On Product of Pseudo-Differential Type Operators

Involving Hankel Type Convolution

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Abstract: In this paper by using Hankel type transform two symbols are defined and pseudo-differential type operators $M(x,D)$ and $N(x,D)$ associated with the Bessel type operator $\Lambda_{\alpha,\beta}$ defined by equation (2.1) in terms of these symbols are defined. Then product of $M(x,D)$ and $N(x,D)$ is defined. It is shown that the pseudo-differential type operator and the product of pseudo-differential type operators are bounded in a certain Sobolev type space associated with the Hankel type transform. Finally, some special cases are studied.

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1. Introduction

In recent years, many authors have extended Hankel transformation

$$\left( h_\mu \phi \right)(x) = \int_0^\infty (xy)^{\frac{\mu}{2}} J_\mu(xy)\phi(y)dy, \phi \in L^1(0,\infty) \quad (1.1)$$

to distributions belonging to $H'_\mu$, the dual of the test function space $H_\mu$ satisfying certain condition on $I = (0,\infty)$. Zemanian[16] has studied these transformation in his monograph. Waphare[11,13] has investigated Hankel type transformation

$$\left( h_{\alpha,\beta} \phi \right)(x) = \int_0^\infty (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\phi(y)dy \quad (1.2)$$
and has been extended to distributions belonging to the dual space $H'_{\alpha,\beta}$. He introduced the space $H_{\alpha,\beta}$ consisting of all complex valued infinitely differentiable functions $\phi$ defined on $I = (0, \infty)$ satisfying

$$\rho_{m,k}^{\alpha,\beta}(\phi) = \sup_{x \in I} |x^m (x^{-1} D_x)^k (x^{2\beta-1} \phi(x))| < \infty$$  \hspace{1cm} (1.3)

for all $m, k \in \mathbb{N}$, and $D_x \equiv \frac{d}{dx}$.

Pseudo-differential operators associated with a numerical valued symbol $a(x, y)$ were discussed by Waphare [11], Waphare [13,14] and Pathak [7]. One formula for such an operator appears as follows

$$h_{\alpha,\beta,a}(\psi)(x,y) = \int_0^\infty (xy)^{x^{\alpha+\beta} J_{\alpha-\beta}(xy)a(x,y)} U_{\alpha,\beta}(y)dy$$ \hspace{1cm} (1.4)

where

$$U_{\alpha,\beta}(y) = \int_0^\infty (xy)^{x^{\alpha+\beta} J_{\alpha-\beta}(xy)a(x,y)}u(x)(\alpha - \beta) \geq -1/2.$$ \hspace{1cm} (1.5)

We define the symbol $a(x, y)$ as the complex valued infinitely differentiable functions on $I \times I$ satisfying

$$(1 + x)^q \left( x^{-1} D_x \right) (y^{-1} D_y)^p a(x, y) \leq K_{p,q,m,q}(1 + y)^{m-p}$$ \hspace{1cm} (1.6)

for all $p, i, q \in \mathbb{N}$, $m$ is a fixed real number, and $D_y \equiv \frac{d}{dy}$. The class of all such symbols is denoted by $H^m$. If $a(x, y)$ satisfies (1.6) with $q = 0$ then the symbol class will be denoted by $H_0^m$.

2. Notations and Terminology

We shall use notations and terminology of Waphare [11,13] and Zemanian [16]. From Waphare [13, equation (2.1)], the differential operator $\Delta_{\alpha,\beta}$ is defined by

$$\Delta_{\alpha,\beta} = \Delta_{\alpha,\beta,x} = x^{2\beta-1} D_x x^{4\alpha} D_x x^{2\beta-1} = (2\beta - 1)(4\alpha + 2\beta - 2)x^{4(\alpha+\beta-1)} + 2(2\alpha + 2\beta - 1)x^{4\alpha+4\beta-3} D_x + x^{2(2\alpha+2\beta-1)} D_x^2$$ \hspace{1cm} (2.1)

If we set $\alpha = \frac{1}{4} + \frac{\mu}{2}, \beta = \frac{1}{4} - \frac{\mu}{2}$ in (2.1), it reduces to

$$x^{-\mu-1/2} D_x x^{2\mu+1} D_x x^{-\mu-1/2} \equiv S_{\mu} \equiv D_x^2 + \frac{1-4\mu^2}{4x^2}$$

which is a differential operator studied by Zemanian [16] and many others later on. From Waphare [11], we have for any $\psi \in H_{\alpha,\beta}$ that

$$h_{\alpha,\beta}(\Delta_{\alpha,\beta}(\psi)) = -y^2 h_{\alpha,\beta}(\psi)$$ \hspace{1cm} (2.2)

$$h_{\alpha,\beta}(x^{-1} D_x)^k (x^{2\beta-1} \psi) = \sum_{i=0}^k \binom{k}{i} (x^{-1} D_x)^i \psi (x^{-1} D_x)^{k-i} (x^{2\beta-1} \psi)$$ \hspace{1cm} (2.3)
where \( b_j \) are constants depending only on \((\alpha - \beta)\). Following Betancor and Marrero [1], we define the Hankel type translation \( \tau_x \phi \) of \( \phi \in H_{a,\beta} \) for \( x \in I \) as:

\[
(\tau_x \phi)(y) = \int_0^\infty \phi(z)D_{a,\beta}(x, y, z)dt, x, y \in I, \tag{2.5}
\]

and

\[
D_{a,\beta}(x, y, z) = \int_0^\infty t^{2\beta-1}i_{a-\beta}(xt)i_{a-\beta}(yt)i_{a-\beta}(z)dt, x, y, z \in I \tag{2.6}
\]

Applying the inversion formula to (2.6), we have

\[
\int_0^\infty t^{2a}i_{a-\beta}(zt)D_{a,\beta}(x, y, z)dz = i_{a-\beta}(xt)i_{a-\beta}(zt). \tag{2.8}
\]

Now we define the space \( L^p_\alpha(I), 1 \leq p \leq \infty \) as the space of those real valued measurable functions on I for which

\[
\|f\|_{L^p_\alpha} = \left[ \int_0^\infty |f(x)|^p d\sigma(x) \right]^{1/p} < \infty, 1 \leq p < \infty, \tag{2.9}
\]

\[
\|f\|_{L^\infty_\alpha} = \text{ess sup}_{0 < x < \infty} |f(x)| < \infty. \tag{2.10}
\]

From Betancor and Marrero [1, p.399] and Marrero and Betancor [6, p.353] we know that for every \( f, g \in L^1_\alpha \),

\[
f \# g(x) = \int_0^\infty f(y)(\tau_y g)(y)dy, x \in I. \tag{2.11}
\]

From Cholewinski [2], Haimo [3] and Hirschman [4], we know that if \( f \) and \( g \) are functions in \( L^1_\alpha(I) \) then

\[
\|f \# g\|_{L^1_\alpha} \leq \|f\|_{L^1_\alpha} \|g\|_{L^1_\alpha}, \tag{2.12}
\]

and

\[
(f \# g)(x) = (g \# f)(x) \text{ a.e.} \tag{2.13}
\]

Note that if \( f \in L^1_\alpha(I) \) and \( g \in L^q_\alpha(I) \) then \( (f \# g)(x) \) is defined for all \( x \) and is continuous (see Hirschman [4, p.308]). A more general result obtained by Hirschman [4, p.311] (see also Waphare [12]) is given in the following lemma.

**Lemma 2.1:** If \( f \in L^r_\alpha(I) \) and \( g \in L^q_\alpha(I) \) with \( 1 \leq p \leq \infty, 1 \leq q \leq \infty \), then for \( 0 < x < \infty, \)

\[
\|f \# g\|_{L^r_\alpha} \leq \|f\|_{L^r_\alpha} \|g\|_{L^q_\alpha} \tag{2.14}
\]

where

\[
1 \leq r \leq \infty, \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 > 0.
\]
One can easily infer that
\[
\left[ h_{\alpha,\beta}(f \ast g)(x) \right](t) = t^{2\beta-1}(h_{\alpha,\beta},f)(t)(h_{\alpha,\beta}g)(t).  \tag{2.15}
\]
and
\[
(f \ast g) \ast h(x) = f \ast (g \ast h)(x), f, g, h \in L^1.  \tag{2.16}
\]

3. Pseudo-differential type operator \( M(x, D) \)

We first define the pseudo-differential type operator \( M(x, D) \) in the following definition.

**Definition 3.1:**

\[
M(x, D) = \int_0^\infty (xz)^{\alpha+\beta} J_{\alpha-\beta}(xz) a(x, z)(h_{\alpha,\beta}u)(z)dz  \tag{3.1}
\]
where \( u \in H_{\alpha,\beta}(I), (\alpha - \beta) \geq -1/2 \) and
\[
a(x, z) = x^{2\beta-1} \int_0^\infty (x\lambda)^{\alpha+\beta} J_{\alpha-\beta}(x\lambda)V(\lambda, z)d\lambda  \tag{3.2}
\]
with the condition that for all \( \lambda \in I, z \in I, \)
\[
|V(\lambda, z)| \leq h(\lambda) \in L^1.  \tag{3.3}
\]
Now we prove a boundedness result for \( M(x, D) \) for which we need the following Sobolev type space (see Waphare \[13, \text{Definition 3.1}\]).

**Definition 3.2:** The space \( G^s_{\alpha,\beta, p}(I), s \in R, (\alpha - \beta) \in R \), is defined to be the set of all those elements \( u \in H'_{\alpha,\beta}(I) \) which satisfy
\[
\|u\|_{G^s_{\alpha,\beta, p}} = \left\| \eta^{s+2\beta-1} h_{\alpha,\beta}u \right\|_{L^p_z}, 1 \leq p < \infty.  \tag{3.4}
\]

**Theorem 3.3** (Boundedness of \( M(x, D) \)):

Let \( (\alpha - \beta) \geq -1/2 \), then
\[
\left\| M(x, D)u(x) \right\|_{G^s_{\alpha,\beta, p}} \leq \left\| h \right\|_{G^{2s}_{\alpha,\beta, p}} \left\| u \right\|_{G^s_{\alpha,\beta, p}}, u \in H_{\alpha,\beta}(I).  \]

**Proof:** From equations (3.1) and (3.2) we have
\[
M(x, D)u(x) = \int_0^\infty (xz)^{\alpha+\beta} J_{\alpha-\beta}(xz) \left[ x^{2\beta-1} \int_0^\infty (x\lambda)^{\alpha+\beta} J_{\alpha-\beta}(x\lambda)V(\lambda, z)d\lambda \right] (h_{\alpha,\beta}u)(z)dz.
\]
Therefore,
\[ h_{\alpha, \beta}(M(x, D)u(x)) \] (y)

\[ \int_0^{\infty} (yx)^{\alpha+\beta} J_{\alpha-\beta}(yx)M(x, D)u(x)dx \]

\[ = \int_0^{\infty} \int_0^{\infty} (yx)^{\alpha+\beta} J_{\alpha-\beta}(yx)(xz)^{\alpha+\beta} J_{\alpha-\beta}(x(z)\lambda^{x2\beta-1}(x\lambda)^{\alpha+\beta} J_{\alpha-\beta}(x\lambda)V(x, z)(h_{\alpha, \beta}u)(z)dzd\lambda dx \]

\[ = \int_0^{\infty} \int_0^{\infty} V(\lambda, z)(h_{\alpha, \beta}u)(z)x^{2\beta-1}(yx)^{\alpha+\beta} J_{\alpha-\beta}(yx)(xz)^{\alpha+\beta} J_{\alpha-\beta}(x(z)\lambda^{x2\beta-1}(x\lambda)^{\alpha+\beta} J_{\alpha-\beta}(x\lambda)dzd\lambda dx \]

Using (2.6) and (2.7), the last expression can be written as

\[ \int_0^{\infty} \int_0^{\infty} V(x, z)(h_{\alpha, \beta}u)(z)dzd\lambda \]

Now using inequality (3.3), we can write

\[ \left\| h_{\alpha, \beta}(M(x, D)u(x)) \right\| (y) \leq \int_0^{\infty} h(\lambda) \left\| h_{\alpha, \beta}(u)(z) \right\| D_{\alpha, \beta}(y, z, \lambda)dzd\lambda . \]

In view of (2.5) and (2.9) we have

\[ \left\| h_{\alpha, \beta}(M(x, D)u(x)) \right\| (y) \leq (h \# h_{\alpha, \beta}u)(z)(y) . \]

Hence

\[ \int_0^{\infty} \left\| h_{\alpha, \beta}(M(x, D)u(x)) \right\| (y)d\sigma(y) \leq \int_0^{\infty} (h \# h_{\alpha, \beta}u)(y)d\sigma(y) . \]

Now using Definition 3.2 and the inequality (2.12) we get

\[ \| M(x, D)u(x) \|_{G_{\alpha, \beta}} \leq \| h \|_{L^1} \| u \|_{G_{\alpha, \beta}} , \quad u \in H_{\alpha, \beta}(I) . \]

4. Pseudo-differential type operator \( N(y, D) \)

**Definition 4.1**: The pseudo-differential type operator \( N(y, D) \) for \( (\alpha - \beta) \geq -1/2 \), is defined by

\[ N(y, D)u(y) = \int_0^{\infty} (yt)^{\alpha+\beta} J_{\alpha-\beta}(yt) b(y, t)(h_{\alpha, \beta}u)(t)dt , \quad u \in H_{\alpha, \beta}(I) , \] (4.1)

where

\[ b(y, t) = y^{2\beta-1} \int_0^{\infty} (ys)^{\alpha+\beta} J_{\alpha-\beta}(ys)W(s, t)ds \] (4.2)

with the condition that for all \( s, t \in I \) and \( (\alpha - \beta) \geq -1/2 \),
\[ |W(s,t)| \leq k(s) \in L^1_{\alpha}(I). \] (4.3)

**Theorem 4.2 (Boundedness of \(N(y, D)\)):** Let \((\alpha - \beta) \geq -1/2\), then
\[
\|N(y, D)u\|_{g^{2\alpha}_{a,b}} \leq \|k\|_{\alpha} \|p\|_{g^{2\beta}_{a,b}}, u \in H_{a,b}(I).
\]

**Proof:** proceeding as in the proof of Theorem 3.3 we find
\[
\left[ h_{a,b}(N(y, D)u(y))(x) \right] = \int \int (xy)^{a+\beta} J_{a-\beta}(xy) [N(y, D)u(y)] dy
\]
\[
= \int \int \int \int (xy)^{a+\beta} J_{a-\beta}(xy)(yt)^{a+\beta} J_{a-\beta}(yt)y^{2\beta-1}(ys)^{a+\beta} J_{a-\beta}(ys)W(s,t)(h_{a,b}u(t))dsdtddy
\]
Now using (2.5), (2.6), (2.7), (2.9) and (4.3), we have
\[
\left[ h_{a,b}(N(y, D)u(y))(x) \right] \leq (k \# h_{a,b}u)(x) \] (4.4)
from which the assertion follows.

### 5. Product of pseudo-differential type operators

**Definition 5.1:** The product of two pseudo-differential operators \(M(x, D)\) and \(N(y, D)\) associated with the symbols \(a(x, z)\) and \(b(y, t)\) respectively is defined by
\[
M(x, D)N(y, D)u(x) = \int (xz)^{a+\beta} J_{a-\beta}(xz)a(x, z)h_{a,b}(N(y, D)u)(z)dz.
\]
From (3.2), (4.1) and (4.2), we have
\[
M(x, D)N(x, D)u(x) = \int \int \int \int (xz)^{a+\beta} J_{a-\beta}(xz)x^{2\beta-1}(x\lambda)^{a+\beta} J_{a-\beta}(x\lambda)V(\lambda, z)
\]
\[
\times (zy)^{a+\beta} J_{a-\beta}(zy)(yt)^{a+\beta} J_{a-\beta}(yt)y^{2\beta-1}(ys)^{a+\beta} J_{a-\beta}(ys)
\times W(s,t)(h_{a,b}u(t))d\lambda dsdtddydz
\]
provided the multiple integral exists.

**Theorem 5.2:** Let \((\alpha - \beta) \geq -1/2\). Then
\[
\|M(x, D)N(x, D)u(x)\|_{g^{2\alpha}_{a,b}} \leq \|h\|_{\alpha} \|k\|_{\beta} \|p\|_{g^{2\beta}_{a,b}}, u \in H_{a,b}(I). \] (5.1)

**Proof:** From Definition 5.1 and (3.2), we have
\[
M(x, D)N(x, D)u(x) = \int (xz)^{a+\beta} J_{a-\beta}(xz)x^{2\beta-1}\int (x\lambda)^{a+\beta} J_{a-\beta}(x\lambda)V(\lambda, z)d\lambda
\]
\[
\times h_{a,b}(N(y, D)u)(z)dz
\]
Therefore,
Product of pseudo-differential type operators

\[ h_{\alpha,\beta}(M(x, D)N(x, D)u(x))(t) = \int_0^\infty (tx)^{\alpha+\beta} J_{\alpha+\beta}(tx)(M(x, D)N(y, D)u(x)dx \]

\[ = \int_0^\infty \int_0^\infty \int_0^\infty (tx)^{\alpha+\beta} J_{\alpha+\beta}(tx)(xz)^{\beta-1}(x\lambda)^{\alpha+\beta} \]

\[ \times J_{\alpha+\beta}(x\lambda)V(\lambda, z)h_{\alpha,\beta}(N(y, D)u(z)dz d\lambda dx \]

\[ = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty (tx)^{\alpha+\beta} J_{\alpha+\beta}(tx) \]

\[ \times (xz)^{\alpha+\beta} J_{\alpha+\beta}(xz)(x\lambda)^{\alpha+\beta} J_{\alpha+\beta}(x\lambda)dz d\lambda dx \]

Using (2.6) and (2.7), we have

\[ h_{\alpha,\beta}(M(x, D)N(x, D)u(x))(t) = \int_0^\infty V(\lambda, z)h_{\alpha,\beta}(N(y, D)u(z)D_{\alpha+\beta}(t, z, \lambda)dz d\lambda \]

Using inequality (3.3) and (4.4), we have

\[ \left| h_{\alpha,\beta}(M(x, D)N(x, D)u(x))(t) \right| \leq \int_0^\infty h(\lambda)(k \# h_{\alpha,\beta}u)(z)D_{\alpha+\beta}(t, z, \lambda)dz d\lambda \]

\[ = (h \#(k \# h_{\alpha,\beta}u))(t) \quad (5.2) \]

Thus

\[ \int_0^\infty \left[ h_{\alpha,\beta}(M(x, D)N(x, D)u(x))(t) \right] d\sigma(t) \leq \int_0^\infty (h \#(k \# h_{\alpha,\beta}u))(t) d\sigma(z). \]

Therefore

\[ \| h_{\alpha,\beta}(M(x, D)N(x, D)u(x)) \|_{L^p} \leq \| h \|_{L^p} \| k \#(h_{\alpha,\beta}u) \|_{L^p} \]

\[ \leq \| h \|_{L^p} \| k \|_{L^p} \| h_{\alpha,\beta}u \|_{L^p}. \]

Now using Definition 3.2, we have

\[ \| M(x, D)N(x, D)u(x) \|_{L^p_{\alpha+\beta}} \leq \| h \|_{L^p} \| k \|_{L^p} \| h_{\alpha,\beta}u \|_{L^p_{\alpha+\beta}}, u \in H_{\alpha,\beta}(I). \]

**Theorem 5.3:** Let \( h \in L^1_{\alpha} \cap L^p_{\tau}, k \in L^1_{\tau} \cap L^s_{\tau} \), then

\[ \| h_{\alpha,\beta} [M(x, D)N(x, D)u(x)] \|_{L^p_{\alpha+\beta}} \leq \| h \|_{L^p} \| k \|_{L^p} \| h_{\alpha,\beta}u \|_{L^p_{\alpha+\beta}}, \]

where \( \frac{1}{r} = \frac{1}{p} + \frac{1}{\tau} + \frac{1}{s} - 2 > 0 \).

**Proof:** From (5.2), we have

\[ \left[ \int_0^\infty \left[ h_{\alpha,\beta}(M(x, D)N(x, D)u(x)) \right] d\sigma(z) \right]^{1/r} \]

\[ \leq \| h \#(k \#(h_{\alpha,\beta}u)) \|_{L^p_{\alpha+\beta}} \]

\[ \leq \| h \|_{L^p} \| k \#(h_{\alpha,\beta}u) \|_{L^p_{\alpha+\beta}}. \]
\[ \leq \|h\|_{\mathcal{L}_p} \|k\|_{\mathcal{L}_q} \|h_{\alpha,\beta}u\|_{\mathcal{L}_r}. \]

Therefore,
\[ \|h_{\alpha,\beta}[M(x, D)N(x, D)u(x)]\|_{\mathcal{L}_r} \leq \|h\|_{\mathcal{L}_p} \|k\|_{\mathcal{L}_q} \|h_{\alpha,\beta}u\|_{\mathcal{L}_r}. \]

6. Some special cases

**Theorem 6.1**: Let \( W(s, t) = W_1(s)W_2(t) \).

Then
\[ \|h_{\alpha,\beta}(N(y, D)u(y))\|_{\mathcal{L}_p} \leq \|W_1\|_{\mathcal{L}_p} \|W_2h_{\alpha,\beta}u\|_{\mathcal{L}_r}, \]

and
\[ N(y, D)u(y) = y^{2\beta-1}(h_{\alpha,\beta}W_1(y)(h_{\alpha,\beta}W_2h_{\alpha,\beta}u)(y). \]

**Proof**: From (4.1), (4.2) and (6.1) we have
\[ N(y, D)u(y) = \int_0^\infty (yt)^{\alpha+\beta} J_{\alpha-\beta}(yt) \left( \int_0^\infty (ys)^{\alpha+\beta} J_{\alpha-\beta}(ys)W_1(s)W_2(t)ds \right) (h_{\alpha,\beta}u)(t)dt \]
\[ = \int_0^\infty \int_0^\infty (yt)^{\alpha+\beta}(ys)^{\alpha+\beta} W_1(s)W_2(t)(h_{\alpha,\beta}u)(t)J_{\alpha-\beta}(yt)J_{\alpha-\beta}(ys)dt ds . \]

Using (2.8), we obtain
\[ N(y, D)u(y) = \int_0^\infty y^{2\beta-1}(yt)^{\alpha+\beta}(ys)^{\alpha+\beta} W_1(s)W_2(t)(h_{\alpha,\beta}u)(t)y^{2\beta-1}(ts)^{-(\alpha+\beta)} \]
\[ \times \int_0^\infty (zy)^{\alpha+\beta} J_{\alpha-\beta}(zy)D_{\alpha,\beta}(t, s, z)dzydzdt . \]
\[ = \int_0^\infty (zy)^{\alpha+\beta} J_{\alpha-\beta}(zy) \left( \int_0^\infty W_1(s)W_2(t)(h_{\alpha,\beta}u)(t)D_{\alpha,\beta}(t, s, z)dzydz \right) . \]
\[ = \int_0^\infty (zy)^{\alpha+\beta} J_{\alpha-\beta}(zy)(W_1 \# W_2h_{\alpha,\beta}u)(z)dz . \] (6.2)

By an application of the inverse Hankel type transform, we obtain
\[ \int_0^\infty (zy)^{\alpha+\beta} J_{\alpha-\beta}(zy)N(y, D)u(y)dy = (W_1 \# W_2h_{\alpha,\beta}u)(z) , \]

Or in the other words
\[ h_{\alpha,\beta}(N(y, D)u(y))(z) = (W_1 \# W_2h_{\alpha,\beta}u)(z) . \] (6.3)

Hence
\[ \int_0^\infty [h_{\alpha,\beta}(N(y, D)u(y))](z)d\sigma(z) \]
From (6.2), it follows that
\[ N(y, D)u(y) = \left[ h_{\alpha, \beta}(W_1 \# W_2 h_{\alpha, \beta}u) \right](y). \]
Lastly using (2.15), we obtain
\[ N(y, D)u(y) = y^{2\beta - 1}(h_{\alpha, \beta}W_1(y))(h_{\alpha, \beta}W_2 h_{\alpha, \beta}u)(y). \]  

**Theorem 6.2:** Let \( W(s, t) = W_1(s)W_2(t) \) and \( V(\lambda, z) = V_1(\lambda)V_2(z) \), where \( V_2 = A, W_2 = B \) are assumed to be constants. Assume further that \( V_1(\lambda) \in L^1_\alpha \) and \( W_1(s) \in L^1_\sigma(I) \).

Then
\[ \|M(x, D)N(x, D)u(x)\|_{\alpha, \beta, 1} \leq \|V_1\|_{\alpha, \sigma, 1} \|W_1\|_{\alpha, \sigma, 1} \|W_2\|_{\alpha, \sigma, 1}, \]
and
\[ M(x, D)N(x, D)u(x) = ABx^{4\beta - 2}(h_{\alpha, \beta}W_1(x))(h_{\alpha, \beta}V_1(x))u(x) \]

**Proof:** From Definition 5.1 and equation (3.2), we have
\[ M(x, D)N(x, D)u(x) \]
\[ = \int_0^\infty (xz)^{\alpha+\beta} J_{\alpha-\beta}(xz) \left[ x^{2\beta-1} \int_0^\infty (x\lambda)^{\alpha+\beta} J_{\alpha-\beta}(x\lambda)V_1(\lambda)V_2(z) d\lambda \right] h_{\alpha, \beta}(N(y, D)u(y))(z) dz \]
\[ = \int_0^\infty (xz)^{\alpha+\beta} x^{2\beta-1}(x\lambda)^{\alpha+\beta} V_1(\lambda)V_2(z) h_{\alpha, \beta}(N(y, D)u(z)) J_{\alpha-\beta}(xz)J_{\alpha-\beta}(x\lambda) d\lambda dz. \]

By using (2.8) we get
\[ M(x, D)N(x, D)u(x) = \int_0^\infty (xz)^{\alpha+\beta} x^{2\beta-1}(x\lambda)^{\alpha+\beta} V_1(\lambda)V_2(z) h_{\alpha, \beta}(N(y, D)u(z)) x^{2\beta-1} \]
\[ \times (z\lambda)^{-(\alpha+\beta)} \int_0^\infty (tx)^{\alpha+\beta} J_{\alpha-\beta}(tx)D_{\alpha, \beta}(z, \lambda, t) dt d\lambda dz \]
which on using (6.3) gives
\[ M(x, D)N(x, D)u(x) = \int_0^\infty (tx)^{\alpha+\beta} J_{\alpha-\beta}(tx) \left[ \int_0^\infty \left[ V_2(z)(W_1 \# W_2 h_{\alpha, \beta}u)(z) \right] V_1(\lambda) \right] dt \]
\[ = \int_0^\infty (tx)^{\alpha+\beta} J_{\alpha-\beta}(tx) \left[ V_2(W_1 \# W_2 h_{\alpha, \beta}u) \# V_1 \right](t) dt. \]

Now an application of the inverse Hankel type transform yields
\[ \int_0^\infty (tx)^{\alpha+\beta} J_{\alpha-\beta}(tx)M(x, D)N(x, D)u(x) dx = \left[ V_2(W_1 \# W_2 h_{\alpha, \beta}u) \# V_1 \right](t) \]
so that
\[ \left[ h_{\alpha, \beta}(M(x, D)N(x, D)u(x)) \right](t) = \left[ A(W_1 \# Bh_{\alpha, \beta}u) \# V_1 \right](t). \]
Hence
\[
\int_{0}^{\infty} \left[ h_{\alpha,\beta}(M(x,D)N(x,D)u(x)) \right](t) d\sigma(t) = AB \int_{0}^{\infty} \left[ \left\| W_{1} \# h_{\alpha,\beta}u \right\| \left\| V_{1} \right\| \right](t) d\sigma(t)
\]

Therefore,
\[
\left\| h_{\alpha,\beta}(M(x,D)N(x,D)u(x)) \right\|_{L^{1}_{\sigma}} = AB \left\| W_{1} \# h_{\alpha,\beta}u \right\| \left\| V_{1} \right\|_{L^{1}_{\sigma}} 
\leq AB \left\| W_{1} \# h_{\alpha,\beta}u \right\| \left\| V_{1} \right\|_{L^{1}_{\sigma}}.
\]

This gives
\[
\left\| M(x,D)N(x,D)u(x) \right\|_{G^{2}_{\alpha,\beta,1}} \leq \left\| V_{1} \right\|_{L^{1}_{\sigma}} \left\| W_{1} \# h_{\alpha,\beta}u \right\| \left\| u \right\|_{G^{2}_{\alpha,\beta,1}}.
\]

From (6.6) and (2.15), we obtain
\[
M(x,D)N(x,D)u(x) = x^{2\beta-1} h_{\alpha,\beta}(V_{2}(W_{1} \# W_{2} h_{\alpha,\beta}u))(x)(h_{\alpha,\beta}V_{1})(x)
\]
\[
= Ax^{2\beta-1} h_{\alpha,\beta}(W_{1} \# W_{2} h_{\alpha,\beta}u))(x)(h_{\alpha,\beta}V_{1})(x)
\]
\[
= ABx^{4\beta-2}(h_{\alpha,\beta}W_{1})(x)(h_{\alpha,\beta}h_{\alpha,\beta}u)(x)(h_{\alpha,\beta}V_{1})(x)
\]
\[
= ABx^{4\beta-2}(h_{\alpha,\beta}W_{1})(x)(h_{\alpha,\beta}V_{1})(x)u(x).
\]

References


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