On the Generalized Struve Transformation

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Abstract

We study the struve transformation on certain class of generalized functions. The generalized Struve transformation is defined and its derivatives are obtained.

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1 Introduction

The classical Struve transformation of a function \( f \) is defined by

\[
(\nu f)(x) = \int_0^\infty (xt)^\frac{\nu}{2} (xt) f(t) \, dt
\]  

(1.1)

where \( \nu(z) \) is the struve function of order \( \nu \), given by [cf. Agarwal and VijayKumar[7] pp.37-38]

\[
\Gamma\left(\nu + \frac{1}{2}\right) \nu(z) = \frac{2 (\frac{z}{2})^\nu}{\sqrt{\pi}} \int_0^{\frac{\pi}{2}} \sin(z \cos \varphi) (\sin \varphi)^{2\nu} d\varphi, \text{Re}(\nu) > -\frac{1}{2}
\]  

(1.2)

or

\[
\Gamma\left(\nu + \frac{1}{2}\right) \nu(z) = \frac{2 (\frac{z}{2})^\nu}{\sqrt{\pi}} \int_0^{\frac{\pi}{2}} \sin(z \cos \varphi) (\sin \varphi)^{2\nu} d\varphi, \text{Re}(\nu) > -\frac{1}{2}
\]  

(1.3)
for all values of \( v \).

Let \( a \) be fixed positive real number and \( I \) be the infinite open interval \((0, \infty)\).

The generalized struve transformation in [7] was investigated on the space \( \hat{H}_a \) of distribution which function as dual of the space \( H_a \) of all in infinitely smooth functions \( \varphi (x) \) on \( I \), such that

\[
\gamma_k (\varphi) = \sup_{t \in I} \left| e^{-at} (tD)^k \varphi (t) \right| < \infty. \tag{1.4}
\]

We have, in this paper, studied the struve transformation on certain space of generalized function which are wider than the Schwartz’ space \( \hat{E} \) of distributions [6, 8].

(see [7] for further properties).

Let \( \Delta \) be a compact subset of \( I \). Denote by \( \mathcal{L} (n_i, n, I) \) the set of all complex valued infinitely smooth functions such that

\[
\sup_{x \in \Delta} \left| D^k \varphi (n) \right| \leq Nn_i n^i,
\]

for some positive constant \( N \) and all \( n_i \).

\[
\eta_{\Delta, k} (\varphi) = \sup_{t \in k} \left| \frac{D^k \varphi (t)}{Nn_i n^i} \right|
\]

We assign to \( \mathcal{L} (n_i, n, I) \) the topology generated by the collection of seminorms \( \{\eta_{\Delta, k}\} \), where \( \Delta \) varies through all compact subset of \( I \) and \( k \) traverses through non-negative integers in \( \mathbb{R}^n \), where

\[
\eta_{\Delta, k} (\varphi) = \sup_{t \in k} \left| \frac{D^k \varphi (t)}{Nn_i n^i} \right|
\]

The space \( \mathcal{L} (n_i, n, I) \), equipped with the topology generated by the countable collection of seminorms \( \{\eta_{\Delta, k}\}_{k=0}^\infty \), is a complete countably multinormed space. A sequence \( \{\varphi_v\} \) in \( \mathcal{L} (n_i, n, I) \) converges to \( \varphi \) if and only if all \( \varphi_v \) and \( \varphi \) are in \( \mathcal{L} (n_i, n, I) \) and for each non-negative integers \( k \), \( \{D^k \varphi\}_{v=1}^\infty \) converges to \( (D^k \varphi) \) uniformly on every compact subset of \( I \) in the sense of topology of \( \mathcal{L} (n_i, n, I) \). \( \hat{\mathcal{E}} (n_i, n, I) \) is the dual space when endowed with its weak topology. However, the space of generalized functions, obtained above, is, indeed, consists of ultradistributions of slow growth similar to that obtained in
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[2, 3, 4, 5], whereas, the ultradistribution space \( Z' \) in [1] functions as a dual of \( Z \) of all Fourier transforms of test functions in \( D \) (the space of test functions of compact support).

If \( \nu (\mathbb{R}^n) \) is a space of test functions \( \varphi \) on \( \mathbb{R}^n \) (\( n \) – dimensional Euclidean space) \( \dot{\nu} (\mathbb{R}^n) \) is called the dual space of \( \nu \) (the set of all linear continuous functional on \( \nu \)), then the elements of \( \dot{\nu} \) are called generalized functions. A generalized function \( f \in \dot{\nu} \) is said to have support in a set \( S \) if \( \langle f, \varphi \rangle = 0 \) whenever \( \varphi \in \nu \) has support in \( \mathbb{R}^n \setminus S \). The smallest closed set having this property is called the support of \( f \), written as \( \text{supp} f \). When \( \text{supp} f \) is compact (a space is compact if and only if each collection of closed sets contained in that space has a nonempty intersection if it has the finite intersection property), the function \( f \) is said to be a generalized function of compact support.

2 Testing function space \( \mathcal{L}(n_i, n, I) \)

**Lemma 2.1.** Let \( \Delta \) be a compact subset of \( I \), for real numbers \( x \) and \( t \) in \( \Delta \) and \( \nu > -\frac{1}{2} \),

(i) \( (xt)^{\frac{\nu}{2}} (xt) \in \mathcal{L}(n_i, n, I) \)

(ii) \( D^m_x \left[ (xt)^{\frac{\nu}{2}} (xt) \right] \in \mathcal{L}(n_i, n, I), m = 1, 2, 3, \ldots \)

**Proof.** \( \Delta \) and \( \nu \) have usual meaning then there be a constant \( C = \inf \Delta \), for the non-negative integers \( k \), such that

\[
\left| D^k_t \left\{ (xt)^{\frac{\nu}{2}} (xt) \right\} \right| \leq \frac{\sum_{j=0}^{k} \alpha_j (\nu) C^{-k} (xt)^{\frac{j}{2}} (xt)}{Nn_in^i} \tag{2.1}
\]

where \( \alpha_j (\nu) \) is a polynomial in \( \nu \).

The right-hand said of (2.1) is finite for all real \( x \) and \( t \), in \( \Delta \), ranges over all compact subset of \( I \). Upon considering the supermum over all \( x, t \in \Delta \), we have

\[ \eta_{\Delta,k} \left[ (xt)^{\frac{\nu}{2}} (xt) \right] < \infty \]

Similarly, for non-negative integers \( k \) and \( m \),

\[
\left| D^k_t \left\{ D^m_x \left[ (xt)^{\frac{\nu}{2}} (xt) \right] \right\} \right| \leq \sum_{j=0}^{k} \sum_{i=0}^{m} \frac{\alpha_{i,j} (\nu) C^{-(m+k)} (xt)^{\frac{j+i}{2}} (xt)}{Nn_in^i} \tag{2.2}
\]

where \( \alpha_{i,j} (\nu) \) is a polynomial in \( \nu \). Multiply both sides by \( \frac{1}{Nn_in^i} \) and let \( \Delta \) vary through compact subsets of \( I \), by considering the supremum, we observe that
\[ \eta_{\Delta,k} \left\{ D^m_x \left[ (xt)^{\frac{1}{\nu}} (xt) \right] \right\} < \infty \]

for all \( m = 1, 2, 3, \ldots \).
This completes the proof of the theorem.

**Definition 2.2** Let \( f \in \mathcal{L} (n_i, n, I) \) be the space of generalized functions supported in \( I = (0, \infty) \).

By virtue of lemma 1, we define the generalized struve transformation of \( f \) as the map of the non-zero real number \( x \), given as

\[ S(x) = \left\langle f(t), (tx)^{\frac{1}{\nu}} (xt) \right\rangle, \quad \nu > -\frac{1}{2}, t > 0 \quad (2.3) \]

**Theorem 1.** Let \( f \in \mathcal{L} (n_i, n, I) \) and \( \Delta \) be a compact subset of \( I \). for every \( x \) and \( t \) in \( \Delta \),

\[ D^m_x S(x) = \left\langle f(t), D^m_x \left\{ (tx)^{\frac{1}{\nu}} (xt) \right\} \right\rangle, \quad m = 1, 2, \ldots \quad (2.4) \]

where \( S(x) \) is defined in (2.3).

**Proof.** In what follows, we attempt to prove this theorem by the method of induction on \( m \).

For \( m = 0 \), (2.4) obviously reduce to (2.3) and that is trivial. Let the relation (2.4) is true for \( (m-1) \) derivatives, let \( \partial x \neq 0 \) and \( x \) be fixed, then

\[
\begin{align*}
\frac{1}{\partial x} \left[ D^{m-1}_x S(x + \partial x) - D^{m-1}_x S(x) \right] - \left\langle f(t), D^m_x \left\{ (tx)^{\frac{1}{\nu}} (xt) \right\} \right\rangle \\
= \frac{1}{\partial x} \left[ D^{m-1}_x \left\langle f(t), ((x + \partial x) t)^{\frac{1}{\nu}} (x + \partial x) t \right\rangle - D^{m-1}_x \left\langle f(t), (tx)^{\frac{1}{\nu}} (xt) \right\rangle \right] \\
- \left\langle f(t), D^m_x \left\{ (tx)^{\frac{1}{\nu}} (xt) \right\} \right\rangle \\
= \left\langle f(t), \frac{1}{\partial x} \left[ D^{m-1}_x h((x + \partial x) t) - D^{m-1}_x h(x t) \right] - D^m_x h(x t) \right\rangle \\
eq \left\langle f(t), \varphi_{\partial x}(t) \right\rangle,
\end{align*}
\]

where

\[ h(x t) = (xt)^{\frac{1}{\nu}} (xt) \quad (2.5) \]

and

\[ \varphi_{\partial x}(t) = \frac{1}{\partial x} \left[ D^{m-1}_x h((x + \partial x) t) - D^{m-1}_x h(x t) \right] - D^m_x h(x t) \quad (2.6) \]
Indeed, Fundamental theorem of calculus and dividing the $m$—derivatives into $(m - 1)$ and first derivatives, from (2.6) we obtain

$$
\varphi_{\partial x} (t) = \frac{1}{\partial x} \int_x^{x+\partial x} D^m_u h(ut) \, du - \frac{1}{\partial x} \int_x^{x+\partial x} D^m_x h(xt) \, du.
$$

$$
= \frac{1}{\partial x} \int_x^{x+\partial x} [D^m_u h(ut) - D^m_x h(xt)] \, du
$$

$$
= \frac{1}{\partial x} \int_x^{x+\partial x} \int_x^u \left[ D^m_u h(\eta t) - D^m_x h(\eta t) \right] d\eta.
$$

Now, to complete the proof of the theorem, it is enough to be shown that $\varphi_{\partial x} (t)$ tends to zero as $\partial x \to 0$ in the topology of $L(n, n, I)$. thus, we have

$$
\left| \frac{D^n \varphi_{\partial x} (t)}{Nn_i n^i} \right| = \left| \frac{1}{\partial x} \int_x^{x+\partial x} \int_x^u D^{m+1}_\eta \{ D^k \eta (\eta t) \} \, d\eta \right| \leq \left| \partial x \right| \left( \inf_{\Delta} \sup_{t \in \Delta} \frac{\left| \sum_{j=0}^{k} \sum_{i=0}^{m+1} \alpha_{j,i} (v) C^{-(m+k+1)} (\eta t)^{j+i} (\eta t) \right|}{Nn_i n^i} \right),
$$

$$
C = \inf \Delta
$$

Summations in the relation (2.7) are bounded and thus, $\sup_{t \in \Delta} \left| \frac{D^n \varphi_{\partial x} (t)}{n n_i n^i} \right|$ tendes to zero as $\partial x \to 0$. Indeed, $\varphi_{\partial x} (t)$ and all of its derivatives converge uniformly to zero on every compact subset of $I, C$ being a constant. This completes the proof of the theorem.

References


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