Positive Solutions of Boundary Value Problems of Nonlinear Third-Order Differential Equations

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Abstract

The paper investigates the problem of existence of positive solutions of nonlinear third-order differential equations. Under the suitable conditions, the existence and multiplicity of positive solutions are established by using Krasnoselskii’s fixed-point theorem of cone.

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1 Introduction

Recently, there are many results on existence of the positive solutions for ODE(see[1][2][3][4][5][6]). The purpose of this paper is to establish the existence of positive solutions for BVP of

\[-u'''(x) = f(x, v),\]
\[-v'''(x) = g(x, u),\]
\[u(0) = u'(0) = 0, \alpha u'(1) + \beta u''(1) = 0,\]
\[v(0) = v'(0) = 0, \alpha v'(1) + \beta v''(1) = 0,\]  

(1)

where \( f, g \in C([0, 1] \times R^+, R^+), f(x, 0) \equiv 0, g(x, 0) \equiv 0, \alpha, \beta \geq 0, \alpha + \beta > 0. \)

Here, a fixed point theorem due to Krasnoselskii is applied to yield the existence of positive solutions of (1). Another fixed point theorem about multiplicity is applied to obtain the multiplicity of positive solutions of (1).
2 Preliminaries

In this section we present some necessary definitions and preliminary lemmas that will be used in the proof of the results.

**Definition 1.** Let $E$ be a real Banach space. A nonempty closed set $P \subset E$ is called a cone of $E$ if it satisfies the following conditions:

1. $x \in P, \lambda > 0$ implies $\lambda x \in P$;
2. $x \in P, -x \in P$ implies $x = 0$. It is easily to know that $(u, v) \in C^3[0, 1] \times C^3[0, 1]$ is the solution of (1) if and only if $(u, v) \in C[0, 1] \times C[0, 1]$ is the solution of the integral equations

\[
\begin{align*}
\begin{cases}
  u(x) = \int_0^1 K(x, y)f(y, v(y))dy, \\
v(x) = \int_0^1 K(x, y)g(y, v(y))dy,
\end{cases}
\end{align*}
\]

where $K(x, y)$ is the Green’s function defined as follows:

\[
K(t, s) = \begin{cases}
  \frac{x^2}{2} - \frac{\alpha x^2 y}{2(\alpha + \beta)}, & \text{if } 0 \leq x \leq y \leq 1, \\
  \frac{y^2}{2} + xy - \frac{\alpha x^2 y}{2(\alpha + \beta)}, & \text{if } 0 \leq y \leq x \leq 1.
\end{cases}
\]

Integral equations (2) can be transferred to the nonlinear integral equation

\[
u(x) = \int_0^1 K(x, y)f(y, \int_0^1 K(y, z)g(z, u(z))dz)dy
\]

It is obvious that $K(x, y) \geq 0$ and $K(x, y) \leq K(1, y)$, for $0 \leq x, y \leq 1$.

**Lemma 1.** $K(x, y) \geq q(x)K(1, y)$ for $0 \leq x, y \leq 1$, where $q(x) = \frac{\beta x^2}{\alpha + 2\beta}$.

**Proof.** If $x \leq y$, then

\[
\frac{K(x, y)}{K(1, y)} = \frac{\frac{x^2}{2} - \frac{\alpha x^2 y}{2(\alpha + \beta)}}{\frac{y^2}{2} + xy - \frac{\alpha x^2 y}{2(\alpha + \beta)}} \geq \frac{\frac{1}{2} - \frac{\alpha y}{2(\alpha + \beta)}}{1 - \frac{\alpha y}{2(\alpha + \beta)}} \geq \frac{\beta x^2}{\alpha + 2\beta}.
\]

If $x \geq y$ and $y \neq 0$, then

\[
\frac{K(x, y)}{K(1, y)} = \frac{\frac{y^2}{2} + xy - \frac{\alpha x^2 y}{2(\alpha + \beta)}}{\frac{y^2}{2} + y - \frac{\alpha y^2}{2(\alpha + \beta)}} = \frac{\frac{y^2}{2} + x - \frac{\alpha x^2 y}{2(\alpha + \beta)}}{\frac{y^2}{2} + 1 - \frac{\alpha y}{2(\alpha + \beta)}}
\geq \frac{\frac{x^2}{2} - \frac{\alpha x^2 y}{2(\alpha + \beta)}}{\frac{y^2}{2} + 1 - \frac{\alpha y}{2(\alpha + \beta)}} = \frac{\beta x^2}{\alpha + 2\beta}x^2.
\]

It is obvious that $K(x, y) \geq q(x)K(1, y)$ when $y = 0$. The proof is complete.

$E = C[0, 1]$ is a Banach space equipped with standard norm $\|u\| = \max_{0 \leq x \leq 1} |u(x)|, u \in E$. Define a cone $P$ by $P = \{u \in E \mid u(x) \geq 0, u(x) \geq q(x) \|u\|, 0 \leq x \leq 1\}$. It is easy to see that if $u \in P$, then $\|Au\| = Au(1)$. Define

\[
Au(x) = \int_0^1 K(x, y)f(y, \int_0^1 K(y, z)g(z, u(z))dz)dy
\]
Theorem 1. Let

suppose either

Proof. From the continuity of \( f \) and \( g \), we know \( Au \in E \) for each \( u \in P \). It follows from Lemma 2, that for \( u \in P \)

Thus \( A(P) \subset P \). Since \( K(x,y), f(x,v) \) and \( g(x,u) \) are continuous. It is easily known that \( A : P \rightarrow P \) is completely continuous. The proof is complete.

Lemma 2. (see [7]) Let \( E \) be a Banach space and \( P \subset E \) is a cone in \( E \). Assume that \( \Omega_1 \) and \( \Omega_2 \) are open subsets of \( E \) with \( 0 \in \Omega_1 \) and \( \overline{\Omega_1} \subset \Omega_2 \). Let \( A : P \cap (\overline{\Omega_2} \setminus \Omega_1) \rightarrow P \) be a completely continuous operator. In addition suppose either

1. \( \| Au \| \leq \| u \|, \forall u \in P \cap \partial \Omega_1 \) and \( \| Au \| \geq \| u \|, \forall u \in P \cap \partial \Omega_2 \) or
2. \( \| Au \| \leq \| u \|, \forall u \in P \cap \partial \Omega_2 \) and \( \| Au \| \geq \| u \|, \forall u \in P \cap \partial \Omega_1 \)

holds. Then \( A \) has a fixed point in \( P \cap (\overline{\Omega_2} \setminus \Omega_1) \).

3 Existence of positive solutions

In this section, we study the existence of positive solutions for BVP (1). we obtain the following existence results.

Following Hu and Wang (see [6]), we give some assumptions.

\( H_1 \) \( \lim_{u \to 0^+} \sup_{0 \leq x \leq 1} \frac{f(x,u)}{u} = 0 \), \( \lim_{u \to 0^+} \sup_{0 \leq x \leq 1} \frac{g(x,u)}{u} = 0 \);

\( H_2 \) \( \lim_{u \to \infty} \inf_{0 \leq x \leq 1} \frac{f(x,u)}{u} = \infty \), \( \lim_{u \to \infty} \inf_{0 \leq x \leq 1} \frac{g(x,u)}{u} = \infty \);

\( H_3 \) \( \lim_{u \to 0^+} \inf_{0 \leq x \leq 1} \frac{f(x,u)}{u} = \infty \), \( \lim_{u \to 0^+} \inf_{0 \leq x \leq 1} \frac{g(x,u)}{u} = \infty \);

\( H_4 \) \( \lim_{u \to \infty} \sup_{0 \leq x \leq 1} \frac{f(x,u)}{u} = 0 \), \( \lim_{u \to \infty} \sup_{0 \leq x \leq 1} \frac{g(x,u)}{u} = 0 \);

Theorem 1. If \( (H_1) \) and \( (H_2) \) are satisfied, then (1) has at least one positive solution \((u,v) \in C^3([0,1], R^+) \times C^3([0,1], R^+)\) satisfying \( u(x) > 0, v(x) > 0 \).

Proof. From \((H_1)\) there is a number \( L_1 \in (0,1) \) such that for each \((x,u) \in [0,1] \times (0,L_1)\), one has \( f(x,u) \leq \eta_1 u \) where \( \eta_1 > 0 \) satisfies \( \eta_1 \int_0^1 K(1,x)dx \leq 1 \) for every \( u \in P \) and \( \| u \| = \frac{L_1}{2} \), note that...
\[ \int_0^1 K(x, y) g(z, u(z)) dz \leq \int_0^1 \eta_1 k(1, z) u(z) dz \leq \| u \| = \frac{L_2}{2}, \text{then} \]

\[
Au(x) \leq \int_0^1 K(1, y) f(y, \int_0^1 K(y, z) g(z, u(z)) dz) dy \\
\leq \eta_1 \int_0^1 K(1, y) \int_0^1 K(1, z) u(z) dz dy \leq \| u \| .
\]

Let \( \Omega_1 = \{ u \in E \mid \| u \| < \frac{L_2}{2} \} \), then

\[ \| Au \| \leq \| u \|, u \in P \bigcap \partial \Omega_1. \] (6)

From \((H_2)\) there is a number \( L_2 > 2L_1 \) for each \((x, u) \in [0, 1] \times (L_2, +\infty)\), one has \( F(x, u) \geq \eta_2 u g(x, u) \geq \eta_2 u \) where \( \eta_2 > 0 \) satisfies \( \eta_2 \int_0^1 K(1, x) q(x) dx \geq 1 \), then, for every \( u \in P \) and \( \| u \| = 2L_2 \), note that \( \int_0^1 K(x, y) g(z, u(z)) dz \geq \int_0^1 \eta_2 K(1, z) u(z) dz \geq \| u \| = 2L_2 > L_2 \), then

\[ \| Au \| = (Au)(1) = \int_0^1 K(1, y) f(y, \int_0^1 K(y, z) g(z, u(z)) dz) dy \\
\geq \eta_2 \int_0^1 K(1, y) \int_0^1 K(y, z) g(z, u(z)) dz dy \\
\geq \eta_2^2 \int_0^1 K(1, y) q(y) \int_0^1 K(y, z) g(z, u(z)) dz dy \geq \| u \| . \]

Let \( \Omega_2 = \{ u \in E \mid \| u \| < 2L_2 \} \), then

\[ \| Au \| \geq \| u \|, u \in P \bigcap \partial \Omega_2. \] (7)

Thus from (6), (7) and Lemma 2, we know that the operator \( A \) has a fixed point in \( P \bigcap (\overline{\Omega_2} \setminus \Omega_1) \). The proof is complete.

**Theorem 2.** If \((H_3)\) and \((H_4)\) are satisfied, then \((1)\) has at least one positive solution \((u, v) \in C^3([0, 1], R^+) \times C^3([0, 1], R^+)\) satisfying \( u(x) > 0, v(x) > 0 \).

**Proof.** From \((H_3)\) there is a number \( L_3 \in (0, 1) \) such that for each \((x, u) \in [0, 1] \times (0, L_3)\), one has \( f(x, u) \geq \eta_3 u \) where \( \eta_3 > 0 \) satisfies \( \eta_3 \int_0^1 K(1, x) q(x) dx \geq 1 \). From \( g(x, 0) \equiv 0 \) and the continuity of \( g(x, u) \), we know that there exists number \( L_3 \in (0, \overline{L_3}) \) small enough such that \( g(x, u) \leq \frac{L_3}{\int_0^1 K(1, x) dx} \) for each \((x, u) \in [0, 1] \times (0, L_3)\). Then for every \( u \in P \) and \( \| u \| = L_3 \), note that \( \int_0^1 K(y, z) g(z, u(z)) dz \leq \int_0^1 K(1, z) \frac{L_3}{\int_0^1 K(1, x) dx} dz \leq \overline{L_3} \). Thus
\[ \| Au(x) \| = (Au)(1) = \int_0^1 K(1, y) f(y, \int_0^1 K(y, z) g(z, u(z)) dz) dy \]
\[ \geq \eta_3 \int_0^1 K(1, y) \int_0^1 K(y, z) g(z, u(z)) dz dy \]
\[ \geq \eta_3^2 \int_0^1 K(1, y) q(y) \int_0^1 K(y, z) q(z) u(z) dz dy \geq \| u \|. \]

Let \( \Omega_3 = \{ u \in E \mid \| u \| < L_3 \} \), then
\[ \| Au \| \geq \| u \|, u \in P \cap \partial \Omega_3. \quad (8) \]

From \((H_4)\), \[ \| Au \| = (Au)(1) \leq \| u \| \text{ as } \| u \| \to \infty. \]
Let \( \Omega_4 = \{ u \in E \mid \| u \| < L_4 \} \). For each \( u \in P \) and \( \| u \| = L_4 > L_3 \) large enough, we have
\[ \| Au \| \leq \| u \|, u \in P \cap \partial \Omega_4. \quad (9) \]

Thus from (8),(9) and Lemma 2, we know that the operator \( A \) has a fixed point in \( P \cap (\Omega_4 \setminus \Omega_3) \). The proof is complete.

4 Examples

In this section, we give three examples to illustrate our results.

**Examples 1.** Let \( f(x, v) = v^3, g(x, u) = u^3 \), then conditions of Theorem 1 are satisfied. From Theorem 1, BVP(1) has at least one positive solution.

**Examples 2.** Let \( f(x, v) = v^{1/3}, g(x, u) = u^{1/3} \), then conditions of Theorem 2 are satisfied. From Theorem 2, BVP(1) has at least one positive solution.

References


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