Composition Operators of Class $Q^*$

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Abstract

In this paper, class $Q^*$ composition operators on $L^2$ space are characterized and the weighted composition operators $C_r = |C|^rU|C|^{1-r}$, $0 < r \leq 1$ and $\tilde{C} = |C|^\frac{1}{2}V|C|^\frac{1}{2}$ are studied.

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1. Introduction and Preliminaries

Let $(X, \Sigma, \lambda)$ be a sigma - finite measure space, a bounded linear operator $Cf = f \circ T$ on $L^2(X, \Sigma, \lambda)$ is said to be a composition operator induced by $T$, a non singular measurable transformation from $X$ into itself when the measure $\lambda T^{-1}$ is absolutely continuous with respect to the measure $\lambda$ and the Radon - Nikodym derivative $d\lambda T^{-1}/d\lambda = f_0$ is essentially bounded. The Radon - Nikodym derivative of the measure $\lambda(T^k)^{-1}$ with respect to $\lambda$ is denoted by $f_0^{(k)}$, where $T^k$ is obtained by composing $T^{-1}$ times. Every essentially bounded complex valued measurable function $f_0$ induces the bounded operator $M_{f_0}$ on $L^2(\lambda)$, which is defined by $M_{f_0}f = f_0f$ for every $f \in L^2(\lambda)$. Further $C^*C = M_{f_0}$ and $C^{*2}C^2 = M_{f_0}^2$ [15]

A weighted composition operator (w.c.o) induced by $T$ is defined as $Wf = w(f \circ T)$, $w$ is a complex valued $\Sigma$ measurable function. Let $w_k$ denote $w(w \circ T)(w \circ T^2)...(w \circ T^{k-1})$ so that $W^k f = w_k(f \circ T)^k$ [13]. Alan Lambert [12] associated conditional expectation operator $E$ with $T$ as $E(\cdot / T^{-1}\Sigma) = E(\cdot)$. $E(f)$ is defined for each non - negative measurable function $f \in L^p(1 \leq p)$ and is uniquely determined by the condition
(i) $E(f)$ is $T^{-1}\Sigma$ measurable.
(ii) if $B$ is any $T^{-1}\Sigma$ measurable set for which $\int_B f d\lambda$ converges we have $\int_B f d\lambda = \int_B E(f) d\lambda$.

For deeper study of the properties of $E$ see [5],[9],[10]. As an operator on $L^p$, $E$ is the projection on to the closure of range of $C$. $E$ is the identity on $L^p$ if and only if $T^{-1}\Sigma = \Sigma$.

Let $B(H)$ denote the Banach Algebra of all bounded linear operators on a Hilbert Space $H$. An operator $T \in B(H)$ is $*$- paranormal if $\|T^* x\|^2 \leq \|T^2 x\| \|x\|$ for every $x \in H$ [2]. Equivalently an operator $T \in B(H)$ is $*$-Paranormal if and only if $T^{*2}T^2 - 2\lambda TT^* + \lambda^2 \geq 0$ for all $\lambda \in \mathbb{R}$ [3]. An operator $T$ is class $Q$ if $T^{*2}T^2 - 2T^*T + 1 \geq 0$. Equivalently $T$ is of class $Q$ if $\|Tx\|^2 \leq \frac{1}{2}(\|T^2x\|^2 + \|x\|^2)$ for every $x \in H$. Class $Q$ operator is studied by B. P Duggal. et al.[7]. Class $Q$ composition operator is studied by S. Panayappan. et al. [14].

2. Class $Q^*$ composition operators

Youngoh Yang and Cheoul Jun Kim [16] introduced a new class ‘class $Q^*$’ operators and studied several properties of class $Q^*$. $*$- paranormal composition operator is studied in [6] by N. Chennappan and S. Karthikeyan. In this article our main aim is to characterize class $Q^*$ composition operator and weighted class $Q^*$ composition operator.

**Definition 2.1** [16]. An operator $T$ is of class $Q^*$ if $T^{*2}T^2 - 2TT^* + I \geq 0$. Equivalently $T \in Q^*$ if $\|T^*x\|^2 \leq \frac{1}{2}(\|T^2x\|^2 + \|x\|^2)$ for every $x \in H$.

The following lemma due to Harrington and Whitely [11] is well known.

**Lemma 2.2.** Let $P$ denote the projection of $L^2$ onto $\overline{R(C)}$
(a) $C^* C f = f_0 f$ and $CC^* f = (f_0 \circ T) P f$ for all $f \in L^2$.
(b) $\overline{R(C)} = \{ f \in L^2 : f$ is $T^{-1}\Sigma$ measurable $\}$.

**Theorem 2.3.** Let $C \in B(L^2(\lambda))$. Then $C$ is of class $Q^*$ if and only if $f_0^{(2)} = 2(f_0 \circ T) P + 1 \geq 0$ a.e., where $P$ is the projection of $L^2$ onto $\overline{R(C)}$.

**Proof.** By definition 2.1, $C$ is of class $Q^*$ if and only if $C^{*2}C^2 - 2CC^* + I \geq 0$. Thus $\langle (C^{*2}C^2 - 2CC^* + I) \chi_E, \chi_E \rangle \geq 0$ for every characteristic function $\chi_E$ of $E$ in $\Sigma$ such that $\lambda(E) < \infty$. This imply that $\langle (M_{f_0^{(2)}} - 2M_{f_0 \circ T} P + 1) \chi_E, \chi_E \rangle \geq 0$. That is $\int_E (f_0^{(2)} - 2(f_0 \circ T) P + 1) d\lambda \geq 0$ for every $E$ in $\Sigma$. Hence $C$ is class $Q^*$ if and only if $f_0^{(2)} - 2(f_0 \circ T) P + 1 \geq 0$ a.e.
Corollary 2.4. If \( C \in B(L^2(\lambda)) \) with dense range then \( C \) is of class \( Q^* \) if and only if \( f_0^{(2)} - 2(f_0 \circ T) + 1 \geq 0 \) a.e.

Proof. Since \( C \) has dense range then we have \( CC^* f = (f_0 \circ T) f \). It follows that \( f_0^{(2)} - 2(f_0 \circ T) + 1 \geq 0 \) a.e.

Example 2.5. Let \( X = N \), the set of all natural numbers and \( \lambda \) be the counting measure on it. Define \( T : N \to N \) by \( T(1)=T(2)=T(3)=1, T(4n+m)=n+1 \), \( m = 0,1,2,3 \) and \( n \in N \). Since \( f_0^{(2)} - 2(f_0 \circ T) + 1 \geq 0 \) for every \( n \), \( C \) is of class \( Q^* \) composition operator.

Theorem 2.6. Let \( C \in B(L^2(\lambda)) \) then \( C \) is \( * \)-paranormal if and only if \( (f_0 \circ T)^2 P \leq f_0^{(2)} \) a.e.

Proof. \( C \) is \( * \)-paranormal if and only if \( C^* C^2 - 2kCC^* + k^2 \geq 0 \). It follows that, \( C \) is \( * \)-paranormal if and only if \( (f_0^{(2)} - 2k(f_0 \circ T)P + k^2 \geq 0 \) a.e. That is \( (f_0^{(2)} \circ T)^2 P \leq f_0^{(2)} \) a.e.

Corollary 2.7. Let \( C \in B(L^2(\lambda)) \) with dense range, if \( C \) is \( * \)-paranormal then \( C \) is of class \( Q^* \).

Proof. \( C \) is \( * \)-paranormal then \( (f_0 \circ T)^2 \leq f_0^{(2)} \) a.e.

Now, \( f_0^{(2)} - 2(f_0 \circ T) + 1 \geq (f_0 \circ T)^2 - 2(f_0 \circ T) + 1 \geq 0 \). That is, \( C \) is of class \( Q^* \).

Theorem 2.8. Let \( C \in B(L^2(\lambda)) \), then \( C^* \in \text{class } Q^* \) if and only if \( (f_0^{(2)} \circ T^2)P_2 - 2f_0 + 1 \geq 0 \), a.e where \( P_2 \) is the projection of \( L^2 \) onto \( \overline{R(C^2)} \).

Proof. \( C^* \) is of class \( Q^* \) if and only if \( C^2C^* - 2C^* C + I \geq 0 \). That is \( \langle (C^2C^* - 2C^* C + I) f, f \rangle \geq 0 \) for every \( f \in L^2 \). We have \( \langle CC^* f, f \rangle = \langle (f_0 \circ T)P_1 f, f \rangle \) and \( \langle C^2C^* f, f \rangle = \langle (f_0^{(2)} \circ T^2)P_2 f, f \rangle \), \( P_1 \) and \( P_2 \) are the projections of \( L^2(\lambda) \) on to \( \overline{R(C)} \) and \( \overline{R(C^2)} \) respectively. Thus \( C^* \) is of class \( Q^* \) if and only if \( \langle (f_0^{(2)} \circ T^2)P_2 f, f \rangle - 2(f_0 f, f) + \langle f, f \rangle \geq 0 \), that is \( (f_0^{(2)} \circ T^2)P_2 - 2f_0 + 1 \geq 0 \), a.e.

3. Weighted Class \( Q^* \) Composition Operator

The following proposition is well known.

Proposition 3.1.\[5\] For \( w \geq 0 \)

i) \( W^* W f = f_0 [E(w^2)] \circ T^{-1} f \).

ii) \( WW^* f = w(f_0 \circ T) E(wf) \).
The following theorem characterize the weighted class $Q^*$ composition operators.

**Theorem 3.2.** If $T^{-1}\Sigma=\Sigma$. Then $W$ is of class $Q^*$ if and only if $f_0^{(2)}(w_2^2) \circ T^{-2} - 2w^2(f_0 \circ T) + 1 \geq 0$ a.e

**Proof.** We have $W^k f = w_k(f \circ T^k)$ and $(W^*)f = f_0^{(k)}E(w_kf) \circ T^{-k}$. Thus $W^*W^k = f_0^{(k)}E(w_k^2) \circ T^{-k}f$. We have $|W^*f| = vE(vf)$ where $v = \frac{w^2\sqrt{f_0T}}{|E(w\sqrt{f_0T})|^4}$. If $T^{-1}\Sigma=\Sigma$ then $E$ becomes identity operator and hence $WW^*f = v^2f = w^2(f_0 \circ T)f; f \in L^2$.

Since $W$ is of class $Q^*$, $W^2W^2 - 2WW^* + 1 \geq 0$. Thus $\langle (W^2W^2 - 2WW^* + 1)f, f \rangle \geq 0$ for all $f \in L^2$. It follows that $\int_E(f_0^{(2)}E(w_k^2) \circ T^{-2} - 2w^2(f_0 \circ T) + 1)d\lambda \geq 0$ for every $E \in \Sigma$. That is $f_0^{(2)}E(w_k^2) \circ T^{-2} - 2w^2(f_0 \circ T) + 1 \geq 0$ a.e.

The Aluthge transform of $T$ is the operator $\tilde{T}$ given by $\tilde{T} = |T|\frac{1}{2}U|T|^\frac{1}{2}$ was introduced in [1] by Aluthge. The idea behind the Aluthge transform is to convert an operator into another operator which shares with the first one some spectral properties but it is closed to being a normal operator. More generally we may form the family of operators $A_r : 0 < r \leq 1$ where $A_r = |A|^{1-r}U|A|^{1-r}$. For a composition operator $C$, the polar decomposition is given by $C = U|C|$ where $|C|f = \sqrt{f_0f}$ and $Uf = \frac{1}{\sqrt{f_0f}} f \circ T$. In [4] Lambert has given general Aluthge transformation for composition operators as $C_r = |C|U|C|^{1-r}C_r f = (f_0 \circ T)^\frac{r}{2} f \circ T$. That is $C_r$ is weighted composition operator with weight $\pi = (f_0 \circ T)^\frac{r}{2}$ where $0 < r < 1$. Since $C_r$ is a weighted composition operator it is easy to show that $|C_r|f = \sqrt{f_0|E(\pi)^2 \circ T^{-1}|f}$ and $|C_r^*|f = vE(vf)$ where $v = \frac{\pi \sqrt{f_0T}}{|E(\pi \sqrt{f_0T})|^4}$. Also we have

$C^k f = \pi_k(f \circ T^k)$,
$C^{*k} f = f_0^{(k)}E(\pi_k f) \circ T^{-k}$,
$C_r^{*k} C^{-k} f = f_0^{(k)}E(\pi_k^2) \circ T^{-k}f$.

**Corollary 3.3.** If $T^{-1}\Sigma=\Sigma$, $C_r$ is class $Q^*$ if and only if $f_0^{(2)}(\pi_2^2) \circ T^{-2} - 2\pi^2(f_0 \circ T) + 1 \geq 0$.

**Proof.** Since $C_r$ is weighted composition operator with weight $\pi = (f_0 \circ T)^\frac{r}{2}$, we get the desired result.

B. P Duggal [8] described the second Aluthge Transformation of $T$ by $\tilde{T} = |T|\frac{1}{2}V|T|^\frac{1}{2}$, where $\tilde{T} = V|T|$ is the polar decomposition of $T$. Now we consider $C_r = |C_r|\frac{1}{2}V|C_r|^\frac{1}{2}$, where $C_r = V|C_r|$ is the polar decomposition of the generalized Aluthge transformation $C_r : 0 < r < 1$. We have $|C_r|f = \sqrt{Jf}$, where $J = f_0E(\pi^2) \circ T^{-1}$. 
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\[ \tilde{C} = |C_r|^{1/2} V |C_r|^{1/2} = \sqrt{J^{1/2}} V (\sqrt{J^{1/2}} f) = \sqrt{J^{1/2}} \pi (\chi_{\sup J^{1/4}} J^{1/4} f) \circ T = J^{1/4} \pi (\chi_{\sup J^{1/4}} J^{1/4} f) \circ T (f \circ T) \]

We see then that \( \tilde{C} \) is a weighted composition operator with weight \( w' = J^{1/4} \pi (\chi_{\sup J^{1/4}} J^{1/4} \circ T) \).

**Corollary 3.4.** If \( T^{-1} \Sigma = \Sigma \), then \( \tilde{C} \) is class \( Q^* \) if and only if \( f_0^{(2)} (w'_{\Sigma}^2) \circ T^{-2} - 2(w')^2 (f_0 \circ T) + 1 \geq 0 \) a.e.

**Proof.** Since \( \tilde{C} \) is weighted composition operator with weight \( w' = J^{1/4} \pi (\chi_{\sup J^{1/4}} J^{1/4} \circ T) \), we get the desired result.

**References**


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