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# A Semi-symmetric Non-metric Connection in a Generalised Co-symplectic Manifold

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**Abstract**

In the present paper, we studied the properties of semi-symmetric non-metric connection in genralised co-symplectic manifold.

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**Keywords:** Semi-symmetric non-metric connection, Almost contact metric manifolds, Generalised co-symplectic manifold, Generalised quasi-Sasakian manifold

## 1 Introduction

Nirmala, Agashe and Chafle [1] defined a semi-symmetric non-metric connection in a Riemannian manifold and studied some of its properties. Recently, Ojha and one of the author [2] defined a new type of semi-symmetric non-metric connection in an almost contact metric manifold and studied its properties. In this paper, we studied the properties of this connection in generalised co-symplectic manifold.

## 2 Preliminaries

An $n$ dimensional differentiable manifold $M_n$ is an almost contact manifold [4] if it admits a tensor field $F$ of type $(1,1)$, a vector field $U$ and a $1-$form $u$

satisfying for arbitrary vector field $X$

$$(a) \quad \overline{\overline{X}} + X = A(X)T, \quad (b) \quad \overline{U} = 0, \tag{1}$$

where

$$\overline{X} \stackrel{def}{=} FX.$$

Again (1) (a) and (1) (b) gives

$$(a) \quad u(\overline{X}) = 0, \quad (b) \quad u(U) = 1. \tag{2}$$

An almost contact manifold $M_n$ in which a Riemannian metric tensor $g$ of type $(0, 2)$ satisfying

$$(a) \quad g(\overline{X}, \overline{Y}) = g(X, Y) - u(X)u(Y), \tag{3}$$

$$(b) \qquad g(X, U) = u(X),$$

for arbitrary vector fields $X$ and $Y$, is called an almost contact metric manifold.

Let us put

$$'F(X, Y) \stackrel{def}{=} g(\overline{X}, Y),$$

then we have

$$(a) \qquad 'F(\overline{X}, \overline{Y}) =' F(X, Y), \tag{4}$$

$$(b) \quad 'F(X, Y) = g(\overline{X}, Y) = -g(X, \overline{Y}) = -'F(Y, X).$$

An almost contact metric manifold satisfying

$$(D_X{}'F)(Y, Z) = u(Y)(D_X u)(\overline{Z}) - u(Z)(D_X u)(\overline{Y}), \tag{5}$$

$$\begin{aligned} &(D_X{}'F)(Y, Z) + (D_Y{}'F)(Z, X) + (D_Z{}'F)(X, Y) \\ &+u(X)[(D_Y u)(\overline{Z}) - (D_Z u)(\overline{Y})] + u(Y)[(D_Z u)(\overline{X}) \\ &-(D_X u)(\overline{Z})] + u(Z)[(D_X u)(\overline{Y}) - (D_Y u)(\overline{X})] = 0 \end{aligned} \tag{6}$$

for arbitrary vector fields $X$, $Y$, $Z$; are respectively called generalised co-symplectic and generalised quasi-Sasakian manifolds [3].

If on any manifold, $U$ satisfies

$$(a) \quad (D_X u)(\overline{Y}) = -(D_{\overline{X}} u)(Y) = (D_Y u)(\overline{X}), \tag{7}$$

$$(b) \quad (D_X u)(Y) = (D_{\overline{X}} u)(\overline{Y}) = -(D_Y u)(X)$$

and

$$(c) \qquad (D_U F) = 0,$$

then $U$ is said to be of the first class and the manifold is said to be of the first class [3].

If on an almost contact metric manifold $U$ satisfies

$$(a) \quad (D_X u)(\overline{Y}) = (D_{\overline{X}} u)(Y) = -(D_Y u)(\overline{X}) \Leftrightarrow \tag{8}$$

$$(b) \quad (D_X u)(Y) = -(D_{\overline{X}} u)(\overline{Y}) = -(D_Y u)(X)$$

and

$$(c) \qquad (D_U F) = 0,$$

then $U$ is said to be of the second class and the manifold is said to be of the second class [3].

The Nijenhuis tensor in generalised co-symplectic manifold is given by

$$(a) \quad N(X,Y) = (D_{\overline{X}} F)(Y) - (D_{\overline{Y}} F)(X) - \overline{(D_X F)(Y)} + \overline{(D_Y F)(X)} \tag{9}$$

$$(b) \quad {}'N(X,Y,Z) = (D_{\overline{X}}{}'F)(Y,Z) - (D_{\overline{Y}}{}'F)(X,Z) + (D_X{}'F)(Y,\overline{Z}) - (D_Y{}'F)(X,\overline{Z}).$$

# 3 Semi-symmetric non-metric connection

Let $D$ be a Riemannian connection, then an affine connection $B$ defined by

$$B_X Y = D_X Y - u(Y)X - g(X,Y)U \tag{10}$$

whose torsion tensor $T$ of $B$

$$T(X,Y) = u(X)Y - u(Y)X, \tag{11}$$

and metric tensor $g$ satisfies

$$(B_X g)(Y,Z) = 2[u(Y)g(X,Z) + u(Z)g(X,Y)] \tag{12}$$

for arbitrary vector fields $X$, $Y$, $Z$; then $B$ is called a semi-symmetric non-metric connection [2].

If we put

$$B_X Y = D_X Y + H(X,Y), \tag{13}$$

where $H$ is a tensor field of type $(1,2)$, then we have

$$(a) \qquad H(X,Y) = -u(Y)X - g(X,Y)U, \tag{14}$$

$$(b) \quad {}'H(X,Y,Z) = -u(Y)g(X,Z) - u(Z)g(X,Y),$$

$$(c) \quad {}'T(X,Y,Z) = u(X)g(Y,Z) - u(Y)g(X,Z),$$

and

$$(d) \quad (B_X u)(Y) = (D_X u)(Y) + u(X)u(Y) + g(X,Y),$$

where

$$'H(X,Y,Z)\overset{\text{def}}{=}g(H(X,Y),Z)$$
$$'T(X,Y,Z)\overset{\text{def}}{=}g(T(X,Y),Z).$$

We have

$$\begin{aligned} X('F(Y,Z)) &= (D_X{}'F)(Y,Z) +' F(D_XY,Z) +' F(Y,D_XZ) \\ &= (\tilde{B}_X{}'F)(Y,Z) +' F(\tilde{B}_XY,Z) +' F(Y,\tilde{B}_XZ) \end{aligned}$$

Using (10) in the last expression, we get

$$(B_X{}'F)(Y,Z) = (D_X{}'F)(Y,Z) + u(Y)'F(X,Z) - u(Z)'F(X,Y). \tag{15}$$

In the almost contact metric manifold with semi-symmetric non-metric connection $B$ it can be seen that [5]

$$\begin{aligned} &(a) \quad 'H(\overline{\overline{X}},Y,Z) =' H(X,\overline{\overline{Y}},Z) +' H(X,Y,\overline{\overline{Z}}), \\ &(b) \quad 'H(X,Y,\overline{\overline{Z}}) =' T(\overline{\overline{X}},Y,Z), \\ &(c) \quad 'T(X,Y,\overline{\overline{Z}}) +' T(X,Y,Z) = 0, \\ &(d) \quad 'T(X,Y,\overline{\overline{Z}}) =' H(X,Y,\overline{\overline{Z}}) -' H(Y,X,\overline{\overline{Z}}) \\ &(e) \quad (B_Xu)(\overline{Y}) = (D_Xu)(\overline{Y}) -' F(X,Y). \end{aligned} \tag{16}$$

The Nijenhuis tensor $N$ in terms of semi-symmetric non-metric connection $B$ is given by

$$\begin{aligned} &(a) \quad N(X,Y) = (B_{\overline{X}}F)(Y) - (B_{\overline{Y}}F)(X) + \overline{(B_XF)(Y)} + \overline{(B_YF)(X)}, \\ &(b) \quad 'N(X,Y,Z) = (B_{\overline{X}}{}'F)(Y,Z) - (B_{\overline{Y}}{}'F)(X,Z) \\ &\qquad + (B_X{}'F)(Y,\overline{Z}) - (B_Y{}'F)(X,\overline{Z}). \end{aligned} \tag{17}$$

**Theorem 3.1** *An almost contact metric manifold with semi-symmetric non-metric connection $B$ satisfies the relation*

$$'T(\overline{X},Y,\overline{Z}) +' T(Y,\overline{Z},\overline{X}) =' H(\overline{X},\overline{Y},Z) -' H(\overline{X},Z,\overline{Y}) \tag{18}$$

**Proof** Barring $X$ and $Z$ in (14) (c) and using (2) (a) and (3) (a), we have

$$'T(\overline{X},Y,\overline{Z}) = -u(Y)g(X,Z) + u(X)u(Y)u(Z) \tag{19}$$

Also by virtue of (2) (a), (3) (a) and (14) (c), we obtain

$$'T(Y,\overline{Z},\overline{X}) = u(Y)g(Z,X) - u(X)u(Y)u(Z) \tag{20}$$

From (2) (a), (3) (a) and (14) (b), we have

$$'H(\overline{X},\overline{Y},Z) = -u(Z)g(X,Y) + u(X)u(Y)u(Z) \tag{21}$$

and

$$'H(\overline{X},Z,\overline{Y}) = -u(Z)g(X,Y) + u(X)u(Y)u(Z) \tag{22}$$

In consequence of (19), (20), (21) and (22), we obtain (18).

**Theorem 3.2** *A generalised co-symplectic manifold with semi-symmetric non-metric connection $B$ satisfies the relations*

$$(a)\quad (B_X{'}F)(Y,\overline{Z}) = u(Y)(B_Xu)(\overline{\overline{Z}}) + 2'H(X,Y,\overline{\overline{Z}}), \tag{23}$$

$$(b)\quad (B_X{'}F)(Y,\overline{Z}) = u(Y)(B_Xu)(\overline{\overline{Z}}) + 2'T(\overline{\overline{X}},Y,Z),$$

$$(c)\quad (B_X{'}F)(\overline{Y},Z) = -u(Z)(B_Xu)(\overline{\overline{Y}}) - 2'H(X,Z,\overline{\overline{Y}}).$$

**Proof** In consequence of (2) (a), (14) (b) and (15), we have

$$(B_X{'}F)(Y,Z) = (D_X{'}F)(Y,Z) +' H(X,Y,\overline{Z}) -' H(X,Z,\overline{Y}) \tag{24}$$

Barring $Z$ in (24) and using (2) (a), (5) and (14) (b), we get

$$(B_X{'}F)(Y,\overline{Z}) = u(Y)(D_Xu)(\overline{\overline{Z}}) +' H(X,Y,\overline{\overline{Z}})$$

Using (14) (b), (14) (d) in the above expression, we get (23) (a). (23) (b) is obtained by using (16) (b) and (23) (a). Now barring $Y$ in (24) and using (2) (a), (5) and (14) (d), we find (23) (c).

**Theorem 3.3** *If $U$ is killing on generalised co-symplectic manifold with semi-symmetric non-metric connection $B$, then*

$$(B_X{'}F)(Y,\overline{Z}) + (B_Y{'}F)(\overline{Z},X) + (B_{\overline{Z}}{'}F)(X,Y) =' N(X,Y,Z) \tag{25}$$

$$-2u(Y)'F(\overline{X},Z) - 2u(Z)(D_{\overline{X}}u)(\overline{Y}) + 2u(X)g(\overline{\overline{Y}},Z).$$

**Proof** From (17) (b), we have

$$'N(X,Y,Z) - (B_X{'}F)(Y,\overline{Z}) - (B_Y{'}F)(\overline{Z},X) - (B_{\overline{Z}}{'}F)(X,Y)$$

$$= (B_{\overline{X}}{'}F)(Y,Z) - (B_{\overline{Y}}{'}F)(X,Z) - (B_{\overline{Z}}{'}F)(X,Y)$$

Using (4) (b), (5) and (15) in the above equation, we get

$$'N(X,Y,Z) - (B_X{'}F)(Y,\overline{Z}) - (B_Y{'}F)(\overline{Z},X) - (B_{\overline{Z}}{'}F)(X,Y)$$

$$= u(Y)[(D_{\overline{X}}u)(\overline{Z}) + (D_{\overline{Z}}u)(\overline{X})] - 2u(X)g(\overline{\overline{Y}},Z) + 2u(Y)'F(\overline{X},Z)$$
$$-u(X)[(D_{\overline{Y}}u)(\overline{Z}) + (D_{\overline{Z}}u)(\overline{Y})] + u(Z)[(D_{\overline{Y}}u)(\overline{X}) - (D_{\overline{X}}u)(\overline{Y})]$$

Since $U$ is killing then putting $(D_Xu)(Y) + (D_Yu)(X) = 0$ in the above equation, we obtain (25).

**Theorem 3.4** *If the generalised co-symplectic manifold is of first class with respect to the Riemannian connection $D$, then it is also first class with respect to the semi-symmetric non-metric connection $B$ and satisfies the equation*

$$(B_X{'}F)(Y,Z) = u(Y)(B_Zu)(\overline{X}) + u(Z)(B_{\overline{X}}u)(Y) \tag{26}$$

**Proof** Barring $X$ and $Y$ in (14) (d) respectively and then using (2) (a), we find

$$(D_{\overline{X}}u)(Y) = (B_{\overline{X}}u)(Y) - g(\overline{X}, Y) \tag{27}$$

and

$$(D_X u)(\overline{Y}) = (B_X u)(\overline{Y}) - g(X, \overline{Y}) \tag{28}$$

Adding (27) and (28), we obtain

$$(D_{\overline{X}}u)(Y) + (D_X u)(\overline{Y}) = (B_{\overline{X}}u)(Y) + (B_X u)(\overline{Y}) \tag{29}$$

In view of (7) (a) and (29), we get

$$(B_{\overline{X}}u)(Y) = -(B_X u)(\overline{Y}) \tag{30}$$

Again in similar way, we have

$$(B_X u)(\overline{Y}) = (B_Y u)(\overline{X}) \tag{31}$$

From (30) and (31), we find

$$(B_X u)(\overline{Y}) = -(B_{\overline{X}}u)(Y) = (B_Y u)(\overline{X}) \tag{32}$$

Taking covariant derivative of $FY = \overline{Y}$ with respect to $B$ and using (1) (b), (2) (a) and (10), we get

$$(B_X F)(Y) + F(D_X Y) - u(Y)\overline{X} = (D_X \overline{Y}) - g(X, \overline{Y})U \tag{33}$$

Now using $(D_X F)(Y) + F(D_X Y) = D_X \overline{Y}$ in (33), we get

$$(B_X F)(Y) = (D_X F)(Y) + u(Y)\overline{X} - g(X, \overline{Y})U \tag{34}$$

Replacing $X$ by $U$ in (34) and using (1) (b), (2) (a), (3) (b) and (7) (c), we get

$$B_U F = 0$$

For (26), using (5), (15) and (16) (e), we obtain

$$(B_X{}'F)(Y, Z) = u(Y)(B_Z u)(\overline{X}) + u(Z)(B_{\overline{X}}u)(Y)$$

Hence the theorem.

**Theorem 3.5** *An almost contact metric manifold admitting a semi-symmetric non-metric connection $B$ is a generalised co-symplectic manifold if*

$$(B_X{}'F)(Y, Z) = u(Y)[(B_X u)(\overline{Z}) + 2'F(X, Z)] \tag{35}$$

$$-u(Z)[(B_X u)(\overline{Y}) + 2'F(X, Y)]$$

**Proof** From (5) and (15), we have

$$(B_X{}'F)(Y,Z) = u(Y)[(D_Xu)(\overline{Z}) +' F(X,Z)] - u(Z)[(D_Xu)(\overline{Y}) +' F(X,Y)]$$

Using (14) (d), we at once obtain (35).

**Theorem 3.6** *A generalised co-symplectic manifold equipped with semi-symmetric non-metric connection $B$ is completely integrable.*

**Proof** In view of (2) (a) and (35), (17) (b) becomes

$$\begin{aligned}'N(X,Y,Z) = u(Y)[(B_{\overline{X}}u)(\overline{Z}) + (B_Xu)(\overline{\overline{Z}})] + u(Z)[(B_{\overline{Y}}u)(\overline{X}) \\ - (B_{\overline{X}}u)(\overline{Y})] - u(X)[(B_{\overline{Y}}u)(\overline{Z}) + (B_Yu)(\overline{\overline{Z}})]\end{aligned}$$

Barring $X$, $Y$ and $Z$ in the last expression and using (2) (a), we get

$$'N(\overline{X},\overline{Y},\overline{Z}) = 0$$

Hence the theorem.

**Theorem 3.7** *If $F$ is killing, then on generalised co-symplectic manifold with semi-symmetric non-metric connection $B$, we have*

$$(B_Xu)(\overline{Z}) + 2'F(X,Z) = 0 \tag{36}$$

**Proof** Since $F$ is killing, therefore

$$(B_X{}'F)(Y,Z) + (B_Y{}'F)(X,Z) = 0 \tag{37}$$

In consequence of (35), (37) becomes

$$\begin{aligned}u(X)[(B_Yu)(\overline{Z}) + 2'F(Y,Z)] + u(Y)[(B_Xu)(\overline{Z}) \\ + 2'F(X,Z)] - u(Z)[(B_Xu)(\overline{Y}) + (B_Yu)(\overline{X})] = 0\end{aligned} \tag{38}$$

Putting $U$ for $Y$ in (38) and then using (1) (b), (2) (a), (2) (b) and (3) (b), we obtain

$$(B_Xu)(\overline{Z}) + 2'F(X,Z) + u(X)(B_Uu)(\overline{Z}) - u(Z)(B_Uu)(\overline{X}) = 0 \tag{39}$$

Again putting $U$ for $X$ and using (1) (b), (2) (a), (2) (b) and (3) (b), we get

$$(B_Uu)(\overline{Z}) = 0 \tag{40}$$

From (39) and (40), we get the result.

**Theorem 3.8** *A generalised co-symplectic manifold of first class with semi-symmetric non-metric connection $B$ satisfy*

$$(B_{\overline{X}}{}'F)(Y,Z) - (B_{\overline{Y}}{}'F)(Z,X) + (B_{\overline{Z}}{}'F)(X,Y) = 0$$

**Proof** By virtue of (5) and (15), we have

$$(B_{\overline{X}}{}'F)(Y,Z) = u(Y)[(D_{\overline{X}}u)(\overline{Z}) - g(\overline{X},\overline{Z})] - u(Z)[(D_{\overline{X}}u)(\overline{Y}) - g(\overline{X},\overline{Y})] \tag{41}$$

Taking cyclic sum of (41) in $X$, $Y$, $Z$ and then using (3) (a), (3) (b) and (7) (a), we obtain the required result.

# 4 Semi-symmetric non-metric connection on quasi-Sasakian manifold

**Theorem 4.1** *A quasi-Sasakian manifold is normal if and only if*

$$(B_X{}'F)(Y,Z) = u(Y)(B_Z u)(\overline{X}) + u(Z)[(B_{\overline{X}}u)(Y) - 2'F(X,Y)], \tag{42}$$

*where $B$ being semi-symmetric non-metric connection.*

**Proof** The necessary and sufficient condition that a quasi-Sasakian manifold to be normal [3] is

$$(D_X{}'F)(Y,Z) = u(Y)(D_Z u)(\overline{X}) + u(Z)(D_{\overline{X}}u)(Y)$$

Using (2) (a), (14) (d) and (17) (e), we get (42).

**Theorem 4.2** *A generalised co-symplectic manifold is quasi-Sasakian manifold if*

$$(B_X{}'F)(U,Y) = (B_Y{}'F)(U,X) + 2'F(X,Y), \tag{43}$$

*where $B$ being a semi-symmetric non-metric connection.*

**Proof** From (15), we have

$$\begin{aligned}
&(D_X{}'F)(Y,Z) + (D_Y{}'F)(Z,X) + (D_Z{}'F)(X,Y) \\
&= (B_X{}'F)(Y,Z) - u(Y)'F(X,Z) + u(Z)'F(X,Y) \\
&+(B_Y{}'F)(Z,X) - u(Z)'F(Y,X) + u(X)'F(Y,Z) \\
&+(B_Z{}'F)(X,Y) - u(X)'F(Z,Y) + u(Y)'F(Z,X)
\end{aligned}$$

Using (35) in the above expression, we find

$$\begin{aligned}
&(D_X{}'F)(Y,Z) + (D_Y{}'F)(Z,X) + (D_Z{}'F)(X,Y) \\
&= u(Y)[(B_X{}'F)(U,Z) - 2'F(X,Z) - (B_Z{}'F)(U,X)] \\
&+u(Z)[(B_Y{}'F)(U,X) - 2'F(Y,X) - (B_X{}'F)(U,Y)] \\
&+u(X)[(B_Z{}'F)(U,Y) - 2'F(Z,Y) - (B_Y{}'F)(U,Z)]
\end{aligned} \tag{44}$$

Since manifold is quasi-Sasakian, therefore

$$(D_X{}'F)(Y,Z) + (D_Y{}'F)(Z,X) + (D_Z{}'F)(X,Y) = 0 \tag{45}$$

From (44) and (45), we get at once (43).

**Theorem 4.3** *Let $D$ be the Riemannian connection and $B$ be a semi-symmetric non-metric connection. Then an almost contact metric manifold is a generalised quasi-Sasakian manifold of the first kind if*

$$(B_X{}'F)(Y,Z) + (B_Y{}'F)(Z,X) + (B_Z{}'F)(X,Y) \tag{46}$$

$$+4[u(X)'F(Y,Z) - u(Y)'F(Z,X) + u(Z)'F(X,Y)] = 0$$

**Proof** From (15), we have

$$(D_X{}'F)(Y,Z) = (B_X{}'F)(Y,Z) - u(Y)'F(X,Z) + u(Z)'F(X,Y) \tag{47}$$

Taking covariant derivative of $u(\overline{Z}) = 0$ with respect to $D$ and using (10), we obtain

$$(D_X u)(\overline{Z}) = (B_X u)(\overline{Z}) - g(X,\overline{Z}) \tag{48}$$

Using (32), (47) and (48) in (6), we get the required result.


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