists \(\lambda_n \in K\) such that \(\lim_i \lambda^i_n = \lambda_n\) in \(K\). In passing to the limit on \(j\), in (1) we obtain

\[(2) \sup_n |\lambda^i_n - \lambda_n| p(y_n) \leq \varepsilon, \text{ for all } i \geq i_0.\]

On the other hand, there exists \(n_0 \in \mathbb{N}\) such that for all \(n \geq n_0\) \(|\lambda^1_n| p(y_n) \leq \varepsilon\). Then there exists \(n_0 \in \mathbb{N}\) such that for all \(n \geq n_0\)

\[|\lambda_n| p(y_n) \leq \max \left( |\lambda^1_n - \lambda_n| p(y_n), |\lambda^0_n| p(y_n) \right) \leq \varepsilon.\]

Therefore \(\sum_n \lambda_n y_n\) converges in \((X, \tau)\).

Let \(y = \sum_n \lambda_n y_n\), then \(y \in E_T\) and by (2) we have \(y = \lim_i y^i\) in \((E_T, \mathfrak{T}_T)\). \(\square\)

Definition 1 \(T\) is called equicontinuous if for every \(p \in (\mathcal{P})\) there exists \(q \in (\mathcal{P})\) such that for all \(y = \sum_i^\infty \lambda_i y_i \in E_T\) and all \(i \geq 1\) \(p(\lambda_i y_i) \leq q(y)\).

Remark 2 Let \(\Psi_n : X \rightarrow E_T, y = \sum_{i=1}^{\infty} \lambda_i y_i \mapsto \lambda_n y_n\) for all \(n \geq 1\), then \(T\) is equicontinuous if and only if \((\Psi_n)_{n=1}^\infty\) is equicontinuous as a linear maps of \((X, \tau)\) in \((E_T, \mathfrak{T}_T)\) i.e. for all \(p \in (\mathcal{P})\) there exists \(q \in (\mathcal{P})\) such that for all \(y = \sum_{i=1}^\infty \lambda_i y_i \in E_T, \mathfrak{T}_T(y) \leq q(y)\). Therefore \(\mathfrak{T}_T = \tau_{E_T}\).

Proposition 4 If \(T\) is equicontinuous, then \(T\) is semi-preserving orthogonally.

Proof. Let \(y \in E_T\), then there exists \((\lambda_i)_{i \in \omega}\) such that \(y = \sum_{i=1}^{\infty} \lambda_i y_i\), then \((y_i)_{i \in \omega}\) is a topological basis of \(E_T\), and for every \(p \in (\mathcal{P}), \mathfrak{T}_T(y) = \sup_i p(\lambda_i y_i) = \sup_i \mathfrak{T}_T(\lambda_i y_i)\), then \((y_i)_{i \in \omega}\) is an orthogonal basis of \((E_T, \mathfrak{T}_T)\). \(\square\)

Proposition 5 Suppose that \((X, \tau)\) is sequentially complete. If \(T\) is equicontinuous then \(E_T\) is closed in \((X, \tau)\).

Proof. By remark 2 \(E_T^{\mathfrak{T}_T} = \overline{E_T}^{\mathfrak{T}_T}\), then \(E_T = \overline{E_T}^{\mathfrak{T}_T}\) because \((E_T, \mathfrak{T}_T)\) is complete, by proposition 3. \(\square\)
**Theorem 1** If \((X, \tau)\) is sequentially complete, \(T\) is complete \((L = X)\) and equicontinuous, then \(T\) is preserving.

**Proof.** For all \(n \geq 1\), \(y_n \in E_T\) then \([y_n] \subset E_T\), so \(L \subset E_T\), \(E_T\) is closed in \((X, \tau)\), therefore \(E_T = X\). Then \((y_n)_n\) is a Schauder basis of \((X, \tau_T)\). Well \(\tau_T = \tau_{E_T}\) (remark 2) then \(\tau_T = \tau\), and so \((y_n)_n\) is a Schauder basis of \((X, \tau)\).

**Remark 3** Under the conditions of theorem 1, \(T\) is orthogonally preserving.

If \((X, \tau)\) is barrelled we have the following theorem which is a converse of theorem 1.

**Theorem 2** Suppose that \((X, \tau)\) is barrelled. Then if \(T\) is topologically preserving, then \(T\) is equicontinuous.

**Proof.** Us \((y_n)_n\) is a topological basis of \((X, \tau)\), then \(E_T = X\), and us \((y_n)_n\) is an orthogonal basis of \((E_T, \tau_T)\), then it is equicontinuous ([2] proposition 5, p. 400). It suffices to show that \(\tau_T = \tau\). We have \(\tau \leq \tau_T\) (by definition of \(\tau_T\) and in fact that \(E_T = X\)). Conversely let \(U = \{x \in X : \overline{\tau_T}(x) \leq 1\} = B_{\tau_T}(0, 1)\), then \(U\) is absolutely \(K\)-convex and closed in \((X, \tau)\); let \(x = \sum_{n=1}^\infty \lambda_n y_n \in X\), then for every \(p \in (P)\) there exists \(n_0 \geq 1\) such that for all \(n \geq n_0\) \(p(\lambda_n y_n) \leq 1\); let \(\lambda \in K\) such that \(|\lambda| \succ \max(1, p(\lambda_1 y_1), p(\lambda_2 y_2), ..., p(\lambda_{n_0} y_{n_0}))\) then \(\overline{\tau_T}(x) \leq \lambda\) so \(x \in \lambda U\) and \(U\) is absorbing; us \((X, \tau)\) is barrelled then \(U\) is a neighbourhood of zero in \((X, \tau)\); therefore \(\tau_T \leq \tau\).

**Remark 4** In a barrelled space, every topological basis is an orthogonal basis (then it is a Schauder basis). N. De Grande-De Kimpe has proved that in a barrelled space, every Schauder basis is an orthogonal basis ([2], proposition 7, p. 401).

**Theorem 3** Let \((X, \tau)\) be a locally \(K\)-convex, sequentially complete and barrelled space and \((z_j)_j\) be a complete sequence in \(X\) such that for all \(j \in \mathbb{N}\) \(z_j \neq 0\); then \((z_j)_j\) is a topological basis (it is a Schauder basis) if and only if for every \(p \in (P)\) there exists \(q \in (P)\) such that for all \(j \in \mathbb{N}\) and all \(z = \sum_{j=1}^\infty \lambda_j z_j \in X\), \(p(\lambda_j z_j) \leq q(z)\).

**Proof.** By theorem 1 and theorem 2.

### 3 Conditions on the matrix \(A\)

In this paragraph \((X, \tau)\) is a space of type of theorem 3 and the family \((P)\) verify \(p(\sum_{i=1}^\infty \lambda_i x_i) = \max_i p(\lambda_i x_i)\) for all \(p \in (P)\).
Theorem 4  Let \((X, \tau)\) be a locally \(K\)-convex, sequentially complete and barrelled space, then \(T\) is semi-preserving if and only if for every \(p \in (\mathcal{P})\) there exists \(q \in (\mathcal{P})\) such that for all \(\lambda = (\lambda_j)_j \in \omega_T\) there exists \(i_0 \geq 1\) such that 

\[
\|p(\lambda)\| \leq \left| \sum_{j=1}^{\infty} a_{i_0j} \lambda_j \right| q(x_{i_0}).
\]

Lemma 1  If \(\sum_i \lambda_i y_i\) converges, then \(\sum_j a_{ij} \lambda_j\) converges for every \(i \in \mathbb{N}\) and in this case 

\[
\sum_{j=1}^{\infty} \lambda_j y_j = \sum_{i=1}^{\infty} \left( \sum_{j=1}^{\infty} a_{ij} \lambda_j \right) x_i.
\]

Proof.  For every \(i \geq 1\), \(f_i\) is continuous, then

\[
\begin{align*}
  f_i \left( \sum_{j=1}^{\infty} \lambda_j y_j \right) &= \sum_{j=1}^{\infty} \lambda_j f_i (y_j) \\
  &= \sum_{j=1}^{\infty} \lambda_j (\sum_{k=1}^{\infty} a_{kj} x_k) \\
  &= \sum_{j=1}^{\infty} \lambda_j \left( \sum_{k=1}^{\infty} a_{kj} f_i (x_k) \right) \\
  &= \sum_{j=1}^{\infty} \lambda_j a_{ij}.
\end{align*}
\]

Then \(\sum_j a_{ij} \lambda_j\) is converging in \(X\) and we have

\[
\sum_{j=1}^{\infty} \lambda_j y_j = \sum_{i=1}^{\infty} \left( \sum_{j=1}^{\infty} a_{ij} \lambda_j \right) x_i.
\]

Proof.  Of theorem. By theorem3, \((y_i)_i\) is a Schauder basis of \(L = [y_i]\) if and only if for all \(p \in (\mathcal{P})\) there exists \(q \in (\mathcal{P})\) such that for all \(y = \sum_{j=1}^{\infty} \lambda_j y_j\) and all \(j \in \mathbb{N}\) we have \(p(\lambda_j y_j) \leq q(y)\). And by lemma1, \(\sum_{j=1}^{\infty} \lambda_j y_j = \sum_{i=1}^{\infty} \left( \sum_{j=1}^{\infty} a_{ij} \lambda_j \right) x_i\), then the theorem holds.

Corollary 1  Under the conditions of theorem4, if \(T\) is complete, then \(T\) is preserving if and only if for every \(p \in (\mathcal{P})\) there exists \(q \in (\mathcal{P})\) such that for every \(\lambda = (\lambda_j)_j \in \omega_T\) there exists \(i_0 \in \mathbb{N}\) such that \(\|p(\lambda)\| \leq \left| \sum_{j=1}^{\infty} a_{i_0j} \lambda_j \right| q(x_{i_0})\).

Theorem 5  Let \((X, \tau)\) be a locally \(K\)-convex, sequentially complete and barrelled space, then \(T\) is preserving if and only if \(A\) maps \(\omega_T\) one-to-one onto \(\Lambda\).

Proof.  The same proof us in ([7], theorem2.1, p. 548).

Theorem 6  Let \((X, \tau)\) be a locally \(K\)-convex, sequentially complete and barrelled space, then

\begin{itemize}
  \item [(i).] \(A\) is injective if and only if \(\Phi_T^{-1}\) is injective,
  \item [(ii).] \(T\) is preserving if and only if \(A\) has an inverse \(B = (b_{ij})_{i,j}\) verifying
  \item [(a).] For every \(i \geq 1\), \((b_{ij})_j \in \Lambda^\beta\) and \((b).\) For every \(\lambda \in \Lambda\), \(B\lambda \in \omega_T\).
\end{itemize}

Where \(B\lambda = \left( \sum_{j=1}^{\infty} b_{ij} \lambda_j \right)_i\) for every \(\lambda = (\lambda_j)_j \in \Lambda\).
Proof. (i) By theorem 5.

(ii) Suppose that $T$ is preserving, then $(y_i)_i$ is a Schauder basis of $(X, \tau)$, so there exists $(b_{ij})_{i,j}$ such that for all $j \in \mathbb{N}$, $x_j = \sum_{i=1}^{\infty} b_{ij} y_i$, thus for every $\lambda = (\lambda_i)_i \in \Lambda$ we have $\sum_j \lambda_j x_j$ converges in $X$ then, by lemma 1, $\sum_j b_{ij} \lambda_j$ converges and $\sum_{j=1}^{\infty} \lambda_j x_j = \sum_{j=1}^{\infty} \left( \sum_{i=1}^{\infty} b_{ij} \lambda_j \right) y_i$, therefore $\left( \sum_{j=1}^{\infty} b_{ij} \lambda_j \right)_i \in \omega_T$. Then for all $i \in \mathbb{N}$ $(b_{ij})_j \in \Lambda$ and $B(\lambda) \subset \omega_T$. On the other hand, for all $j \in \mathbb{N}$ $x_j = \sum_{i=1}^{\infty} b_{ij} y_i$ so for all $j \in \mathbb{N}$ and all $k \in \mathbb{N}$ $\sum_i a_{ki} b_{ij}$ converges and for all $j \in \mathbb{N}$ $x_j = \sum_{i=1}^{\infty} b_{ij} y_i = \sum_{k=1}^{\infty} \left( \sum_{i=1}^{\infty} a_{ki} b_{ij} \right) x_k$, then $\sum_{k=1}^{\infty} a_{ki} b_{ij} = 1$ if $k = j$ and $\sum_{k=1}^{\infty} a_{ki} b_{ij} = 0$ if $k \neq j$. Then for all $j \in \mathbb{N}$ $A(b_{ij})_i = e^j$. At the same we show that for all $j \in \mathbb{N}$ $B(a_{ij})_i = e^j$. Then $A$ has an inverse $B$ verifying (a) and (b).

Conversely, suppose that $A$ has an inverse $B$ verifying (a) and (b), then $A(B e^i) = e^i$ and $B(A e^i) = e^i$ for all $i \in \mathbb{N}$, moreover $(e^i)_i$ is a Schauder basis of $\Lambda$ and $E_T$. On the other hand, for every $\lambda = (\lambda_j)_j \in \omega_T$ and for every $p \in (P)$ we have $\overline{p}(A \lambda) = p \left( \sum_{j=1}^{\infty} a_{ij} \lambda_j \right) x_i = p \left( \sum_{j=1}^{\infty} \lambda_j y_i \right) \leq \overline{p}(\lambda)$.

Then $A$ is continuous of $(\omega_T, \tau_T)$ towards $(\Lambda, \tau)$. At the same, $B$ is continuous because for all $\lambda = (\lambda_j)_j \in \Lambda$ and all $p \in (P)$ $\overline{p}(B \lambda) = \sup_i p \left( \sum_{j=1}^{\infty} b_{ij} \lambda_j y_i \right)$, but $\tau_T = \tau_{j/E_T}$ ($(X, \tau)$ is barrelled), then there exists $q \in (P)$ such that $\sup_i p \left( \sum_{j=1}^{\infty} b_{ij} \lambda_j y_i \right) \leq q \left( \sum_{j=1}^{\infty} \lambda_j y_i \right)$, but $\sum_{i=1}^{\infty} \left( \sum_{j=1}^{\infty} b_{ij} \lambda_j \right) y_i = \sum_{j=1}^{\infty} \lambda_j x_j \left( A^{-1} = B \right)$, so $\overline{p}(B \lambda) \leq q \left( \sum_{j=1}^{\infty} \lambda_j x_j \right) \leq \overline{p}(\lambda)$.

Then $A$ maps $\omega_T$ one-to-one onto $\Lambda$ and theorem 5 gives the conclusion. □

Theorem 7 Let $(X, \tau)$ be a locally $K$-convex, sequentially complete and barrelled space, then $T$ is preserving if and only if $A$ has an inverse $B = (b_{ij})_{i,j}$ and (3) : for every $p \in (P)$ there exists $q \in (P)$ such that for all $i,j \in \mathbb{N}$ $p(b_{ij} y_j) \leq q(x_i)$ and $p(a_{ij} x_j) \leq q(y_i)$ where $y_i = T(x_i)$ for all $i \geq 1$.

Proof. Suppose that $T$ is preserving and let $B = (b_{ij})_{i,j}$, the inverse of $A$, then for all $p \in (P)$ there exists $q \in (P)$ such that for all $i,j \in \mathbb{N}$ $p(b_{ij} y_j) \leq q(x_i)$ (theorem 4) $x_i = \sum_{j=1}^{\infty} b_{ij} y_j$ for all $i, j \in \mathbb{N}$. At the same we have the second inequality of (3).

Conversely, suppose that $A$ has an inverse $B = (b_{ij})_{i,j}$ and (3) holds; let $p \in (P)$ so there exists $q \in (P)$ such that for all $i,j \in \mathbb{N}$ $p(b_{ij} y_j) \leq q(x_i)$. Let $\lambda = (\lambda_j)_j \in \omega_T$ then $\sum_{j=1}^{\infty} \lambda_j y_j = \sum_{j=1}^{\infty} \left( \sum_{i=1}^{\infty} a_{ij} \lambda_j \right) x_i$ (lemma 1); for all $i \in \mathbb{N}$ let $\mu_i = \sum_{j=1}^{\infty} a_{ij} \lambda_j$, then $\mu = (\mu_i)_i \in \Lambda$ and $\lambda = B \mu$, so for all $i \in \mathbb{N}$ $\lambda_i = \sum_{j=1}^{\infty} b_{ij} \mu_j$ and for every $i \in \mathbb{N}$ we have

\[
\begin{align*}
p(\lambda_i y_i) &= p \left( \sum_{j=1}^{\infty} b_{ij} \mu_j y_i \right) \\
&\leq \max_j p(b_{ij} \mu_j y_j) \\
&\leq \max_j |\mu_j| q(x_i) \quad \text{(by (3))} \\
&\leq \max_j q(\mu_j x_i)
\end{align*}
\]
\[ \leq \max_j q (\sum_{k=1}^{\infty} a_{jk} \lambda_k x_j). \]

On the other hand, for all \( i \in \mathbb{N} \) if \( x_i = \sum_{j=1}^{\infty} b_{ij} y_j \) then \( (x_i)_i \subset E_T \) and by the same us before we obtain \( \lambda = (\lambda_j)_j \in \Lambda \) \( p (\lambda_i x_i) = \max_j q (\sum_{k=1}^{\infty} b_{jk} \lambda_k y_j) \), so \( (x_i)_i \) is an orthogonal basis of \( E_T \) (remark3). Then \( T \) is complete.

Finally \( T \) is complete and verifying the conditions of theorem 4, then \( T \) is preserving (corollary1).

**Corollary 2** Under the conditions of theorem before, \( T \) is preserving if and only if \( A \) has an inverse \( B = (b_{ij})_{i,j} \) and for all \( i,j \in \mathbb{N} \)

\[ |b_{ij}| \leq \inf_p \in (\mathcal{P}) \frac{\lambda_{i,m(p)}}{\mu_{j,p}} \quad \text{and} \quad |a_{ij}| \leq \inf_p \in (\mathcal{P}) \frac{\mu_{i,m(p)}}{\lambda_{j,p}}, \]

where \( \lambda_{i,p} = p (x_i) \), \( \mu_{i,m(p)} = m (p) (y_i) \) for all \( i \in \mathbb{N} \) and all \( p \in (\mathcal{P}) \).

**References**


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