Regional Boundary Observability for Semi-Linear Systems Approach and Simulation

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Abstract

The paper aims at extending the notion of regional boundary observability developed for linear systems to the semilinear parabolic case. We give definitions and some properties of this notion and we show that under some hypothesis, the regional boundary observability guaranteed. We describe two approaches where the first is based on HUM method combined with fixed point techniques and the second uses sectorial property. The obtained results are illustrated through numerical examples and simulations.

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1 Introduction

The theory of the distributed parameter systems is sufficiently developed to be able to deal with many questions arise in real problems, examples of a thermal nature are furnaces for heating metal slabs or heat exchangers. Many problems can be reformulated as problem of analysis for infinite-dimensional systems. The existing results of control theory of linear differential parameters system (DPS) is classical and of rather limited use in the nonlinear area. New semi-linear and nonlinear methodology for control stabilization, observability and optimization needs to be developed for such systems. During the past thirty years, several methods in theory for nonlinear partial differential equations (PDEs) have been made especially for semilinear ones. For an extensive list of publications, see (Khapalov, [7]) and (Klamka, [8]), where the considered assumption on the nonlinear term and the used method to solve constrained exact controllability, cover a wide class of semilinear systems. For a controllability problem when one is faced with the question of steering a system from an initial state to a prescribed one, it is important to take into account the effects of nonlinearity. For example, see (Fabre et al, [2]) prove approximate controllability in $L^p(\Omega)$ for $1 \leq p < +\infty$ by means of a control which can be internal or on the boundary and when the right term in the considered partial differential equation is expressed by a nonlinear operator $\mathcal{N}$ assumed to be globally Lipschitz. That is achieved for some $\sigma > 0$ and $\beta > 0$ such that

$$||\mathcal{N}(s)|| \leq \sigma |s| + \beta$$

Moreover, in the case of the interior control, they prove approximate controllability in $C_0(\Omega)$. The applied technique combines a variational approach to the controllability problem for a linear equation and fixed point methods.

Null-controllability was also proved in (Fernàndez- Cara, [3]) for semilinear distributed parabolic systems when the nonlinear term $\mathcal{N}(s)$ grows slower than $s \log |s|$ as $|s| \to 0$, or

$$||\mathcal{N}(s)|| \leq \epsilon |s| \log |s| \quad \text{for large } |s|,$$

where $\epsilon$ depends on the system domain, the final time and the geometric support of the control. Fixed point theorems and Gronwall’s inequality remain important tools used by mathematicians to solve the various questions, particularly the controllability problem for semilinear systems, see (Zuazua et al.,[19]; Kassara et al, [5], [6]) and the references therein.

For example the exact controllability of abstract semilinear equations was considered by (I. Laseika and R. Triggiani, [9]), where the applications involve boundary control problems for the wave and plate equations. The stabilization problem for a semilinear parabolic system was also considered by (Y. Yan D.
Coca and V. Barbu, [14]) using finite-dimensional feedback controllers with support in an arbitrary open subset which are active in one equation only and under some assumption on the linearized system they explore an optimal design methodology based on the finite element approximation of the semilinear parabolic system which is illustrated through numerical simulations. Recently regional controllability for a clase of distributed semilinear systems has been introduced by (Zerrik et al. [17], [18]). They prove regional controllability when the system is asymptotically linear, (i.e) for some $\alpha > 0$, the nonlinear term $N(s)$ satisfies

$$\lim_{|s| \to +\infty} \frac{N(s)}{s} = \alpha \quad \text{and} \quad N' \in L^\infty(\mathbb{R})$$

the analytical case is then considered using generalized inverse techniques. In the two cases the control achieving a regional target is characterized via fixed point theorems and depends on the final time $T$.

We note that the semilinear systems have occupied an important place in control and systems theories. The analysis of distributed parameter for semilinear systems is related to a set of concept such as controllability and observability which is one of the most important notion in systems analysis. Many work have been devoted to the observation problem in the whole domain (see [1]), but some delicate ones need to be studied only in some subregion of the system evolution domain. This is the subject of regional controllability and observability which was pioneered by A. El Jai and his co-workers (see [15]) especially for linear systems and studied the possibility to reconstruct a state only on a subregion $\omega$ of the system evolution domain $\Omega$. This notion was extended to various kind of linear system which constituted an extensive literature on exact and approximate regional controllability and observability but very little has been done for semilinear ones. The momentum of differential parameters system research is now visibly moving toward the study of regional controllability and observability of semilinear and nonlinear partial differential equation which take up a prominent place in control theory of distributed parameter systems.

Here we are interested on regional boundary observability of semilinear parabolic systems. More precisely the question concerns the possibility of regional observability for semilinear system in the case where the target subregion is located in the boundary of the system domain. There are many applications of these notions and the interesting one may be the problem of determination of laminar boundary flux conditions developed in steadystate by a vertical heated plate and consists of the study of thermal transfer by natural convection generated by a uniformly heated plate located in a small enclosure. Inside that enclosure, differences in wall surface produce natural convection movements. The heat exchanger maintains a prescribed temperature on the
back face of the plate by means of hot water circulation. All the faces of this active wall are insulated except for the front face. The objective is to find the unknown boundary convective condition on a part of the front face of the active plate using measurements given by internal thermocouples, see (Aparron [13]) for more details. The aim is to carry out some methods for regional reconstruction of the initial state of semilinear parabolic systems in a part of the boundary of the system evolution domain and which can be exploitable from practical point of view. This is the subject of this paper which is organized as follows Section 2 is devoted to the presentation of the system under consideration and characterization of the approximate regional boundary observability for semilinear systems. Section 3 concerns a reconstruction approach using an extension of the Hilbert Uniqueness method (HUM). The obtained results are performed through numerical example and simulations. Section 4 we develop a new fruitful approach based on a sectorial property and combines the fixed point techniques leading to a second algorithm performed through numerical example and simulation.

2 Problem statement and preliminaries

Let $\Omega$ be an open bounded domain of $\mathbb{R}^n (n = 1, 2, 3)$ with sufficiently smooth boundary $\partial \Omega$. Let’s denote $Q = \Omega \times [0,T]$ and $\Sigma = \partial \Omega \times [0,T]$. We consider a semilinear parabolic system described by the evolution equation

$$
\begin{cases}
\frac{\partial y(x,t)}{\partial t} + Ay(x,t) = Ny(x,t) & \text{in } Q \\
y(x,0) = y_0(x) & \text{in } \Omega \\
\frac{\partial y(\xi,t)}{\partial \nu_A} = 0 & \text{on } \Sigma
\end{cases}
$$

(1)

where $y_0$ is the initial state assumed to be defined in the whole domain $\Omega$ and the function of measurement is given by the output function

$$z(t) = Cy(.,t)$$

(2)

where $A$ is a second order linear differential operator which indicates the adjoint operator of $A$ and $\frac{\partial y}{\partial \nu_A}$ denotes the co-normal derivative associated with $A$ generates a strongly continuous semigroup $(S(t))_{t \geq 0}$ on the state space space $L^2(\Omega)$, $N$ is a nonlinear operator assumed to be defined so that to ensure the existence and uniqueness of the solution of system (1) (see [4] and [12]) and $C$ is a linear and bounded operator defined from $L^2(\Omega)$ to the space $\mathbb{R}^q$ where $q$ depends on the structure of the considered sensors. System (1) can be inter-
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interpreted in the mild sense as follows:

\[
\begin{align*}
    \begin{cases}
        y(.,t) = S(t)y_0(.) + \int_0^t S(t - \tau)Ny(.,\tau)d\tau \\
        y(.,0) = y_0(.)
    \end{cases}
\end{align*}
\]  

(3)

For \( \Gamma \subset \partial \Omega \), regular enough, we introduce the following notation:

- \( \chi_\Gamma : H_{1/2}^+(\partial \Omega) \rightarrow H_{1/2}^+(\Gamma) \) the restriction operator to \( \Gamma \) and \( \chi_\Gamma^* \) denotes its adjoint.

- \( \gamma_0 : H^1(\Omega) \rightarrow H_{1/2}^+(\partial \Omega) \) the trace operator of order zero while \( \gamma_0^* \) denotes its adjoint.

- \( y_1^0 = \chi_\Gamma \gamma_0 y_0 \), the restriction of the initial state to \( \Gamma \), \( \bar{y}_0 \) is the residual part and let us \( \bar{y}_0^0 \) such that \( \bar{y}_0^0 = \chi_\Gamma \bar{y}_0^1 \) so that \( y_0 = R\bar{y}_0^1 + \bar{y}_0 \), where \( R : H_{1/2}^+(\partial \Omega) \rightarrow H^1(\Omega) \) satisfying \( \gamma_0 Rg = g, \forall g \in H_{1/2}^+(\partial \Omega) \).

- \( H_\Gamma = \chi_\Gamma \gamma_0 K^* \), where \( K \) is the observation operator defined from \( H^1(\Omega) \) to \( Y = L^2(0,T;\mathbb{R}^q) \) (the observation space) by \( (K\xi)(t) = CS(t)\xi \).

To system (1) we associate the linear one

\[
\begin{align*}
    \begin{cases}
        \frac{\partial y(x,t)}{\partial t} + A y(x,t) = 0 \quad \text{in} \quad Q \\
        y(x,0) = y_0(x) \quad \text{in} \quad \Omega \\
        \frac{\partial y(\xi,t)}{\partial \nu_A} = 0 \quad \text{on} \quad \Sigma
    \end{cases}
\end{align*}
\]

(4)

augmented with the output equation

\[ z(t) = C y(.,t) \]  

(5)

Without loss of generation, we note \( y(.,t) \) by \( y(t) \) and we recall the following definition

**Definition 2.1**

System (4) augmented with the output (5) is said to be regionally exactly (resp. approximately) boundary observable on \( \Gamma \) if \( \text{Im}(H_\Gamma) = H_{1/2}^+(\Gamma) \) (resp. \( \overline{\text{Im}}(H_\Gamma) = H_{1/2}^+(\Gamma) \)).

The notion of regional observability considered as a particular case of output observability was introduced and developed for linear systems (see [?]).
Remark It is clear that:

1. if the system (4) is regionally boundary observable on $\Gamma$, then it is regionally boundary observable on any $\Gamma_1 \subset \Gamma$.

2. in the linear case, one can find states which are regionally boundary observable on $\Gamma$ but not observable on the whole boundary domain $\partial \Omega$ (see [15]).

When a feedback control law has to be implemented one needs knowledge of the current state, the observability problem occurs when one has to reconstruct the state of the system. The problem is whether knowledge of the dynamics of system together with the output function, is sufficient to reconstruct the initial state $y(0)$.

Definition 2.2 System (1) augmented with the output (2) is said to be regionally observable on $\Gamma$ if it is possible to reconstruct the state $y(t)|_{\Gamma}$ at any time $t$ and the reconstructed state depends continuously on $z$.

We assume that the solution of (1) is in $X = L^2(0,T;H^1(\Omega))$ and we define, for $t \in ]0,T[$, the operator $L(.) : X \rightarrow X$ as follows

$$L(t)x = \int_0^t S(t-\tau)x(\tau)d\tau$$

Then the solution of (1) can be expressed by

$$y = S(.)y_0 + S(.)\bar{y}_0 + L(.)\bar{N}y$$

The problem of regional observability of the system (1) on $\Gamma$, can be stated as follows:

Problem Given system (1) and the output (2), is it possible to reconstruct $y_0^1$ which is the initial state of the system (1) on $\Gamma$?

3 HUM approach

The aim of this section is to give an extension of the Hilbert uniqueness method introduced in the linear case by Lions [10] which allows the determination of the regional boundary initial state on $\Gamma$. The approach uses the internal regional observability and leads to an algorithm which is performed through a numerical example.

The solvability of the problem (2.2) on the whole domain ($\omega = \Omega$) has been considered using fixed point theorem and several papers have been published.
(see [5]). Here the subject is to give a regional reconstruction method. Let’s decompose the initial state as follows

\[ y_0 = \begin{cases} y_0^1 & \text{in } \omega_r \\ y_0^2 & \text{in } \Omega \setminus \omega_r \end{cases} \]  

(6)

With \( r > 0 \) small enough, \( F_r = \bigcup_{z \in \Gamma} B(z,r) \) and \( \omega_r = F_r \cap \Omega \) with \( B(z,r) \) is the ball of radius \( r \) centered in \( z \), then we have the result

**Proposition 3.1** If the system (4) is regionally observable in \( \omega_r \), then it is boundary regionally observable on \( \Gamma \), and then the system (1) is boundary regionally observable on \( \Gamma \) as the restriction of the trace of the fixed point of

\[ y(t) = \phi(y(.)(t)) \]

at \( t = 0 \), where the function \( \phi : \mathcal{X} \rightarrow \mathcal{X} \) is defined by

\[ \phi(y(.)(t)) = S(t)\gamma_0^\ast \chi_\Gamma^\ast H_\Gamma^\dagger(z(.)) - K(.)\bar{y}_0 - CL(.)\mathcal{N}y(.) + S(t)\bar{y}_0 + L(t)\mathcal{N}y(.) \]

where \( H_\Gamma^\dagger = [H_\Gamma^\ast H_\Gamma]^{-1}H_\Gamma^\ast \) is the pseudo-inverse of \( H_\Gamma \) and \( \chi_\Gamma^\gamma_0\bar{y}_0 = 0 \).

**Proof.**

If we suppose that the system (4) is regionally observable in \( \omega_r \), then it is boundary regionally observable on \( \Gamma \), see for details ([17]).

Now, we show that the system (1) is boundary regionally observable on \( \Gamma \):

We consider the system (1) augmented with the output (2), we have

\[ CS(t)\gamma_0^\ast \chi_\Gamma^\ast y_0^1 = z - CS(t)\bar{y}_0 - CL(t)\mathcal{N}y(.) \]

Since the system (4) is approximately regionally observable on \( \Gamma \), the operator \( [H_\Gamma^\ast H_\Gamma]^{-1} \) exists and hence

\[ y_0^1 = H_\Gamma^\dagger(z - CS(t)\bar{y}_0 - CL(t)\mathcal{N}y(.)) \]

The problem is reduced to the problem of finding the restriction of the fixed point of \( \phi \).

In the sequel our subject is the reconstruction of the component \( y_0^1 \). We consider the system (1) supposed observed by an internal zone sensor \((D,f)\) with \( D \subset \Omega \) and \( f \in L^2(D) \) and let \( G \) be defined by

\[ G = \{ h \in H^1(\Omega) \mid h = 0 \text{ in } \Omega \setminus \omega_r \} \]
We consider the semilinear system
\[
\begin{align*}
\frac{\partial \varphi(x, t)}{\partial t} + A \varphi(x, t) &= N \varphi(x, t) \quad \text{in } Q \\
\varphi(x, 0) &= \varphi_0(x) \quad \text{in } \Omega \\
\frac{\partial \varphi(\xi, t)}{\partial \nu_A} &= 0 \quad \text{on } \Sigma
\end{align*}
\]
which can be decomposed as follows
\[
\begin{align*}
\frac{\partial \varphi_1(x, t)}{\partial t} &= -A \varphi_1(x, t) \quad \text{in } Q \\
\varphi_1(x, 0) &= \varphi_0(x) \quad \text{in } \Omega \\
\frac{\partial \varphi_1(\xi, t)}{\partial \nu_A} &= 0 \quad \text{on } \Sigma
\end{align*}
\]
and
\[
\begin{align*}
\frac{\partial \theta(x, t)}{\partial t} &= -A \theta(x, t) + N(\theta(x, t) + \varphi_1(x, t)) \quad \text{in } Q \\
\theta(x, 0) &= 0 \quad \text{in } \Omega \\
\frac{\partial \theta(\xi, t)}{\partial \nu_A} &= 0 \quad \text{on } \Sigma
\end{align*}
\]
System (8) has a unique solution \( \varphi \in L^2(0, T; H^1(\Omega)) \cap C^0(0, T; L^2(\Omega)) \) (see [11]) and the mapping
\[
\varphi_0 \in G \mapsto \|\varphi_0\|_G = \left[ \int_0^T \langle \varphi_1(t), f \rangle_{L^2(D)}^2 \right]^{\frac{1}{2}}
\]
induces a semi-norm on \( G \).
If the system (8) is approximately observable in \( \omega_r \), the semi-norm defines a norm on \( G \) (see [15]). We also denote by \( G \) the completion of \( G \).
We define the auxiliary system as
\[
\begin{align*}
\frac{\partial \tilde{\psi}(x, t)}{\partial t} &= A^* \tilde{\psi}(x, t) + N(\tilde{\psi}(x, t)) - \langle \varphi(t), f \rangle_{L^2(D)(\chi_D f)(x)} \quad \text{in } Q \\
\tilde{\psi}(x, T) &= 0 \quad \text{in } \Omega \\
\frac{\partial \tilde{\psi}(\xi, t)}{\partial \nu_{A^*}} &= 0 \quad \text{on } \Sigma
\end{align*}
\]
This allows to consider the application
\[
\mu : G \rightarrow G^* \\
\varphi_0 \mapsto P \tilde{\psi}(0)
\]
\( P \) is the projection on \( G^* \) and we decompose \( \tilde{\psi} \) as
\[
\tilde{\psi} = \psi_0 + \psi_1
\]
with $\psi_0$ and $\psi_1$ are solutions of the following systems

$$
\begin{cases}
\frac{\partial \psi_0(x,t)}{\partial t} = A^* \psi_0(x,t) - \langle \varphi_1(t), f \rangle_{L^2(D)}(x_D f)(x) \quad \text{in} \quad Q \\
\psi_0(x,T) = 0 \quad \text{in} \quad \Omega \\
\frac{\partial \psi_0(\xi,t)}{\partial \nu_{A^*}} = 0 \quad \text{on} \quad \Sigma
\end{cases}
$$

which allows to consider the operator defined by

$$
\mu(\varphi_0) = P\psi_0(0) + P\psi_1(0)
$$

we can define an operator $\Lambda$ from $G$ to $G^*$ as follows

$$
\Lambda \varphi_0 = P\psi_0(0)
$$

Then we have

$$
\mu(\varphi_0) = \Lambda \varphi_0 + K \varphi_0
$$

While $K$ is a nonlinear operator given by

$$
K : G \longrightarrow G^* \\
\varphi_0 \longrightarrow P\psi_1(0)
$$

It is assumed that the system (4) is regionally observable on $\Gamma$ then $\Lambda$ is invertible and finally we obtain

$$
\varphi_0 = \Lambda^{-1}P\tilde{\psi}(0) - \Lambda^{-1}K \varphi_0
$$

Considering the system

$$
\begin{cases}
\frac{\partial \tilde{\psi}(x,t)}{\partial t} = A^* \tilde{\psi}(x,t) + N(\tilde{\psi}(x,t)) - z(t)(x_D f)(x) \quad \text{in} \quad Q \\
\tilde{\psi}(x,T) = 0 \quad \text{in} \quad \Omega \\
\frac{\partial \tilde{\psi}(\xi,t)}{\partial \nu_{A^*}} = 0 \quad \text{on} \quad \Sigma
\end{cases}
$$

If $\varphi_0$ is chosen such that $\tilde{\psi}(0) = \bar{\psi}(0)$ in $\omega_r$ then the system (13) can be seen as the adjoint of the system (1) and our problem of observability turn up to solve the equation

$$
\varphi_0 = \Theta(\varphi_0)
$$
where $\Theta(\varphi_0)$ is solution of the equation

$$\Lambda (\Theta(\varphi_0)) = P\tilde{\psi}(0) - K(\varphi_0) \quad (15)$$

then we have the following result

**Proposition 3.2** If the system (4) is regionally exactly observable in $\omega_r$ and if there exists $c > 0$ such that $||N(x)|| \leq c||x||$ then the equation (15) admits a unique fixed point.

**Proof.** We show that under some hypothesis that the operator $\Theta$ has a unique fixed point which corresponds to the initial state to be observed in $\omega_r$. We show that

$$\Lambda (\Theta(\varphi_0)) = P\tilde{\psi}(0) - K(\varphi_0)$$

has a fixed point considering two steps:

**Step 1 :** Let’s consider $p > 0$, and consider the closed ball $B_p = B(0, p)$, we have

$$K(B_p) = \{P\psi_1(T) \mid \varphi_0 \in B_p\}$$

Let’s

$$\tilde{B}_p = \{P\psi_1(t) \mid \varphi_0 \in B_p, \ t \in [0, T]\}$$

we note that $K(B_p) \subset \tilde{B}_p$ then it is sufficient to show that $\tilde{B}_p$ is relatively compact.

We have $\psi_1(.)$ is a solution of (12) then

$$\psi_1(t) = \int_T^t S(t-\tau) [N(\psi_0(\tau) + \psi_1(\tau)) - \langle \theta(\tau), f \rangle (\chi_D f)(.)] d\tau$$

$\psi_1(.) \in C(0, T; L^2(\Omega))$ then

$$\exists c_1 > 0 \mid ||P\psi_1(t)||_{C^*} \leq c_1 ||\psi_1(t)||$$

Since $S(.)$ is a strongly continuous semigroup in $[0, T]$, then

$$\exists M > 0 \mid ||S(t)|| \leq M, \ \forall t \in [0, T]$$

According to the expression of $\psi_1(t)$ we have

$$||\psi_1(t)|| \leq M \int_T^t \left[ c(\|\psi_0(\tau)\| + \|\psi_1(\tau)\|) + \|\theta(\tau)\| \|f\| \right] d\tau$$

Since $\psi_0(.)$ is a solution of (11) then

$$||\psi_0(t)|| \leq M ||f||^2 \int_T^t \|\varphi_1(\tau)\| d\tau$$
So \( \varphi_1(.) \) is a solution of (8) gives
\[
\| \psi_0(t) \| \leq TM^2 \| f \|^2 \| \varphi_0 \|
\]
and then
\[
\int_t^T \| \psi_0(\tau) \| d\tau \leq T^2 M^2 \| f \|^2 \| \varphi_0 \|
\]
On the other hand we have \( \theta(.) \) is a solution of (9) then
\[
\theta(t) = \int_0^t S(t-\tau) \left[ N(\theta(\tau) + \varphi_1(\tau)) \right] d\tau
\]
and \( \| \theta(t) \| \leq tcM^2 \| \varphi_0 \| + Mc \int_0^t \| \theta(\tau) \| d\tau \)

Since \( t \mapsto tcM^2 \| \varphi_0 \| \) is nondecreasing, using the Gronwall theorem we obtain
\[
\| \theta(t) \| \leq TM^2 c \exp(McT) \| \varphi_0 \|
\]
and then
\[
\int_t^T \| \theta(\tau) \| d\tau \leq T^2 M^2 c \exp(McT) \| \varphi_0 \|
\]
Finally we obtain
\[
\| \psi_1(t) \| \leq (M^4 c^2 T^2 \| f \|^2 [1 + \exp(McT)]) \| \varphi_0 \| + Mc \int_t^T \| \psi_1(\tau) \| d\tau
\]
By Gronwall theorem
\[
\| \psi_1(t) \| \leq (M^4 c^2 T^3 \| f \|^2 [1 + \exp(McT)]) \| \varphi_0 \|
\]
and then \( \tilde{B}_P \) is uniformly bounded.

Now let’s show that \( \tilde{B}_P \) is equicontinu, indeed, we have
\[
\psi_1(t_2) - \psi_1(t_1) = B_1 + B_2
\]
Where
\[
B_1 = \int_{t_1}^{t_2} [S(t_2 - \tau) - S(t_1 - \tau)] \left[ N(\psi_0(\tau) + \psi_1(\tau)) \right] d\tau
\]
\[
B_2 = \int_{t_1}^{t_2} S(t_1 - \tau) \left[ N(\psi_0(\tau) + \psi_1(\tau)) - \langle \theta(\tau), f \rangle(\chi_D f) \right] d\tau
\]
but we have
\[
(\forall \varepsilon_1 > 0)(\exists \alpha > 0) \quad |t_2 - t_1| < \alpha \implies \| S(t_2 - \tau) - S(t_1 - \tau) \| \leq \varepsilon_1 (\forall \tau \in [0, T])
\]
which gives
\[
\| B_1 \| \leq \varepsilon_1 c \left[ T^2 M^2 \| f \|^2 + T^4 M^4 c^2 \| f \|^2 (1 + \exp(MTc)) \right] \| \varphi_0 \|
\]
and

\[ \| \mathbf{B}_2 \| \leq \alpha \left[ cTM^3 \| f \|^2 + c^3M^5T^3 \| f \|^2(1 + \exp(Mc)) + TM^3c \exp(McT) \right] \| \varphi_0 \| \]

Then

\[ \| P\psi_1(t_2) - P\psi_1(t_1) \|_{G^*} \leq \| \psi_1(t_2) - \psi_1(t_1) \| \leq \varepsilon_1A_1 + \alpha A_2 \]

with

\[
\begin{cases}
A_1 = [cT^2M^2 \| f \|^2 + T^4M^4c^3 \| f \|^2(1 + \exp(Mc))] \| \varphi_0 \| \\
A_2 = [cTM^3 \| f \|^2 + c^3M^5T^3 \| f \|^2(1 + \exp(Mc)) + TM^3c \exp(McT)] \| \varphi_0 \| 
\end{cases}
\]

For \( \varepsilon \leq \frac{\varepsilon}{2A_1} \) and \( \alpha \leq \frac{\varepsilon}{2A_2} \), we obtain

\[ \| P\psi_1(t_2) - P\psi_1(t_1) \|_{G^*} \leq \varepsilon \]

Consequently \( \Theta : B_p \rightarrow G^* \) is compact.

**Step 2:** Using the Schauder theorem, we show that \( \Theta \) enforces the ball in its self.

Since the linear system (8) is exactly regionally observable in \( \omega_r \), then \( \Lambda^{-1}P \) is bounded and then we have

\[ \| \Theta(\varphi_0) \| \leq \| \Lambda^{-1}P\| (\| \bar{\psi}(0) \| + \| \psi_1(0) \|) \]

which show that \( \Theta \) admits a unique fixed point and the initial state to be observed on \( \Gamma \) is given by \( y_0^* = \chi_{\gamma} \gamma_0 \varphi_0 \).

**Algorithm:** With the same hypothesis as in the last section, we have the following algorithm

**Step 1:**
- The initial state \( y_0 \), the subregion \( \omega_r \), the location of the sensor \( D \) and the function \( f \).
- Threshold accuracy \( \varepsilon \).

**Step 2:**

\[
\text{Repeat}
\]

- Resolution of (8) and obtaining \( \varphi_1 \).
- Resolution of (9) and obtaining \( \theta \).
- Resolution of (11) and obtaining \( \psi_0 \).
- Resolution of (12) and obtaining \( \psi_1 \).
- Resolution of (15) and obtaining \( \Theta(\varphi_0) \).
- Resolution of \( \varphi_0 = \Theta(\varphi_0) \) and obtaining \( \varphi_0 \).

\text{Until} \quad \| \varphi_0 - y_0 \|_{L^2(\omega_r)} \leq \varepsilon.

**Step 3:** The solution \( \varphi_0 \) corresponds to regional initial state to be observed in the subregion \( \omega_r \).
4 Simulations results

In this section, we present a numerical example which illustrate the previous algorithm. The obtained results are related to the considered subregion and the location of the sensor.

Let’s consider the system described in $\Omega = ]0, 1[ \times ]0, 1[$ by the following equation:

$$\begin{align*}
\frac{\partial y}{\partial t}(x, t) &= 0.01 \sum_{i=1}^{2} \frac{\partial^2 y}{\partial x_i^2}(x, t) + \sum_{k,l=0}^{\infty} \langle y(t), \varphi_{kl} \rangle \langle \varphi(t), \varphi_{kl} \rangle \varphi_{kl}(x) \quad \text{in} \quad \Omega \times ]0, T[ \\
y(x, 0) &= y_0(x) \quad \text{in} \quad \Omega \\
\frac{\partial y(\xi, t)}{\partial \nu} &= 0 \quad \text{on} \quad \partial \Omega \times ]0, T[ \\
\end{align*}$$

(17)

where $x = (x_1, x_2)$ and $(\varphi_{kl})_{kl}$ is a complete set of $H^1(\Omega)$.

The system (17) is augmented with the output function described by a pointwise sensor located in $(b_1, b_2)$ where $b_1 = 0.11, b_2 = 0.27$ and $T = 2$

$$z(t) = y(b_1, b_2, t), \quad t \in ]0, T[. \quad \text{(18)}$$

Let’s consider $\omega_r = ]0, 0.18[ \times ]0, 1[$ the subregion target and

$$y_0(x_1, x_2) = 2 \left( \frac{x_1^3}{3} + \frac{x_2^2}{2} + \alpha \right) \left( \frac{x_3^3}{3} + \frac{x_2^2}{2} + \beta \right)$$

be the initial state to be observed on $\Gamma = \{0\} \times [0, 1]$, with $\alpha$ and $\beta$ are chosen for numerical considerations. Using the previous algorithm, we obtain the following results:

Figure1: The desired initial state in $\omega_r$. Figure2: The estimated initial state in $\omega_r$. 
The initial state is obtained with reconstruction error $||y_0 - y_{re}||^2_{L^2(\Gamma)} = 6.81 \times 10^{-7}$. We note that the estimated initial state is very close to the exact one, which show the efficiently of the considered approach.

4.1 Reconstruction Error - Pointwise Location Sensor

The following simulations results show the evolution of the estimated state error with respect to the sensor location. For $b_2 = 0.27$ and $b_1 \in [0,1]$, we obtain:

For $b_1 = 0.11$ and $b_2 \in [0,1]$, we obtain:
From figure 4 and 5, we reveal the following facts:

- For a given subregion $\Gamma$, there is an optimal sensor location (optimal in the sense that it leads to a minimum error corresponding to best solution very close to the initial boundary state).
- When a sensor is located sufficiently far from the subregion $\Gamma$, the estimated error is constant for any locations.
- The worst locations correspond to a great error which correspond to the non $\Gamma$-strategic sensor as developed in the linear case ([16]).

### 4.2 Relation Subregion Area - Estimated State Error

Here we show that the reconstruction error depends on the subregion area. The table 1 show that both the error and the subregion area increase or decrease. The results are similar for other types of sensors.

| Subregion $\Gamma$ | $||y_0 - y_{oe}||^2_{L^2(\Gamma)}$ |
|--------------------|-------------------------------|
| $\{0\} \times ]0,1[$ | $1.80 \times 10^{-3}$ |
| $\{0\} \times ]0,0.1[$ | $4.89 \times 10^{-6}$ |
| $\{0\} \times ]0,0.15[$ | $6.56 \times 10^{-6}$ |
| $\{0\} \times ]0,0.16[$ | $7.12 \times 10^{-6}$ |
| $\{0\} \times ]0,0.17[$ | $6.00 \times 10^{-7}$ |
| $\{0\} \times ]0,0.18[$ | $6.81 \times 10^{-7}$ |

*Table 1: The evolution of the boundary reconstruction error with respect to the subregion $\Gamma$ area.*
5 Sectorial approach

The results given here for observability of semilinear systems are considered under the same hypothesis as in the last section concerning $\omega_r$, we have the following result:

**Proposition 5.1** If the linear part of system (1) is approximately $\omega_r$-observable then (1) is $\Gamma$-observable.

**Proof.**
Let’s consider the following operators: $\tilde{\gamma} : H^1(\omega_r) \rightarrow H^\frac{3}{2}(\partial \omega_r)$ the trace operator and $\tilde{\chi}_\Gamma : H^\frac{3}{2}(\partial \omega_r) \rightarrow H^\frac{3}{2}(\Gamma)$ the restriction operator on $\Gamma$, which allow to consider the decomposition of $y_0^1$ as follows:

$$y_0^1 = \tilde{\chi}_\Gamma \tilde{\gamma} \tilde{y}_0$$

Where $\tilde{y}_0$ is the initial state in $\omega_r$ and $y_0^1$ is the initial state to be observed on $\Gamma$. In the sequel we study our problem in the following space $E = \text{Im} R + \text{Ker}(\chi_\Gamma \gamma_0)$.

We have

$$y_0 = \chi_{\omega_r}^* \tilde{y}_0 + \tilde{y}_0 = R\tilde{y}_0^1 + \tilde{y}_0^2$$

while $\tilde{y}_0$ and $\tilde{y}_0^2$ are the residual parts of $y_0$ and $R$ is defined like in section 2, then we obtain

$$\chi_{\omega_r}^* \tilde{y}_0 = R\tilde{y}_0^1 + (\tilde{y}_0^2 - \tilde{y}_0)$$

On the other hand, the mild solution of (1) is given by

$$y = S(t)y_0 + L(t)N y$$
$$= S(t)\chi_{\omega_r}^* \tilde{y}_0 + S(t)\tilde{y}_0 + L(t)N y.$$

which gives

$$CS(t)\chi_{\omega_r}^* \tilde{y}_0 = z - CS(t)\tilde{y}_0 - CL(t)N y.$$

Since the linear part of (1) is approximately $\omega_r$-observable, then

$$\tilde{y}_0 = [K\chi_{\omega_r}^*]^\dagger (z - CS(t)\tilde{y}_0 - CL(t)N y).$$

Where $[K\chi_{\omega_r}^*]^\dagger = [(K\chi_{\omega_r}^* \chi_{\omega_r}^* [K\chi_{\omega_r}^*])^{-1} [K\chi_{\omega_r}^*]^*].$

Then

$$\chi_{\omega_r}^* \tilde{y}_0 = \chi_{\omega_r}^* [K\chi_{\omega_r}^*]^\dagger (z - CS(t)\tilde{y}_0 - CL(t)N y)$$

which gives

$$R\tilde{y}_0^1 + (\tilde{y}_0^2 - \tilde{y}_0) = \chi_{\omega_r}^* [K\chi_{\omega_r}^*]^\dagger (z - CS(t)\tilde{y}_0 - CL(t)N y)$$

then

$$\chi_{\Gamma} \gamma_0 R\tilde{y}_0^1 = \chi_{\Gamma} \gamma_0 \left(\chi_{\omega_r}^* [K\chi_{\omega_r}^*]^\dagger (z - CS(t)\tilde{y}_0 - CL(t)N y) - (\tilde{y}_0^2 - \tilde{y}_0)\right)$$
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Hence

\[ y_0^1 = \chi_r \gamma_0 \left( \chi_{\omega_r}^* \left[ K \chi_{\omega_r}^* \right]^\dagger (z - CS(t)\bar{y}_0 - CL(t)\mathcal{N}y) - (\bar{y}_0^2 - \bar{y}_0) \right) \]

Which gives the initial state to be observed on \( \Gamma \).

5.1 Algorithm

Let’s consider the mapping \( f : \tilde{y}_0 \in B(0,m) \mapsto y(.) \in B(0,a) \) where \( y(.) \) is the solution of system (1) corresponding to the initial state \( \tilde{y}_0 \) in \( \omega_r \) and the residual part \( \bar{y}_0 \) in \( \Omega \setminus \omega_r \), satisfies the Lipschitz condition, we have

**Proposition 5.1** The sequence of regional initial state

\[
\begin{cases}
\tilde{y}_{0,0} = 0 \\
\tilde{y}_{0,n+1} = K_{\omega_r}^\dagger (z - K(.)\bar{y}_0 - CL(.)\mathcal{N}f(\tilde{y}_{0,n}))
\end{cases}
\]  

(19)

where \( K_{\omega_r} = K \chi_{\omega_r}^* \), converges to the desired initial state \( \tilde{y}_0^* \) to be observed in \( \omega_r \).

**Proof.** For more details we refer the reader to [17].

Let’s consider \( y_n = f(\tilde{y}_{0,n}) \) and \( r_{n+1} = z - CS(.)\bar{y}_0 - CL(.)\mathcal{N}y_n \)

Then

\[ z_n = Cy_n = r_n + z - r_{n+1} \]

We have the following algorithm:

Step 1: • The initial state \( y_0 \), the subregion \( \omega_r \) and the location of the sensor
  • \( r_1 = z - CS(.)\bar{y}_0 - CL(.)\mathcal{N}f(0) \).
  • Threshold accuracy \( \varepsilon \).

Step 2: Repeat
  • Resolution of \( \tilde{y}_{0,n} = K_{\omega_r}^\dagger \quad n \geq 1 \).
  • Resolution of \( \dot{y}_n = Ay_n + \mathcal{N}y_n, \quad y_n(0) = \chi_{\omega_r}^* \tilde{y}_{0,n} + \bar{y}_0 \)
  • \( z_n = Cy_n \)
  Until \( |z_n - z| \leq \varepsilon \).

Step 3: The solution \( \tilde{y}_{0,n} \) corresponds to regional initial state to be observed in the subregion \( \omega_r \).

Using the previous proposition, we deduce the initial state on \( \Gamma \).
5.2 Simulations results

Here we give a numerical example illustrating the previous algorithm. The obtained results are related to the considered subregion and the considered sensor location (see [15]).

We consider the semilinear system described in $\Omega = [0,1] \times [0,1]$ by

$$\begin{align*}
\frac{\partial y}{\partial t}(x_1, x_2, t) &= 0.01 \sum_{i=1}^{2} \frac{\partial^2 y}{\partial x_i^2}(x_1, x_2, t) + \sum_{k,l=0}^{\infty} |\langle y(t), \varphi_{kl} \rangle| \langle y(t), \varphi_{kl} \rangle \varphi_{kl}(x_1, x_2) \text{ in } \Omega \times ]0,T[ \\
y(x_1, x_2, 0) &= y_0(x_1, x_2) \text{ in } \Omega \\
\frac{\partial y(\xi, t)}{\partial \nu} &= 0 \text{ on } \partial \Omega \times ]0,T[ 
\end{align*}$$

Augmented with the output function given by

$$z(t) = y(b_1, b_2, t), \quad t \in ]0,T[$$

where $b_1 = 0.69$, $b_2 = 0.60$ and $T = 2$. For $\omega_r = [0,0.16] \times [0,1]$ be the subregion target, and

$$y_0(x_1, x_2) = \left( \ln(x_1 + 1)^2 + \frac{x_1^2}{2} - 2x_1 + \alpha \right) \left( \ln(x_2 + 1)^2 + \frac{x_2^2}{2} - 2x_2 + \beta \right)$$

the initial state to be observed on the subregion $\Gamma = \{0\} \times [0,1]$, where $\alpha$ and $\beta$ are chosen for numerical consideration. Using the previous algorithm, we obtain the following results:

*Figure 6: The desired initial state in $\omega_r$. Figure 7: The estimated initial state in $\omega_r*.}
Figure 8: The exact (continuous line) and estimated (dashed line) initial state on $\Gamma$.

The initial state is obtained with error $\| y_0 - y_{oe} \|_{L^2(\Gamma)}^2 = 1.22 \times 10^{-7}$.

Remark If we consider the example used in HUM approach and we apply the obtained algorithm of the sectorial case, we obtain the following figures:

Figure 9: The desired initial state in $\omega_r$. Figure 10: The estimated initial state in $\omega_r$.

Figure 11: The exact (continuous line) and estimated (dashed line) initial state on $\Gamma$.

From the Figure 9, we note that the estimated initial state is very close to
the exact one in the subregion $\Gamma$, the initial state is obtained with error $\|y_0 - y_{\omega}\|^2_{L^2(\Gamma)} = 8.25 \times 10^{-7}$.

6 Conclusion

The regional boundary observability for distributed parabolic semilinear systems is considered. The regional internal and boundary observability of linear systems was explored to solve the problems related to boundary observability of semilinear one which constitutes a natural extension. We explored two reconstruction approaches where the first is based on the Hilbert uniqueness method and the second is based on a sectorial property combined with the fixed point techniques. The two approaches lead to algorithms which are successfully performed through numerical examples and simulations. The problem of regional gradient observability of semilinear systems is of great interest and the work is under consideration and will be the subject of the feature paper.

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References


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