Multiresolution Analysis on the
Hardy Space of Analytic Functions

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Abstract

The main aim of this paper is to generalize the concepts of wavelet
and multiresolution analysis from $L^2(\mathbb{R})$ to $H^2(\mathbb{D})$, the Hardy space of
analytic functions on the open unit disc with square summable Taylor
coefficients.

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1 Introduction

The theory of wavelet is a relatively new area of mathematics. In order
to construct orthonormal wavelets, Mallat [10] established the theory of the
classical multiresolution analysis (MRA). However, indeed there exist or-
thonormal wavelets which can not be derived by MRAs, one calls them non-
MRA wavelets. The first example of an object now called a wavelet is the
Haarfunction introduced by Haar in 1910. Haar showed that the appro-
priate translates and dilates of Haarfunction form an orthonormal basis of
$L^2([0,1])$. (see [8])

Wavelet turn out to have many advantages in studying various function
spaces. They form an unconditional bases for bases for variety of function
spaces, e.g. $L^p(\mathbb{R})$, Hardy and Sobolov spaces. (see [1], [2], [9])
In [6], Goh give a general approach of constructing orthonormal wavelet for a separable Hilbert space of complex-valued functions, and his construction is applied to several Hilbert spaces like the space of analytic functions on the unit disk.

In this paper we introduce new concepts of wavelet and multiresolution analysis on $H^2(\mathbb{D})$, the Hardy space of analytic functions on the unit disc with square summable Taylor coefficients. We generalize the concepts of wavelet and multiresolution analysis from $L^2(\mathbb{R})$ to $H^2(\mathbb{D})$. It is necessary to notice that our method is different from that in [6].

This paper is organized as follows: Section 2 contains some notations and preliminaries which is needed during the paper. In section 3, we explore new concepts of multiresolution analysis on the Hardy space and by using this definition, we find an example of an orthonormal wavelet on the Hardy space.

2 Notations and Preliminaries

We denote by $L^2(\mathbb{R})$ the Hilbert space of all square-integrable functions with the inner product

$$\langle f, g \rangle = \int_{\mathbb{R}} f(t)\overline{g(t)}dt,$$

for $f, g \in L^2(\mathbb{R})$.

Many examples of wavelets have been produced using the related concept of a multiresolution analysis.

**Definition 2.1** A multiresolution analysis (MRA) on $L^2(\mathbb{R})$ consists of sequence of closed subspaces $\{V_j\}_{j \in \mathbb{Z}}$ of $L^2(\mathbb{R})$ satisfying

i) $V_j \subseteq V_{j+1}$ for $j \in \mathbb{Z}$

ii) $f \in V_j$ if and only if $f(2 \cdot) \in V_{j+1}$

iii) $\bigcup_{j \in \mathbb{Z}} V_j = L^2(\mathbb{R})$ and $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$

iii) There exists $\phi \in V_0$ such that $\{\phi(\cdot - l) : l \in \mathbb{Z}\}$ is an orthonormal basis for $V_0$. Herein, $\phi$ is called a scaling function of the MRA.

Let $\phi_{j,l}(\cdot) = 2^{j/2}\phi(2^j \cdot - l)$; if $j \in \mathbb{Z}$, condition (ii) implies that

$$\{\phi_{j,l} : l \in \mathbb{Z}\}$$

is an orthonormal basis for $V_j$. 


D.Gabor considered this type of system in 1946. (see [4])

Another way of producing an orthonormal basis from a single function involves translations and modulations. For example, a basis for $L^2(\mathbb{R})$ is the following: let $g = \chi_{[0,1]}$ and for $m, n \in \mathbb{Z}$

$$g_{m,n}(t) = e^{2\pi imx}g(t-n).$$

D.Gabor considered this type of system in 1946. (see [4])

A wavelet is a special case of a vector in a separable Hilbert space that generates a basis under the action of a collection, or system, of unitary operators defined in terms of translations and dilation operation. This approach to wavelet theory goes back, in particular, to earlier work of Goodman, Lee and Tang [7] in the context of multiresolution analysis.

Let $H$ be a separable Hilbert space. Fix a countable abelian group $\Gamma$ of unitary operators on $H$, which we call translations and a unitary operator $\delta$ on $H$, which we call a dilation. We assume that the dilation is commensurate with the translations in the sense that the group $\delta^{-1}\Gamma\delta$ is a subgroup of finite index $N$ in $\Gamma$.

Definition 2.4 A multiresolution analysis (MRA) on $H$ consists of a sequence of closed subspaces $\{V_j\}_{j \in \mathbb{Z}}$ of $H$ satisfying:

i) $V_j \subseteq V_{j+1}$ for all $j \in \mathbb{Z}$

ii) $\delta(V_j) = V_{j+1}$ for all $j \in \mathbb{Z}$

iii) $\bigcup_{j \in \mathbb{Z}} V_j$ is dense in $H$ and $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$
There exists \( \phi \in V_0 \) such that \( \{ g\phi : g \in \Gamma \} \) is an orthonormal basis for \( V_0 \). Herein, \( \phi \) is called a scaling function of the MRA on \( H \).

We have \( \{ \delta^j g\phi : g \in \Gamma \} \) as an orthonormal basis for \( V_j \) for all \( j \in \mathbb{Z} \).

**Definition 2.5** Given a MRA \( \{ V_j \}_{j \in \mathbb{Z}} \) on \( H \), and let for every \( j \in \mathbb{Z}, W_j = V_{j+1} \ominus V_j \). If there exists \( \psi \in W_0 \), such that \( \{ g\psi : g \in \Gamma \} \) is an orthonormal basis for \( W_0 \), we call \( \psi \) an orthonormal wavelet of the MRA. In fact \( \{ \delta^j g\psi : g \in \Gamma \} \) is an orthonormal basis for \( W_j \) for all \( j \in \mathbb{Z} \) and hence \( \{ \delta^j g\psi : g \in \Gamma, j \in \mathbb{Z} \} \) is an orthonormal basis for \( H \).

In the case \( H = L^2(\mathbb{R}) \), if let \( \delta = D \) and \( \Gamma = \{ T^l : l \in \mathbb{Z} \} \) we see that the classic definition of wavelet on \( L^2(\mathbb{R}) \) is the same of the definition on abstract Hilbert space.

## 3 wavelet on hardy space

The theory of Hardy spaces is very rich with many highly developed branches and originated in the context of complex function theory and Fourier analysis in the beginning of twentieth century.

Let \( \mathbb{D} \) denote the open unit disc in the complex plane, and the Hardy space \( H^2(\mathbb{D}) \) be the set of all functions \( f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n \) holomorphic in \( \mathbb{D} \) such that

\[
\sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty,
\]

where \( \hat{f}(n) \) denoting the \( n \)th Taylor coefficient of \( f \) (see [11]). For systematic exposition of the subject see books by Duren[3] and Garnet[5].

**Theorem 3.1** Let \( S \) be an isometric isomorphism from \( L^2(\mathbb{R}) \) onto \( H^2(\mathbb{D}) \), \( \{V'_j\}_{j \in \mathbb{Z}} \) be a MRA on \( L^2(\mathbb{R}) \), with the scaling function \( \phi' \) and an orthonormal wavelet \( \psi' \). Put \( V_j = S(V'_j), \phi = S(\phi'), \psi = S(\psi'), \delta = SDS^{-1}, \Gamma = \{ ST^l S^{-1} : l \in \mathbb{Z} \} \). Then \( \{V_j\}_{j \in \mathbb{Z}} \) is a multiresolution analysis(MRA) on Hardy space \( H^2(\mathbb{D}) \) and \( \{ \delta^j g\psi : g \in \Gamma, j \in \mathbb{Z} \} \) is an orthonormal basis for \( H^2(\mathbb{D}) \), so \( \psi \) is an orthonormal wavelet on \( H^2(\mathbb{D}) \).

**Proof.** We have \( \psi' \) is an orthonormal wavelet of the MRA on \( L^2(\mathbb{R}) \) if and only if \( \{ D^j T^l \psi' : j,l \in \mathbb{Z} \} \) is an orthonormal basis for \( L^2(\mathbb{R}) \). For any \( g,h \in \Gamma \) we have \( g = ST^l S^{-1} \) and \( h = ST^n S^{-1} \), for some \( l,n \in \mathbb{Z} \).Since \( S \) is an isometric isomorphism, for each \( j \) and \( m \) we have

\[
\langle \delta^j g\psi, \delta^m h\psi \rangle = \langle SD^j T^l \psi', SD^m T^n \psi' \rangle = \langle D^j T^l \psi', D^m T^n \psi' \rangle = \langle \psi'_j, \psi'_{m,n} \rangle
\]
Since $S$ is onto, for any $f \in H^2(\mathbb{D})$ there is $g \in L^2(\mathbb{R})$ such that $f = S(g)$, and so

$$\langle \delta^j g \psi, f \rangle = \langle SD^j T^l \psi', S(g) \rangle = \langle D^j T^l \psi', g \rangle = \langle \psi_{j,l}', g \rangle.$$ 

Also

$$\delta(V_j) = SDS^{-1}(V_j) = SD(V_j') = S(V_{j+1}) = V_{j+1},$$

and the proof is complete now.

Now let $W$ be the set of all nonnegative integers, $\varphi : \mathbb{Z} \longrightarrow W$, $\eta : \mathbb{Z} \times \mathbb{Z} \longrightarrow W \times W$ and $\upsilon : W \times W \longrightarrow W$ are defined as

$$\varphi(n) = \begin{cases} 2n & \text{if } n \geq 0; \\ -(2n + 1) & \text{if } n < 0. \end{cases}$$

$$\eta(m, n) = (\varphi(m), \varphi(n)) \quad \upsilon(m, n) = 2^n(2n + 1) - 1$$

Therefore the function $r : \mathbb{Z} \times \mathbb{Z} \longrightarrow W$ with $r(m, n) = \upsilon \circ \eta(m, n)$ is a bijective function.

Since $\{g_{m,n} : m, n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$, for given $f \in L^2(\mathbb{R})$ we have $f(t) = \sum_{m,n \in \mathbb{Z}} \langle f, g_{m,n} \rangle g_{m,n}(t)$. Define an operator $S : L^2(\mathbb{R}) \longrightarrow H^2(\mathbb{D})$ by

$$S(f) = S\left( \sum_{m,n \in \mathbb{Z}} \langle f, g_{m,n} \rangle g_{m,n} \right) = \sum_{m,n \in \mathbb{Z}} \langle f, g_{m,n} \rangle z^{r(m,n)}.$$

Since $\{z^n : n \in \mathbb{N}_0\}$ is an orthonormal basis for $H^2(\mathbb{D})$ and $r$ is a bijection, $S$ is an isometric isomorphism.

**Lemma 3.2** $S(\chi_{[k,k+1]}) = z^{r(0,k)}$.

**Proof.** By a simple computation we have

$$\langle \chi_{[k,k+1]}, g_{m,n} \rangle = \int_{\mathbb{R}} \chi_{[k,k+1]}(t) \overline{g_{m,n}(t)} \, dt$$

$$= \int_{\mathbb{R}} e^{-2\pi i m t} \chi_{[k,k+1]}(t) \chi_{[n,n+1]}(t)$$

$$= \begin{cases} 0 & \text{if } n \neq k \text{ or } m \neq 0; \\ 1 & \text{if } n = k, m = 0. \end{cases}$$

Hence

$$S(\chi_{[k,k+1]}) = \sum_{m,n \in \mathbb{Z}} \langle \chi_{[k,k+1]}, g_{m,n} \rangle z^{r(m,n)} = z^{r(0,k)}.$$

**Lemma 3.3** The function $\phi = 1$ is a scaling function on $H^2(\mathbb{D})$ and $V_0 = \{z^l : l \in \mathbb{Z}, l \geq 0\} \cup \{z^{-l-2} : l \in \mathbb{Z}, l < 0\}$. 
Hence by simple computation we have

\[ S(\phi') = \chi_{[0,1]}(x) \text{ a scaling function for } L^2(\mathbb{R}) \text{ and } \{\phi'(\cdot - l) : l \in \mathbb{Z}\} \text{ is an orthonormal basis for } V_0, \]

by using Theorem 3.1, we conclude that \( S\phi' \) is a scaling function on \( H^2(\mathbb{D}) \), \( \{S(\phi'(\cdot - l)) : l \in \mathbb{Z}\} \) is an orthonormal basis for \( V_0 \) and by the previous Lemma \( S(\phi') = z^{r(0,0)} = 1 \). Hence

\[
V_0 = \text{span}\{S\phi'(\cdot - l) : l \in \mathbb{Z}\} = \{S(\chi_{[l,l+1]}): l \in \mathbb{Z}\} = \{z^{r(0,l)} : l \in \mathbb{Z}\} = \{z^{4l} : l \geq 0\} \cup \{z^{-4l-2} : l < 0\}.
\]

**Lemma 3.4** The function \( \psi(z) = \frac{2(z^2-1)}{\pi iz^3} \left( \sum_{k \text{ odd}, k \geq 1} z^k \right) \) is an orthonormal wavelet on \( H^2(\mathbb{D}) \).

**Proof.** We know that \( S \) is an isometric isomorphism from \( L^2(\mathbb{R}) \) onto \( H^2(\mathbb{D}) \) and \( \psi(t) = \chi_{[0,\frac{1}{2}]}(t) - \chi_{[\frac{1}{2},1]}(t) \) is an orthonormal wavelet for \( L^2(\mathbb{R}) \).

So, by using Theorem 3.1, \( S(\psi') \) is an orthonormal wavelet on \( H^2(\mathbb{D}) \). Also

\[
\langle \chi_{[0,\frac{1}{2}]} - \chi_{[\frac{1}{2},1]}, g_{m,n} \rangle = \int_{\mathbb{R}} \chi_{[0,\frac{1}{2}]}(t) - \chi_{[\frac{1}{2},1]}(t) g_{m,n}(t) dt
\]

\[
= \int_0^{\frac{1}{2}} g_{m,n}(t) dt - \int_{\frac{1}{2}}^1 g_{m,n}(t) dt
\]

\[
= \int_0^{\frac{1}{2}} e^{-2\pi int} \chi_{[m,n+1]}(t) - \int_{\frac{1}{2}}^1 e^{-2\pi int} \chi_{[n,n+1]}(t)
\]

\[
= \begin{cases} 
0 & \text{if } n \neq 0 \text{ or } m \text{ is even;} \\
\frac{2}{\pi im} & \text{if } n = 0, m \text{ is odd.}
\end{cases}
\]

Hence by simple computation we have

\[
S(\psi') = \sum_{m,n\in\mathbb{Z}} \langle \psi', g_{m,n} \rangle z^{r(m,n)}
\]

\[
= \sum_{k=-\infty}^{\infty} \frac{\langle \psi', g_{2k+1,0} \rangle}{\pi i(2k+1)} z^{r(2k+1,0)}
\]

\[
= \sum_{k=-\infty}^{\infty} \frac{2}{\pi i(2k+1)} z^{r(2k+1,0)}
\]

\[
= \sum_{k=-\infty}^{\infty} \frac{2}{\pi i(2k+1)} z^{2^{4k+2} - 1} + \sum_{k=0}^\infty \frac{2}{\pi i(2k+1)} z^{2^{4k+2} - 1}
\]

\[
= \sum_{k=0}^\infty \frac{2}{\pi i(2k+1)} (z^{2^{4k+2} - 1} - z^{2^{4k+1} - 1})
\]

\[
= \frac{2(z^2 - 1)}{\pi iz^3} \left( \sum_{k \text{ odd}, k \geq 1} z^k \right).
\]
and the proof is complete now.

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References


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