Controllability of Fractional Semilinear Mixed
Volterra-Fredholm Integrodifferential Equations
with Nonlocal Conditions

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Abstract
In this article, the controllability problem of fractional semilinear mixed Volterra-
Fredholm integrodifferential equations in Banach space together with nonlocal con-
ditions is investigated. An example is given to illustrate the theory.

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 calculus

1 Introduction
Fractional differential equations are emerged as a new branch of applied mathematics by
which many physical and engineering approaches can be modelled [5]. The fact that frac-
tional differential equations are considered as alternative models to nonlinear differential
equations which induced extensive researches in various fields including the theoretical
part [8],[9],[13]. The existence of fractional semilinear integrodifferential equations are
one of the theoretical fields that investigated by many authors [11]. On the other hand,
the controllability is one concept of control dynamic systems that some classes of such sys-
tems can be represented by nonlinear differential equations [2], [3], [6]. So it is naturally to
extend this concept of such systems to be represented by fractional differential equations
[1], [7], [10], [14]. Recently, controllability of fractional nonlinear differential systems in
infinite dimensional spaces has been investigated by many authors [4], [15]. Motivated
by these works, we show that a particular class of fractional integrodifferential systems in
Banach spaces is controllable provided that some conditions have to be satisfied.

The paper is organized as follows. In section 2, some definitions, lemmas, and as-
sumptions are introduced to be used in the sequel. Section 3 and 4 will involve the main
results and proofs of controllability problems of the given system with classical initial and nonlocal conditions respectively. Finally, in section 5, an example is introduced to illustrate the theory.

2 Preliminaries

We need some basic definitions and properties of fractional calculus which will be used in this paper.

**Definition 2.1** A real function \( f(t), t \geq t_0 \) is said to be in the space \( C_\mu \), \( \mu \in \mathbb{R} \), if there exists a real number \( p > \mu \), such that \( f(t) = (t - t_0)^p f_1(t) \), where \( f_1 \in C[t_0, \infty) \), and it is said to be in the space \( C^m_\mu \) if and only if \( f^{(n)} \in C_\mu, n \in \mathbb{N} \).

**Definition 2.2** A function \( f \in C_\mu, \mu \geq -1 \) is said to be fractional integrable of order \( \alpha > 0 \) if

\[
I^\alpha f(t) = (I^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - s)^{\alpha - 1} f(s) ds < \infty,
\]

where \( t_0 \geq 0 \); and if \( \alpha = 0 \), then \( I^0 f(t) = f(t) \). Moreover, \( I^\alpha (C_\mu) \) denotes the space of all fractional integrable functions of order \( \alpha \).

Next, we introduce the Caputo fractional derivative.

**Definition 2.3** The fractional derivative in the Caputo sense is defined as

\[
\frac{d^\alpha f(t)}{dt^\alpha} = I^{1-\alpha} \left( \frac{df(t)}{dt} \right) = \frac{1}{\Gamma(1 - \alpha)} \int_{t_0}^{t} (t - s)^{-\alpha} \left( \frac{df(s)}{ds} \right) ds
\]

for \( 0 < \alpha \leq 1, t_0 \geq 0, f \in C_{-1} \).

The properties of the above operators and the general theory of fractional differential equations can be found in [12] and [13].

Consider the following fractional semilinear integrodifferential system

\[
\begin{align*}
\frac{d^\alpha x(t)}{dt^\alpha} &= Ax(t) + Bu(t) + f(t, x(t), \int_{t_0}^{t} k(t, s, x(s)) ds, \int_{t_0}^{T} h(t, s, x(s)) ds), \\
x(t_0) &= x_0 \in X.
\end{align*}
\]  

(1)

where \( t \in J = [t_0, T], t_0 \geq 0, 0 < \alpha < 1 \), \( x \in Y = C(J, X) \) is a continuous function on \( J \) with values in the Banach space \( X \) and \( \|x\|_Y = \max_{t \in J} \|x(t)\|_X \). The control function
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$u \in L_2(J, U)$, a Banach space of admissible control functions with $U$ as a Banach space. $B : U \to X$ is a bounded linear operator, and the nonlinear functions $f : J \times X \times X \times X \to X$, $k : D \times X \to X$, and $h : D_0 \times X \to X$ are continuous. Here $D = \{(t, s) \in \mathbb{R}^2 : t_0 \leq s \leq t \leq T\}$, and $D_0 = J \times J$. The operator $\frac{d^\alpha}{dt^\alpha}$ denotes the Caputo fractional derivative of order $\alpha$. For brevity let us take

$$Kx(t) = \int_{t_0}^t k(t, s, x(s))ds, \quad \text{and} \quad Hx(t) = \int_{t_0}^T h(t, s, x(s))ds.$$ 

The common norm $\|\cdot\|$ is used throughout the sequel.

**Definition 2.4 ([4])** A continuous solution $x(t)$ of the integral equation

$$x(t) = S(t - t_0)x_0 + \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} S(t-s) \left( Bu(s) + f(s, x(s), Kx(s), Hx(s)) \right) ds$$

is called a mild solution of equation (1), where $S(\cdot)$ is a strongly continuous semigroup of bounded linear operator generated by $A$.

If $\alpha \to 1$, the mild solution (2) becomes the standard mild solution given by

$$x(t) = S(t - t_0)x_0 + \int_{t_0}^t S(t-s) \left( Bu(s) + f(s, x(s), Kx(s), Hx(s)) \right) ds$$

which is the mild solution of the system

$$\begin{align*}
\frac{dx(t)}{dt} &= Ax(t) + Bu(t) + f(t, x(t), \int_{t_0}^t k(t, s, x(s))ds, \int_{t_0}^T h(t, s, x(s))ds), \\
x(t_0) &= x_0.
\end{align*}$$

(3)

The non-fractional system (3) was consider by [6] in the inclusion form.

Analogous to the conventional controllability concept, the controllability of fractional dynamical system (1) is defined as in [7].

**Definition 2.5** The fractional system (1) is said to be controllable on the interval $J$ if for every $x_0, x_1 \in X$, there exists a control $u \in L_2(J, U)$ such that the solution $x(\cdot)$ of (1) satisfies $x(T) = x_1$.

To proceed, we need the following assumptions:

**(A1)** $S(\cdot)$ is a $C_0$—semigroup generated by the operator $A$ on $X$ which satisfies $M = \max_{t \in J} \|S(t)\|$.
There exist positive continuous real-valued functions \( f_1, k_1, \) and \( h_1 \) defined respectively on \( J, D, \) and \( D_0, \) such that
\[
\begin{align*}
\| f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2) \| &\leq f_1(t)(\| x_1 - x_2 \| + \| y_1 - y_2 \| + \| z_1 - z_2 \|) \\
\| k(t, s, x_1) - k(t, s, x_2) \| &\leq k_1(t, s) \| x_1 - x_2 \| \\
\| h(t, s, x_1) - h(t, s, x_2) \| &\leq h_1(t, s) \| x_1 - x_2 \|
\end{align*}
\]
for all \( x_1, y_1, z_1, x_2, y_2, z_2 \in Y. \) Moreover, assume that \( k_2(t) = \sup_{s \in J} \{ k_1(t, s), \| k(t, s, 0) \| \} \) and \( h_2(t) = \sup_{s \in J} \{ h_1(t, s), \| h(t, s, 0) \| \} \) for any \( t \in J. \)

The real-valued functions \( g(t) = f_1(t)(1 + (t - t_0)k_2(t) + (T - t_0)h_2(t)) \) and \( b(t) = (t - t_0)f_1(t)k_2(t) + (T - t_0)f_1(t)h_2(t) + \| f(t, 0, 0) \| \) are fractional integrable of order \( \alpha \) for any \( t \in J. \) Moreover, The linear operators \( F, \) and \( G \) are defined on \( I^\alpha(C_\mu) \) such that
\[
\begin{align*}
F(g) &= \frac{1}{\Gamma(\alpha)} \int_{t_0}^{T} (T - s)^{\alpha - 1} g(s) ds \\
G(b) &= \frac{1}{\Gamma(\alpha)} \int_{t_0}^{T} (T - s)^{\alpha - 1} b(s) ds
\end{align*}
\]
are also fractional integrable of order \( \alpha. \)

The linear operator \( W : L_2(J, U) \to X \) defined by
\[
W u = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{T} (T - s)^{\alpha - 1} S(T - s) Bu(s) ds
\]
induces an invertible operator \( \hat{W} \) defined on \( L_2(J, U)/\ker W, \) and there exists a positive constant \( K \) such that \( \| B\hat{W}^{-1} \| \leq K. \)

**Remark 2.6** In view of (A2), one can get the following estimates
\[
\begin{align*}
\| K x(t) \| &\leq (t - t_0)k_2(t)(\| x \| + 1) \\
\| H x(t) \| &\leq (T - t_0)h_2(t)(\| x \| + 1)
\end{align*}
\]
and
\[
\begin{align*}
\| K x_1(t) - K x_2(t) \| &\leq (t - t_0)k_2(t) \| x_1 - x_2 \| \\
\| H x_1(t) - H x_2(t) \| &\leq (T - t_0)h_2(t) \| x_1 - x_2 \|
\end{align*}
\]
for any \( t \in J. \)

In this section, we prove the main results for controllability of the system (1).
Theorem 2.7 If the hypotheses (A1)-(A4) are satisfied, and if
\[ q^{-1}M^\alpha [MKF(g(t) + g(t))] < 1, \quad 0 < q < 1, \quad \text{for any } t \in J, \]
then the fractional integro-differential system (1) is controllable on J.

Proof. Let \( B_r = \{ x \in Y : ||x|| \leq r \} \) be a closed subspace of Y, where r is any fixed finite real number which satisfies the inequality
\[ (1 - q)r \geq \left( M ||x_0|| + \frac{MK}{\Gamma(\alpha + 1)} (||x_1|| + M ||x_0||) (T - t_0)^\alpha + MI^\alpha (MKG(b)(t) + b(t)) \right). \]

Define the control \( u \in L_2(J, U) \) by
\[
u(t) = \tilde{W}^{-1} \left[ x_1 - S(T - t_0)x_0 - \frac{1}{\Gamma(\alpha)} \int_{t_0}^{T} (T - s)^{\alpha-1} S(T - s) f(s, x(s), Kx(s), Hx(s)) ds \right] (t). \]

We shall show that when using the above control, the operator \( \Psi : Y \rightarrow Y \) defined by
\[
\Psi x(t) = T(t - t_0)x_0 + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - s)^{\alpha-1} S(t - s) B\tilde{W}^{-1} \left[ x_1 - S(T - t_0)x_0 
- \frac{1}{\Gamma(\alpha)} \int_{t_0}^{T} (T - r)^{\alpha-1} S(T - r) f(r, x(r), Kx(r), Hx(r)) dr \right] (s) ds 
+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - s)^{\alpha-1} S(t - s) f(s, x(s), Kx(s), Hx(s)) ds.
\]
has a Banach fixed point which is a solution of the system (1). It is clear that if \( \Psi \) has a fixed point \( x(\cdot) \), then \( \Psi x(T) = x_1 \). Hence the system (1) is controllable on J. It remains to prove the existence of such fixed point. Firstly, we show that the operator \( \Psi \) maps \( B_r \) into itself. For this, by using assumptions (A1), (A2), and triangle inequality, we have
\[
||\Psi x(t)|| \leq M ||x_0|| + \frac{1}{\Gamma(\alpha)} \left| \left| \int_{t_0}^{t} (t - s)^{\alpha-1} S(t - s) B\tilde{W}^{-1} \left[ x_1 - S(T - t_0)x_0 
- \frac{1}{\Gamma(\alpha)} \int_{t_0}^{T} (T - r)^{\alpha-1} S(T - r) f(r, x(r), Kx(r), Hx(r)) dr \right] (s) ds 
+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - s)^{\alpha-1} S(t - s) f(s, x(s), Kx(s), Hx(s)) ds \right| \right|.
\]
therefore
\[
\Psi(x(t)) \leq M \|x_0\| + \frac{MK}{\Gamma(\alpha + 1)} (\|x_1\| + M \|x_0\|) (t - t_0)^\alpha
\]
\[+ \frac{M^2 K}{(\Gamma(\alpha))^2} \int_{t_0}^{t} (t - s)^{\alpha - 1} \left( \int_{t_0}^{T} (T - r)^{\alpha - 1} f_1(r)(\|x(r)\| + \|Kx(r)\| + \|Hx(r)\|) dr \right) (s) ds
\]
\[+ \|f(r, 0, 0, 0)\| dr) (s) ds + \frac{M}{\Gamma(\alpha)} \int_{t_0}^{t} (t - s)^{\alpha - 1} \times
\]
\[f_1(s)(\|x(s)\| + \|Kx(s)\| + \|Hx(s)\| + \|f(s, 0, 0, 0)\|) ds,
\]
by Remark (2.6), we have
\[
\|\Psi(x(t))\| \leq M \|x_0\| + \frac{MK}{\Gamma(\alpha + 1)} (\|x_1\| + M \|x_0\|) (t - t_0)^\alpha
\]
\[+ \frac{M^2 K}{(\Gamma(\alpha))^2} \int_{t_0}^{t} (t - s)^{\alpha - 1} \left( \int_{t_0}^{T} (T - r)^{\alpha - 1} f_1(r)(\|x(r)\| + \|Kx(r)\| + \|Hx(r)\|) dr \right) (s) ds
\]
\[+ \|f(r, 0, 0, 0)\| dr) (s) ds + \frac{M}{\Gamma(\alpha)} \int_{t_0}^{t} (t - s)^{\alpha - 1} \times
\]
\[f_1(s)(\|x(s)\| + \|Kx(s)\| + \|Hx(s)\| + \|f(s, 0, 0, 0)\|) ds
\]
therefore
\[
\|\Psi(x(t))\| \leq M \|x_0\| + \frac{MK}{\Gamma(\alpha + 1)} (\|x_1\| + M \|x_0\|) (t - t_0)^\alpha
\]
\[+ \frac{M^2 K}{(\Gamma(\alpha))^2} \int_{t_0}^{t} (t - s)^{\alpha - 1} \left( \int_{t_0}^{T} (T - r)^{\alpha - 1} f_1(r)(1 + (r - t_0)k_2(r) + (T - t_0)h_2(r)) \|x(r)\| dr \right) (s) ds
\]
\[+ \frac{M^2 K}{(\Gamma(\alpha))^2} \int_{t_0}^{t} (t - s)^{\alpha - 1} \times
\]
\[\left( \int_{t_0}^{T} (T - r)^{\alpha - 1} (f_1(r)k_2(r)(r - t_0) + f_1(r)h_2(r)(T - t_0)) + \|f(r, 0, 0, 0)\| dr \right) (s) ds
\]
using (A2) we have 

\[ + \frac{M}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f_1(s)(1 + (s-t_0)k_2(s) + (T-t_0)h_2(s)) \|x(s)\| \, ds \]

\[ + \frac{M}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} (f_1(s)k_2(s)(s-t_0) + f_1(s)h_2(s)(T-t_0) + \|f(s,0,0,0)\|) \, ds \]

using the hypothesis (A3), we get

\[ \|\Psi x(t)\| \leq M \|x_0\| + \frac{MK}{\Gamma(\alpha+1)} (\|x_1\| + M \|x_0\|) (T-t_0)^{\alpha} \]

\[ + \frac{M^2K}{\Gamma(\alpha)} \|x\| \int_{t_0}^{t} (t-s)^{\alpha-1} F(g)(s)ds + \frac{M^2K}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} G(b)(s)ds \]

\[ + \frac{M}{\Gamma(\alpha)} \|x\| \int_{t_0}^{t} (t-s)^{\alpha-1} g(s)ds + \frac{M}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} b(s)ds \]

\[ = M \|x_0\| + \frac{MK}{\Gamma(\alpha+1)} (\|x_1\| + M \|x_0\|) (T-t_0)^{\alpha} \]

\[ + MI^\alpha (MKG(b)(t) + b(t)) + MI^\alpha (MKF(g)(t) + g(t)) \|x\|. \]

Hence, if \(x \in B_r\), then \(\|\Psi x(t)\| \leq (1-q)r + qr = r\), which proves that \(\Psi(B_r) \subset B_r\). Next, we prove that \(\Psi\) is a contraction mapping. For this, pick any \(x_1, x_2 \in B_r\), we have

\[ \|\Psi x_1(t) - \Psi x_2(t)\| \]

\[ = \left\| \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} S(t-s)BW^{-1} \left( \frac{1}{\Gamma(\alpha)} \int_{t_0}^{T} (T-r)^{\alpha-1} S(T-r) \times \right. \]

\[ [f(r, x_2(r), Kx_2(r), Hx_2(r)) - f(r, x_1(r), Kx_1(r), Hx_1(r))] \, dr \right) \, ds \]

\[ + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} S(t-s) [f(s, x_1(s), Kx_1(s), Hx_1(s)) - f(s, x_2(s), Kx_2(s), Hx_2(s))] \, ds \]

\[ \leq \frac{M^2K}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \left( \frac{1}{\Gamma(\alpha)} \int_{t_0}^{T} (T-r)^{\alpha-1} \times \right. \]

\[ \|f(r, x_2(r), Kx_2(r), Hx_2(r)) - f(r, x_1(r), Kx_1(r), Hx_1(r))\| \, dr \right) \, ds \]

\[ + \frac{M}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \|f(s, x_1(s), Kx_1(s), Hx_1(s)) - f(s, x_2(s), Kx_2(s), Hx_2(s))\| \, ds \]

using (A2) we have

\[ \|\Psi x_1(t) - \Psi x_2(t)\| \]
\[
\begin{align*}
&\leq \frac{M^2K}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \left( \frac{1}{\Gamma(\alpha)} \int_{t_0}^{T} (T-r)^{\alpha-1} f_1(r) \times \\
&\quad \left[ \|x_2(r) - x_1(r)\| + \|Kx_2(r) - Kx_1(r)\| + \|Hx_2(r) - Hx_1(r)\| \right] dr \right) (s)ds \\
&\quad + \frac{M}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f_1(s) \times \\
&\quad \left[ \|x_2(s) - x_1(s)\| + \|Kx_2(s) - Kx_1(s)\| + \|Hx_2(s) - Hx_1(s)\| \right] ds \\
&\leq \frac{M^2K}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \left( \frac{1}{\Gamma(\alpha)} \int_{t_0}^{T} (T-r)^{\alpha-1} f_1(r) \times \\
&\quad \left[ \|x_2(r) - x_1(r)\| + \int_{t_0}^{r} k_1(r, \theta) \|x_1(\theta) - x_2(\theta)\| d\theta + \int_{t_0}^{T} h_1(r, \theta) \|x_1(\theta) - x_2(\theta)\| d\theta \right] dr \right) (s)ds \\
&\quad + \frac{M}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f_1(s) \times \\
&\quad \left[ \|x_2(s) - x_1(s)\| + \int_{t_0}^{s} k_1(s, r) \|x_1(r) - x_2(r)\| dr + \int_{t_0}^{s} h_1(s, r) \|x_1(r) - x_2(r)\| dr \right] ds \\
\end{align*}
\]

and by (A3) we have

\[
\|\Psi x_1(t) - \Psi x_2(t)\| \\
\leq \frac{M^2K}{\Gamma(\alpha)} \|x_2 - x_1\| \times \\
\int_{t_0}^{t} (t-s)^{\alpha-1} \left( \frac{1}{\Gamma(\alpha)} \int_{t_0}^{T} (T-r)^{\alpha-1} f_1(r) \times \\
&\quad \left[ 1 + \int_{t_0}^{r} k_1(r, \theta)d\theta + \int_{t_0}^{T} h_1(r, \theta)d\theta \right] dr \right) (s)ds \\
&\quad + \frac{M}{\Gamma(\alpha)} \|x_2 - x_1\| \int_{t_0}^{t} (t-s)^{\alpha-1} f_1(s) \left[ 1 + \int_{t_0}^{s} k_1(s, r)dr + \int_{t_0}^{T} h_1(s, r)dr \right] ds \\
\leq \frac{M^2K}{\Gamma(\alpha)} \|x_2 - x_1\| \times \\
\int_{t_0}^{t} (t-s)^{\alpha-1} \left( \frac{1}{\Gamma(\alpha)} \int_{t_0}^{T} (T-r)^{\alpha-1} f_1(r) \left[ 1 + (r-t_0)k_2(r) + (T-t_0)h_2(r) \right] dr \right) (s)ds \\
&\quad + \frac{M}{\Gamma(\alpha)} \|x_2 - x_1\| \int_{t_0}^{t} (t-s)^{\alpha-1} f_1(s) \left[ 1 + (s-t_0)k_2(s) + (T-t_0)h_2(s) \right] ds
\]
\[
\leq \frac{M^2 K}{\Gamma(\alpha)} \|x_2 - x_1\| \int_{t_0}^{t} (t - s)^{\alpha - 1} F(g)(s)ds + \frac{M}{\Gamma(\alpha)} \|x_2 - x_1\| \int_{t_0}^{t} (t - s)^{\alpha - 1} g(s)ds
\]
\[
= MI^\alpha (g(t) + MKF(g)(t))\|x_2 - x_1\| \leq q \|x_2 - x_1\|.
\]

Hence, \(\Psi\) has a unique fixed point \(x = \Psi(x) \in B_r\), which is a solution of (2), and hence is a mild solution of (1).

### 3 Nonlocal problem

We discuss in this section, the controllability problem of the fractional integrodifferential system (1) with a nonlocal condition of the form

\[x(t_0) + g(x) = x_0,\]  \hspace{1cm} (4)

where \(g : Y \to Y\) is a given function that satisfies the following condition

\[(A5)\ g\ is\ a\ continuous\ function\ and\ there\ exists\ a\ positive\ constant\ G\ such\ that\]

\[\|g(x) - g(y)\| \leq G\|x - y\|, \text{ for } x, y \in Y.\]

**Theorem 3.1** If the hypotheses (A1)-(A5) are satisfied, and if

\[q^{-1} M \left( I^\alpha [MKF(g)(t) + g(t)] + G \left( 1 + \frac{MK}{\Gamma(\alpha + 1)} (T - t_0)^\alpha \right) \right) < 1, \ 0 < q < 1,
\]

then the fractional integrodifferential system (1) with nonlocal condition (4) is controllable on \(J\).

**Proof.** Let \(B_r = \{x \in Y : \|x\| \leq r\}\) be a closed subspace of \(Y\), where \(r\) is any fixed finite real number which satisfies the inequality

\[r \geq (1 - q) \left( -1 M (\|x_0\| + \|g(0)\|) + \frac{MK}{\Gamma(\alpha + 1)} (\|x_1\| + M \|x_0\| + M \|g(0)\|) (T - t_0)^\alpha \\
+ MI^\alpha (MKG(b)(t) + b(t)) \right).\]

Define the control \(u \in L_2(J, U)\) by

\[u(t) = \tilde{W}^{-1} [x_1 - S(T - t_0) (x_0 - g(x)) - \frac{1}{\Gamma(\alpha)} \int_{t_0}^{T} (T - s)^{\alpha - 1} S(T - s)f(s, x(s), Kx(s), Hx(s))ds] (t).\]
Applying the same technique as in proof of Theorem 2.7, on the operator \( \Phi : Y \rightarrow Y \) defined by

\[
\Phi x(t) = T(t - t_0) (x_0 - g(x)) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{T} (t - s)^{\alpha-1} S(t - s) B \tilde{W}^{-1} \left[ x_1 - S(T - t_0) (x_0 - g(x)) \right] ds
\]

\[
- \frac{1}{\Gamma(\alpha)} \int_{t_0}^{T} (T - r)^{\alpha-1} S(T - r) f(r, x(r), Kx(r), Hx(r)) dr \right] (s) ds
\]

\[
+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{T} (T - s)^{\alpha-1} S(t - s) f(s, x(s), Kx(s), Hx(s)) ds.
\]

to show that, this operator has a fixed point \( x = \Phi(x) \in B_r \). This fixed point is then a solution of the system (1) and (4). On the other hand, by using this fixed point of the operator \( \Phi \), it is clear that \( \Phi x(T) = x_1 \). Hence the system (1-4) is controllable on \( J \). The rest of the details of the proof is similar to Theorem 2.7 and hence it is omitted.

### 4 Example

Consider the following nonlinear partial integrodifferential system

\[
\begin{cases}
\frac{\partial^\alpha}{\partial t^\alpha} \omega(t, y) = \frac{\partial^2}{\partial y^2} \omega(t, y) + \mu(t, y) + f(t, \omega(t, y), \int_{t_0}^{t} k(t, s, \omega(s, y)) ds, \int_{t_0}^{T} h(t, s, \omega(s, y)) ds), \\
\omega(t, 0) = \omega(t, \pi) = 0, \quad t \in J = [0, \pi], \\
\omega(0, y) + \sum_{k=1}^{m} c_k \phi(t_k, y) = \omega_0(y), \quad \phi \in B_r, \quad 0 \leq y \leq \pi,
\end{cases}
\]

where \( 0 < \alpha < 1, c_k > 0 \), and \( \mu : J \times [0, \pi] \rightarrow [0, \pi] \) is continuous. Let \( X = U = L^2[0, \pi], Y = C(J, L^2[0, \pi]), B_r = \{ x \in Y : \| x \| \leq r \} \) for some \( r \). Put \( x(t) = \omega(t, \cdot), (Bu)(t)(\cdot) = \mu(t, \cdot), \) and \( g(\phi(t, \cdot)) = \sum_{k=1}^{m} c_k \phi(t_k, y) \). Let \( A : D(A) \subset X \rightarrow X \) defined by \( Ax = x'' \), where

\[
D(A) = \left\{ x \in X : x, \ x' \text{ absolutely continuous, } x'' \in X, \ x(0) = x(\pi) = 0 \right\}.
\]

Therefore

\[
Ax = \sum_{n=1}^{\infty} n^2(x, x_n)x_n, \quad x \in D(A),
\]
where \( x_n(s) = \sqrt{\frac{2}{\pi}} \sin(ns), \ n = 1, 2, 3, \ldots \) is the orthogonal set of eigenfunctions of \( A \). It can be easily shown that \( A \) is the infinitesimal generator of an analytic semigroup \( S(t), \ t \geq 0 \) in \( X \) and is given by

\[
S(t)x = \sum_{n=1}^{\infty} \exp(-n^2t)(x,x_n)x_n, \ x \in X,
\]

and \( S(t) \) satisfies (A1). With the above choices, we see that the system (5) is the abstract formulation of (1) and (4). Assume that the operator \( W \) defined by

\[
(W\mu)(y) = \frac{1}{\Gamma(\alpha)} \sum_{n=1}^{\infty} \int_{0}^{T} (T-s)^{\alpha-1} \exp(-n^2(T-s))(\mu(t,y),x_n)x_n ds
\]

has a bounded invertible operator \( \tilde{W}^{-1} \) in \( L_2(J,U)/\ker W \). Moreover, if the functions \( f, k, \) and \( h \) satisfy the hypotheses (A2) and (A3), then we deduce that the system (5) is controllable on \( J \).

References


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